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Hasimoto surfaces in Galilean space G_3

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Abstract

In this article Hasimoto surfaces in Galilean space G_3 will be considered, Gauss curvature (K) and Mean curvature (H) of Hasimoto surfaces $\chi = \chi(s, t)$ will be investigated, some characterization of s-curves and t-curves of Hasimoto surfaces in Galilean space G_3 will be introduced. Example of Hasimoto surfaces will be illustrated.

Keywords: Galilean geometry, Hasimoto surface, Smoke ring equation

Mathematics Subject Classification: 53A35, 53B25, 53C42

Introduction

The geometry of Galilean is one of the Non Euclidean geometry which is very important in special Relativity. For more about Galilean geometry one can read [1–4].

The Galilean geometry is the geometry that is transferred from Euclidean geometry to special relativity. A long time ago curves and surfaces in Euclidean space were studied. Recently, mathematicians have begun to introduce curves and surfaces in Galilean spaces G_3 and G_4 the reader can see the following references [5–11].

Hasimoto surfaces are obtained when the motions of local speed of the curve is proportional to the local curvature of the curve. Hasimoto surfaces is studied in Minkowski 3-space reader can see [12]. Generated surfaces via inextensible flows of curves in R^3 are studied by Rawa and Samah [13]. Hasimoto surfaces were constructed by many mathematicians [3, 12, 14].

The position vector of the surface $\chi = \chi(s, t)$ is called Hasimoto surface if the relation $\chi_t = \chi_s \times \chi_{ss}$ hold.

In this article Hasimoto surfaces $\chi = \chi(s, t)$ in Galilean space G_3 will be introduced, Gauss curvature (K) and the Mean curvature (H) of Hasimoto surfaces will be obtained. Some conditions for the *s-parameter curves* and *t-parameter curves* of Hasimoto surfaces to be geodesic curves, or asymptotic lines in Galilean space G_3 will be given. Finally the necessary and sufficient conditions for the curves to be principal curves on the Hasimoto surfaces in G_3 will be introduced. Example of Hasimoto surfaces $\chi = \chi(s, t)$ in Galilean space G_3 will be illustrated.

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Preliminaries

Galilean space of dimension three (G_3), is defined to be the space due to Cayley–Klein model, equipped with the metric of signature $(0, 0, +, +)$ which is called projective metric. The triple (ω, f, I) are called The absolute of Galilean geometry where ω is defined to be the ideal plane (sometimes called the absolute plane), f is a line in the absolute plane ω which is called the absolute line and I is defined to be the elliptic involution point $(0, 0, x_2, x_3) \rightarrow (0, 0, x_3, -x_2)$.

If the plane contains f, it is called the Euclidean plane, if the plane does not contain f it is called isotropic plane, this means that planes $x =$ constant are Euclidean planes, i.e. the plane ω is Euclidean plane. A vector $v = (v_1, v_2, v_3)$ is called non-isotropic vector if the first component v_1 is not equal to zero. All vectors $v = (1, v_2, v_3)$ are unit non-isotropic vectors. The vectors $v = (0, v_2, v_3)$ are isotropic vectors.

In Galilean space G_3 we have four types of lines [1]:

1. Lines, which do not cross the absolute line f is called proper non-isotropic lines.
2. The lines, which not belong to the ideal plane ω but intersect the absolute line f is called the proper isotropic lines.
3. All lines of the ideal plane ω except f are called proper non-isotropic lines.
4. The absolute line f.

Suppose that $\overrightarrow{u} = (u_1, u_2, u_3)$ and $\overrightarrow{v} = (v_1, v_2, v_3)$ are two vectors in Galilean space G_3. Galilean scalar product in G_3 is

$$\langle \overrightarrow{u}, \overrightarrow{v} \rangle_{G3} = \begin{cases} u_1 v_1 & \textit{if } u_1 \neq 0 \textit{ or } v_1 \neq 0 \\ u_2 v_2 + u_3 v_3 & \textit{if } u_1 = 0 \textit{ and } v_1 = 0 \end{cases}$$

The norm of the vector $\overrightarrow{u} = (u_1, u_2, u_3)$ can be written as $\| \overrightarrow{u} \|_{G_3} = \sqrt{\langle \overrightarrow{u}, \overrightarrow{u} \rangle_{G_3}}$.

The vector product of $\overrightarrow{u} = (u_1, u_2, u_3)$ and $\overrightarrow{v} = (v_1, v_2, v_3)$ in Galilean space G_3 is defined by

$$\overrightarrow{u} \times \overrightarrow{v} = \begin{cases} \begin{vmatrix} 0 & e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} & \text{if } x_1 \neq 0 \text{ or } y_1 \neq 0. \\ \begin{vmatrix} e_1 & e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} & \text{if } x_1 = 0 \text{ and } y_1 = 0. \end{cases}$$

The curve $r(s) = (s, y(s), z(s))$ is called the admissible curve. The associated invariant trihedron (Frenet invariant) **T**, **N**, and **B** for $r(s)$ is given by the following equations.

$$\begin{aligned} \mathbf{T} &= \left(1, y', z'\right) \\ \mathbf{N} &= \frac{1}{k}\left(0, y'', z''\right) \\ \mathbf{B} &= \frac{1}{k}\left(0, -z'', y''\right) \end{aligned}$$

where **T** is the Tangent vector to $r(s)$, **N** is the Normal vector to $r(s)$, and **B** is the Binormal of $r(s)$.

Also $k(s)$ is called the *curvature function* of the admissible curve $r(s)$, and is denoted by the relation

$$k(s) = \sqrt{y''^2 + z''^2}$$

and $\tau(s)$ is the *torsion function* of the admissible curve $r(s)$ and is given by the following equation

$$\tau(s) = \frac{1}{k^2} \det\left(r'(s), r''(s), r'''(s)\right).$$

The Frenet equations in Galilean space G_3 for the a admissible curve $r(s)$ can be written as

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{bmatrix} = \begin{bmatrix} 0 & k & 0 \\ 0 & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix} \tag{1}$$

A C^n-*surface* M, $n \geq 1$, immersed in Galilean space $r : U \to M, U$ belongs to R^2, is denoted by $\chi(s,t) = (x(s,t), y(s,t), z(s,t))$.

First fundamental form for the surface $\chi(s,t)$ is denoted by I and is given by the following equation.

$$I = \left(g_1 ds + g_2 dt\right)^2 + \epsilon\left(h_{11} ds^2 + 2h_{12} ds dt + h_{22} dt^2\right)$$

where the symbols $g_i = x_i$ is the derivatives of the first coordinates function $x(s, t)$ with respect to s and t, and $h_{ij} = \tilde{r}_i.\tilde{r}_j$ the Euclidean inner product of the projection $\tilde{r}_k$ onto *yz-plane*. Furthermore,

$$\epsilon = \begin{cases} 0, & \text{if } ds : dt \text{ is non-isotropic} \\ 1, & \text{if } ds : dt \text{ is isotropic} \end{cases}$$

Gauss curvature K is denoted by

$$K = \frac{L_{11}L_{22} - L_{12}^2}{W^2} \tag{2}$$

Mean curvature H is given by

$$H = \frac{g_2^2 L_{11} - 2g_1 g_2 L_{12} + g_1^2 L_{22}}{2W^2} \tag{3}$$

where

$$W = \sqrt{(x_t z_s - x_s z_t)^2 + \left(x_s y_t - x_t y_s\right)^2}$$

and

$$L_{ij} = \frac{x_s r_{ij} - x_{ij} r_s}{x_s} .N, x_s = g_1 \neq 0, \tag{4}$$

The vector $N = \frac{1}{W}(0, x_t z_s - x_s z_t, x_s y_t - x_t y_s)$ is the normal vector to the surface $\chi(s,t)$.

$$S = \frac{1}{W}\left(0, x_t y_s - x_s y_t, x_t z_s - x_s z_t\right)$$

is called a side tangential vector which is tangent plane to surface M.

Main results

In this section we will introduce Frenet equations of curves in both directions s, and t parameters. For Hasimoto surface $\chi(s,t)$, we will obtain Gauss Curvature (K), Mean Curvature (H), and we will prove that Hasimoto surfaces are Weingarten surfaces. Also we obtain the necessary and sufficient conditions for the *t-curves* of Hasimoto surface $\chi(s,t)$ to be geodesic curves, or to be asymtotic curves. Also, we give conditions of the parameter curves to be lines of curvature. Finally, we give characterization for the *s-parameter curves* to be principal direction for Hasimoto surface $\chi(s,t)$. At the end of this section example of Hasimoto surface in Galilean space G_3 is introduced.

Theorem 1 *Let $\chi = \chi(s,t)$ be Hasimoto surface in Galilean space G_3 where $\chi = \chi(s,t)$ is admissible curve with unit speed for all t. The Frenet equations* $\mathbf{T}', \mathbf{N}'$ **and** $\mathbf{B}'$ *with respect to the parameter s is given by the following equations*

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \end{bmatrix} = \begin{bmatrix} 0 & k & 0 \\ 0 & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix} \tag{5}$$

The Frenet Equations $\mathbf{T}^{\cdot}, \mathbf{N}^{\cdot}$ and $\mathbf{B}^{\cdot}$ *with respect to the parameter t, is obtained by the following equations*

$$\begin{bmatrix} \mathbf{T}^{\cdot} \\ \mathbf{N}^{\cdot} \\ \mathbf{B}^{\cdot} \end{bmatrix} = \begin{bmatrix} 0 & -\tau k & 0 \\ 0 & 0 & -\tau^2 \\ 0 & \tau^2 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix} \tag{6}$$

where $k \neq 0$ is the curvature and τ is the torsion for the curve $\chi = \chi(s,t)\, \forall t$.

Proof

Frenet equations $\mathbf{T}', \mathbf{N}'$ **and** $\mathbf{B}'$ *with respect to s is given directly from Frenet equation in Galilean space G_3 (1). Suppose that we have the differentiable functions α, β, γ and η where*

$$\begin{bmatrix} \mathbf{T}^{\cdot} \\ \mathbf{N}^{\cdot} \\ \mathbf{B}^{\cdot} \end{bmatrix} = \begin{bmatrix} 0 & \alpha & \gamma \\ \beta & 0 & \eta \\ -\gamma & -\eta & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \end{bmatrix} \tag{7}$$

Our aim is to find α, β, γ and η functions interms of the curvature and torsion functions for the *s-curve* $\chi = \chi(s,t)$ for all t.

Using the conditions $\mathbf{T}_{ts} = \mathbf{T}_{st}$ and $\mathbf{N}_{ts} = \mathbf{N}_{st}$ we obtain

$$(\alpha_s - \gamma\tau)\mathbf{N} + (\alpha\tau + \gamma_s)\mathbf{B} = (k\beta)\mathbf{T} + (k_t)\mathbf{N} + k\eta\mathbf{B} \tag{8}$$

i.e.

$$\beta = 0, \alpha_s = \gamma\tau + k_t, \gamma_s = k\eta - \alpha\tau \tag{9}$$

From the condition $\chi_{st} = \chi_{ts}$ we get the the following equations

$$\gamma = 0, \eta = -\tau^2 \tag{10}$$

substituting from Eqs. (8, 9) we give the system in (6). □

In the following theorem we will prove that Gaussian curvature K for Hasimoto surface equal to zero and the mean curvature H is equal to $\dfrac{-(x_t^2 + 2\tau x_s x_t + \tau^2 x_s^2)}{2k}$.

Theorem 2 *Let $\chi = \chi(s,t) = (x(s,t), y(s,t), z(s,t))$ be a Hasimoto surface in Galilean space G_3 where s-curves of the Hasimoto surfaces $\chi(s,t)$ is curves with unit norm of the speed for all t, then the Gauss curvature K of $\chi(s,t)$ will be given form the relation*

$$K = 0 \tag{11}$$

and the Mean curvature H of $\chi(s,t)$ will be obtained from the relation

$$H = \frac{-\left(x_t^2 + 2\tau x_s x_t + \tau^2 x_s^2\right)}{2k} \tag{12}$$

k is the curvature function of *s-curves* of $\chi(s,t)$ for all t and $\tau(s)$ is the torsion function of *s-curves* of $\chi(s,t)$ for all t.

Proof

Suppose that $\chi(s,t) = (x(s,t), y(s,t), z(s,t))$ is a parametrization of the surface $\chi(s,t)$ where the parameters $s,t \in R$, and $x(s,t), y(s,t), z(s,t) \in C^3$. The normal of the surface is given by $N = -\mathbf{N}$

since $\chi_s = \mathbf{T}$ we obtain $\chi_{st} = -k\tau\mathbf{N}$ from the property of Hasimoto surfaces $r_t = k\mathbf{B}$, we have $r_{ts} = k_s\mathbf{B} - k\tau\mathbf{N}$ therefore $k_s = 0$. By using the statement (4) of the second fundamental form we give

$$L_{ij} = \begin{pmatrix} -k & k\tau \\ k\tau & -k\tau^2 \end{pmatrix}$$

hence, Gauss curvature K of Hasimoto surfaces $\chi(s,t)$ identically zero.

Mean curvature H of Hasimoto surface is given by

$$H = \frac{g_2^2 L_{11} - 2g_1 g_2 L_{12} + g_1^2 L_{22}}{2W^2} = \frac{-kx_t^2 - 2k\tau x_s x_t - k\tau^2 x_s^2}{2k^2}$$
$$= \frac{-\left(x_t^2 + 2\tau x_s x_t + \tau^2 x_s^2\right)}{2k}$$

□

Since Gauss Curvature of Hasimoto surfaces in Galilean space G_3 equal zero the following corollary is true.

Corollary 1 *Hasimoto surface $\chi(s,t)$ is a Weingarten surface in G_3.*

Proof
The identically Jacobi equation

$$\Phi(H,K) = K_t H_s - H_t K_s = 0$$

Therefore, Hasimoto surface $\chi(s,t)$ is Weingarten surface. □

The curve $r(s)$ is a geodesic curve if and only if it has geodesic curvature equal to zero ($k_g = 0$), the curve is called asymptotic is its normal curvature $k_n = 0$

In the following theorems we give some properties for the *s-curves* and *t-curves* of Hasimoto surface $\chi(s,t)$ to be geodesic curves and asymptotic curves in G_3.

Theorem 3 *Let $\chi(s,t)$ be a Hasimoto surface in G_3. Then the following statements are satisfied*

1. The *s-curves* of $\chi(s,t)$ are geodesic curves.
2. The *t-curves* of $\chi(s,t)$ are geodesic curves, $\Longleftrightarrow$ the curvature of the *t-curves* of $\chi(s,t)$ *equal to zero for all s* ($k_t = 0$).

Proof
1. For the s-curves of the Hasimoto $\chi(s,t)$ for all t, the geodesic curvature is obtain from the following relation

$k_g = S \cdot \chi_{ss} = (N \times \mathbf{T}) \cdot (k\mathbf{N}) = \mathbf{0}$, which proof the statement 1.

2. The geodesic curvature for the *t-curves* of the Hasimoto surface$\chi(s,t)$ for all s is $k_g = S \cdot \chi_{tt} = (-n \times \mathbf{T}) \cdot (k_t\mathbf{B} + k\tau^2\mathbf{N}) = k_t$. □

Theorem 4 *Suppose that $\chi(s,t)$ is Hasimoto surface in Galilean space G_3. Then the following statement are satisfied.*

1. *s-curves are asymptotic* $\Longleftrightarrow$ if $k = 0$ (*which means that s-curves not asymptotic curves*).
2. *t-curves are asymptotic curves of Hasimoto surface* $\chi(s,t) \Longleftrightarrow \tau = 0$.

Proof

1. Let $\chi(s,t)$ *be Hasimoto surface in Galilean space* G_3*. Since normal curvature* $k_n = N \cdot \chi_{ss} = -\mathbf{N} \cdot k\mathbf{N} = -k$*, then s-curves are asymptotic curves* $\Leftrightarrow k = 0$ *(imposble).*

2. For *t-curves* we have $\chi_{tt} = k_t\mathbf{B}+k\tau^2\mathbf{N}$, $k_n = -\mathbf{N} \cdot (k_t\mathbf{B}+k\tau^2\mathbf{N}) = -k\tau^2$ i.e. *t-curves* are asymptotic curves of Hasimoto surface $\Longleftrightarrow k\tau^2 = 0$ but $k \neq 0$ therefore $\tau^2 = 0$ this means that τ must equal zero. □

Corollary 2 *s-curves and t-curves of Hasimoto surface* $\chi = \chi(s,t)$ in G_3 *are said to be lines of curvature if and only if* $k\tau = 0$.

Proof

$F = M = \mathbf{0} \Leftrightarrow k\tau = 0$. □

Corollary 3 *If s-curves and t-curves of Hasimoto surfaces* $\chi(s,t)$ *in* G_3 *are asymptotic curves then s-curves and t-curves are lines of curvatures.*

Proof

From Theorem 4 above t-curves are asymptotic curves of Hasimoto surfaces $\Leftrightarrow \tau = 0$*. This implies* $k\tau = 0$ *which means that t-curves are lines of curvatures.* □

Principal direction are tangent directions of a curve $r(s)$ on a surface if the normal field of the surface satisfy $\det(\alpha^{\cdot}, N, N^{\cdot}) = 0$ this condition essential for principal directions in Euclidean space [15].

Theorem 5 *Let,* $\chi(s,t)$ *be Hasimoto surfaces in* G_3*, then*

1. *s-curves of Hasimoto surface* $\chi(s,t)$ *are principal direction for all t if and only if* $\tau = 0$.
2. *t-curves of Hasimoto surface* $\chi(s,t)$ *are principal direction.*

Proof

1. For s-parameter curves $\det(\chi_s, N, N_s) = \det(\mathbf{T}, -\mathbf{N}, -\mathbf{N}_s) = \tau \det(\mathbf{T}, \mathbf{N}, \mathbf{B})$.

Hence, $\det(\chi_s, N, N_s) = 0 \Longleftrightarrow \tau = 0$.

2. For t-parameter curves $\det(\chi_t, N, N_t) = \det(k\mathbf{B}, -\mathbf{B}, \tau^2\mathbf{B}) = 0$. □

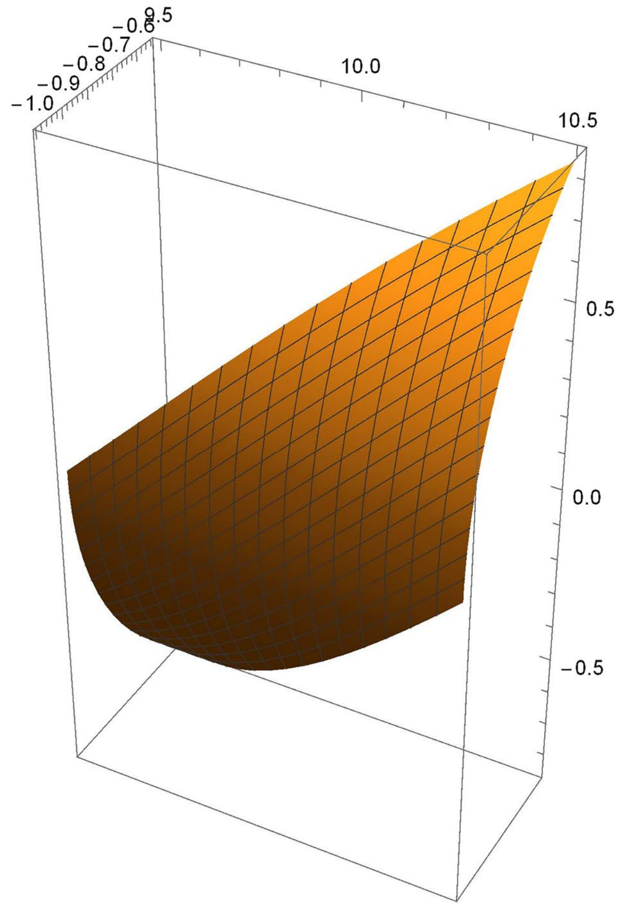

Fig. 1 Hasimoto surface $\chi(s,t) = (s+10, -\cos(s+t), \sin(s+t))$

Example 1

Consider Hasimoto surface (Fig. 1) $\chi(s,t)$ *where*

$\chi(s,t) = (s+10, -\cos(s+t), \sin(s+t)), -0.5 \leq s, t \leq 0.5$, then

The tangent vector for the curve is

$\mathbf{T} = (1, \sin(s+t), \cos(s+t))$

The normal vector for the curve is

$\mathbf{N} = (0, \cos(s+t), -\sin(s+t))$

The binormal vector for the curve is

$\mathbf{B} = (0, \sin(s+t), \cos(s+t))$

the curvature function $k = 1$, the torsion function $\tau = -1$

Mean curvature for $\chi(s,t)$ is $H = -1$

Abbreviations

G_3: Galilean space of dimension three; $k(s)$: Curvature function; $\tau(s)$: Torsion function; K: Gauss curvature; H: Mean curvature; N: The normal of the surface; k_g: The geodesic curvature.

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Authors' contributions
The author collected the data, performed the calculation, and was a major contributor in writing the manuscript. The author read and approved the final manuscript.

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