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Convolution conditions for two subclasses of analytic functions defined by Jackson q -difference operator

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Abstract

By using Jackson q -derivative, some characterizations in terms of convolutions for two classes of analytic functions in the open unit disc are given. Also, coefficient conditions and inclusion properties for functions in these classes are found.

Keywords: Jackson q -derivative, q -starlike and convex functions, Convolution, Inclusion relations, Salagean q -differential operator

Mathematics Subject Classification: Primary 30C45, 30C55

Introduction

Let A be the class of functions of the form:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (1)$$

which are analytic in the open unit disc

$$\mathcal{U} = \{z : z \in \mathcal{C}, |z| < 1\}.$$

Jackson q -derivative of a function f is given by (see [1, 2])

$$D_q f(z) = \begin{cases} \frac{f(qz) - f(z)}{(q-1)z} & (z \neq 0), \\ f'(0) & (z = 0). \end{cases} \quad (2)$$

If $f(z) = z^n$, then

$$D_q f(z) = D_q(z^n) = \frac{q^n - 1}{q - 1} z^{n-1} = [n]_q z^{n-1},$$

where

$$[n]_q := \frac{1 - q^n}{1 - q}$$

Using the observation

$$\lim_{q \rightarrow 1} \frac{1 - q^n}{1 - q} = n$$

we have

$$\lim_{q \rightarrow 1^-} D_q f(z) = \lim_{q \rightarrow 1^-} [n]_q z^{n-1} = n z^{n-1} = f'(z).$$

If the function f defined by (1), then

$$D_q f(z) = 1 + \sum_{k=2}^{\infty} [k]_q a_k z^{k-1}.$$

Also, we have the following q -derivative rules

$$D_q [f(z).g(z)] = g(z)D_q f(z) + f(qz)D_q g(z)$$

and

$$D_q \frac{f(z)}{g(z)} = \frac{g(z)D_q f(z) - f(z)D_q g(z)}{g(z)g(qz)}.$$

The q -difference operator D_q has been extensively investigated in the field of geometric function theory by many authors. For some recent works related to this operator on the classes of analytic functions, we refer to [3–7].

The Hadamard product (or convolution) of two functions $f, g \in A$, denoted by $f * g$, is

$$(f * g)(z) = f(z) * g(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k,$$

where f is given by (1) and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$. Govindaraj and Sivasubramanian [8] defined the differential operator $S_q^n f(z) : A \rightarrow A$ by:

$$\begin{aligned} S_q^0 f(z) &= f(z), \\ S_q^1 f(z) &= z D_q f(z) \end{aligned}$$

and for $n \in N_0 = \{0, 1, \dots\}$

$$\begin{aligned} S_q^n f(z) &= z D_q (S_q^{n-1} f(z)) \\ &= z + \sum_{k=2}^{\infty} [k]_q^n a_k z^k \\ &= (f * G_q^n)(z), \end{aligned} \tag{3}$$

where

$$G_q^n(z) = z + \sum_{k=2}^{\infty} [k]_q^n z^k.$$

The differential operator S_q^n is called Salagean q -differential operator.

Definition 1 For $f \in A$, we say f belongs to class $S_{\lambda, \zeta}^*(A, B)$, if and only if

$$\frac{(1 - \lambda)zD_\zeta f(z) + \lambda zD_\zeta(zD_\zeta f(z))}{(1 - \lambda)f(z) + \lambda zD_\zeta f(z)} \prec \frac{1 + Az}{1 + Bz} \quad (z \in U), \tag{4}$$

where $0 < \zeta < 1, 0 \leq \lambda \leq 1, -1 \leq B < A \leq 1, D_\zeta$ is Jackson q -derivative with $q = \zeta$ and \prec denotes the usual subordination (see [9–11]).

It is noticed that, by giving specific values to A, B and λ we obtain the following important subclasses studied by various authors in earlier works:

1. $S_{0,q}^*(1 - 2\alpha, -1) \equiv S_q^*(\alpha)$ and $S_{1,q}^*(1 - 2\alpha, -1) \equiv C_q(\alpha)$ are, respectively, the classes of q -starlike and q -convex functions (see Seoudy and Aouf [12] and Ramachandran et al. [13])
2. $S_{0,q}^*(A, B) \equiv S_q^*[A, B]$ and $S_{1,q}^*(A, B) \equiv C[A, B]$ the classes of q -starlike and q -convex functions which are associated with the Janowski functions (see Srivastava et al. [14])
3. $S_{0,q}^*(\frac{b^2 - a^2 + a}{b}, \frac{1 - a}{b}) \equiv S_q^*(\alpha)$, where $a = \frac{1 - \alpha q}{1 - q}$ and $b = \frac{1 - \alpha}{1 - q}$ (see Polatoglu et al. [15])
4. $\lim_{\zeta \rightarrow 1^-} S_{\alpha, \zeta}^*([b(1 + m) - m], -m) \equiv S_M^*(\alpha, b)$, where $b \in \mathbb{C}, m = 1 - \frac{1}{M}$ and $M > \frac{1}{2}$ (see Lashin [16])
5. $\lim_{\zeta \rightarrow 1^-} S_{0, \zeta}^*([b(1 + m) - m], -m) \equiv F^*(b, M)$, where $m = 1 - \frac{1}{M}$ and $M > \frac{1}{2}$ (see Nasr and Aouf [17])
6. $\lim_{\zeta \rightarrow 1^-} S_{0, \zeta}^*(1 - 2\alpha, -1) \equiv S^*(\alpha)$ and $\lim_{\zeta \rightarrow 1^-} S_{1, \zeta}^*(1 - 2\alpha, -1) \equiv C(\alpha)$ ($0 \leq \alpha < 1$) the classes of starlike and convex functions of order α (see Robertson [18]).
7. $\lim_{\zeta \rightarrow 1^-} S_{0, \zeta}^*(1, -1) \equiv S^*$ and $\lim_{\zeta \rightarrow 1^-} S_{1, \zeta}^*(1, -1) \equiv C$ the usual classes of starlike, convex and spirallike functions (see Goodman [19]).

With the help of the Salagean ζ -differential operator S_ζ^n given by (3), we say that a function $f \in A$ is in the class $S_{\lambda, \zeta}^*(n, A, B)$ if and only if

$$\frac{(1 - \lambda)S_\zeta^{n+1}f(z) + \lambda S_\zeta^{n+2}f(z)}{(1 - \lambda)S_\zeta^n f(z) + \lambda S_\zeta^{n+1}f(z)} \prec \frac{1 + Az}{1 + Bz} \quad (z \in U), \tag{5}$$

where $0 < \zeta < 1, -1 \leq B < A \leq 1, 0 \leq \lambda \leq 1$ and $n \in N_0$.

We note that

1. $\lim_{\zeta \rightarrow 1^-} S_{\lambda, \zeta}^*(n, 1 - 2\alpha, -1) \equiv S_n(\lambda, \alpha)$ (see Wang and Aghalary [20]).
2. $\lim_{\zeta \rightarrow 1^-} S_{\lambda, \zeta}^*(0, 1 - 2\alpha, -1) \equiv T(\lambda, \alpha)$ (see Altintas [21])
3. $\lim_{\zeta \rightarrow 1^-} S_{\lambda, \zeta}^*(1, 1 - 2\alpha, -1) \equiv C(\lambda, \alpha)$ (see Kamali and Akbulut [22])

Methods

The aim of the present paper is to obtain convolution properties, necessary and sufficient conditions, coefficient estimates, and inclusion properties of functions belonging to the class $S_{\lambda, \zeta}^*(n, A, B)$ with techniques similar to those used by Silverman et al. [23].

Results and discussion

Proposition 2 *If $f \in A$, then*

$$\frac{z}{(1-z)(1-\zeta z)} = z + \sum_{k=2}^{\infty} [k]_{\zeta} z^k \tag{6}$$

and

$$\frac{z}{(1-z)(1-\zeta z)(1-\zeta^2 z)} = z + \sum_{k=2}^{\infty} \frac{[k]_{\zeta} [k+1]_{\zeta}}{[2]_{\zeta}} z^k \tag{7}$$

Proof To prove the first part, we write

$$\frac{z}{(1-z)(1-\zeta z)} = z[1 + (1 + \zeta)z + (1 + \zeta + \zeta^2)z^2 + (1 + \zeta + \zeta^2 + \zeta^3)z^3 + \dots].$$

It is well Known that

$$\frac{1 - \zeta^n}{1 - \zeta} = 1 + \zeta + \dots + \zeta^{n-1}$$

then

$$\frac{z}{(1-z)(1-\zeta z)} = z + \frac{1 - \zeta^2}{1 - \zeta} z^2 + \frac{1 - \zeta^3}{1 - \zeta} z^3 + \dots = z + \sum_{k=2}^{\infty} [k]_{\zeta} z^k.$$

To prove the second part, we write

$$\begin{aligned} \frac{z}{(1-z)(1-\zeta z)(1-\zeta^2 z)} = & z + [1 + \zeta(1 + \zeta)]z^2 + [1 + \zeta(1 + \zeta) + \zeta^2(1 + \zeta + \zeta^2)]z^3 \\ & + [1 + \zeta(1 + \zeta) + \zeta^2(1 + \zeta + \zeta^2) + \zeta^3(1 + \zeta + \zeta^2 + \zeta^3)]z^4 + \dots \end{aligned}$$

Now, we will prove that $\mu_k = \frac{[k]_{\zeta} [k+1]_{\zeta}}{[2]_{\zeta}}$, where μ_k is the coefficient of z^k i.e. will prove

$$[2]_{\zeta} \mu_k = \frac{1 - \zeta^2}{1 - \zeta} [1 + \zeta \frac{1 - \zeta^2}{1 - \zeta} + \zeta^2 \frac{1 - \zeta^3}{1 - \zeta} + \dots + \zeta^{k-1} \frac{1 - \zeta^k}{1 - \zeta}] = [k]_{\zeta} [k + 1]_{\zeta}. \tag{8}$$

From (8), we find that $[2]_{\zeta} \mu_1 = [1]_{\zeta} [2]_{\zeta}$. Next assume that (8) is true for $k = m$. Then

$$\begin{aligned}
 [2]_{\zeta} \mu_{m+1} &= \frac{1-\zeta^2}{1-\zeta} \left[1 + \zeta \frac{1-\zeta^2}{1-\zeta} + \zeta^2 \frac{1-\zeta^3}{1-\zeta} + \dots + \zeta^{m-1} \frac{1-\zeta^m}{1-\zeta} + \zeta^m \frac{1-\zeta^{m+1}}{1-\zeta} \right] \\
 &= [m]_{\zeta} [m+1]_{\zeta} + \frac{1-\zeta^2}{1-\zeta} \zeta^m \frac{1-\zeta^{m+1}}{1-\zeta} \\
 &= [m+1]_{\zeta} \left\{ \frac{1-\zeta^m}{1-\zeta} + \zeta^m \frac{1-\zeta^2}{1-\zeta} \right\} = [m+1]_{\zeta} [m+2]_{\zeta}.
 \end{aligned}$$

Hence, by induction, the result is valid for all k , which ends the proof. □

Theorem 3 A function f of the form (1) in the class $S_{\lambda, \zeta}^*(A, B)$ if and only if

$$\frac{1}{z} \left[f(z) * \left\{ (1-\lambda) \frac{z - \zeta Cz^2}{(1-z)(1-\zeta z)} + \lambda \frac{z + \zeta(1-(1+\zeta)C)z^2}{(1-z)(1-\zeta z)(1-\zeta^2 z)} \right\} \right] \neq 0 (z \in \mathbb{U}) \tag{9}$$

where $C = C_{\theta} = \frac{e^{-i\theta} + A}{A - B}$, $\theta \in [0, 2\pi)$.

Proof The function f belongs to the class $S_{\lambda, \zeta}^*(A, B)$ if and only if

$$\frac{(1-\lambda)zD_{\zeta}f(z) + \lambda zD_{\zeta}(zD_{\zeta}f(z))}{(1-\lambda)f(z) + \lambda zD_{\zeta}f(z)} = \frac{1 + A\omega(z)}{1 + B\omega(z)}.$$

Thus $f \in S_{\lambda, \zeta}^*(A, B)$ is equivalent to

$$\frac{(1-\lambda)zD_{\zeta}f(z) + \lambda zD_{\zeta}(zD_{\zeta}f(z))}{(1-\lambda)f(z) + \lambda zD_{\zeta}f(z)} \neq \frac{1 + Ae^{i\theta}}{1 + Be^{i\theta}}, \quad (z \in \mathbb{U}, \theta \in [0, 2\pi)).$$

This simplifies to

$$zD_{\zeta} \left[(1-\lambda)f(z) + \lambda zD_{\zeta}f(z) \right] \left(1 + Be^{i\theta} \right) - \left[(1-\lambda)f(z) + \lambda zD_{\zeta}f(z) \right] \left(1 + Ae^{i\theta} \right) \neq 0. \tag{10}$$

Since

$$f(z) = f(z) * \frac{z}{(1-z)}. \tag{11}$$

From (6), It follows that

$$zD_{\zeta}f(z) = f(z) * \frac{z}{(1-z)(1-\zeta z)} \tag{12}$$

(see also Piejko et al. [24]). Using (11) and (12), we obtain

$$(1-\lambda)f(z) + \lambda zD_{\zeta}f(z) = f(z) * \frac{z - (1-\lambda)\zeta z^2}{(1-z)(1-\zeta z)} \tag{13}$$

Using the identity $zD_{\zeta}(f * g) = f * zD_{\zeta}g$, we conclude that

$$zD_{\zeta} [(1 - \lambda)f(z) + \lambda zD_{\zeta} f(z)] = f(z) * \frac{z + \zeta [1 - (1 - \lambda)(1 + \zeta)]z^2}{(1 - z)(1 - \zeta z)(1 - \zeta^2 z)}. \tag{14}$$

Substituting (13) and (14) into (10), we get

$$\begin{aligned} & \frac{1}{z} [f(z) * \{- (1 - \lambda)(1 - \zeta^2 z)[(A - B)e^{i\theta} z - \zeta (1 + Ae^{i\theta})z^2] \\ & - \lambda(1 - \zeta^2 z)(A - B)e^{i\theta} z + \lambda \zeta (1 + \zeta)(1 + Be^{i\theta})z^2\} / (1 - z)(1 - \zeta z)(1 - \zeta^2 z)] \neq 0. \end{aligned} \tag{15}$$

So that the left hand side of (15) may be expressed as

$$\begin{aligned} & \frac{1}{z} [f(z) * \{- (1 - \lambda)(1 - \zeta^2 z)[(A - B)e^{i\theta} z - \zeta (1 + Ae^{i\theta})z^2] \\ & - \lambda[(A - B)e^{i\theta} z + \{\zeta(A - B)e^{i\theta} \\ & - [\zeta(1 + \zeta)(A - B)e^{i\theta} + \zeta(1 + \zeta)(1 + Be^{i\theta})]z^2\}] / (1 - z)(1 - \zeta z)(1 - \zeta^2 z)] \end{aligned}$$

or, equivalently,

$$\begin{aligned} & \frac{1}{z} [f(z) * \{- (1 - \lambda)(1 - \zeta^2 z)[(A - B)e^{i\theta} z - \zeta (1 + Ae^{i\theta})z^2] \\ & - \lambda[(A - B)e^{i\theta} z + \zeta[(A - B)e^{i\theta} - (1 + \zeta)(1 + Ae^{i\theta})]z^2\}] / (1 - z)(1 - \zeta z)(1 - \zeta^2 z)] \end{aligned}$$

Then (15) can be rewritten as the following

$$\frac{1}{z} \left[f(z) * \left\{ (1 - \lambda) \frac{z - \zeta \frac{(e^{-i\theta} + A)}{(A - B)} z^2}{(1 - z)(1 - \zeta z)} + \lambda \frac{z + \zeta \left(1 - (1 + \zeta) \frac{(e^{-i\theta} + A)}{(A - B)} \right) z^2}{(1 - z)(1 - \zeta z)(1 - \zeta^2 z)} \right\} \right] \neq 0$$

where $z \in \mathbb{U}$, $\theta \in [0, 2\pi)$. Hence the proof of Theorem 3 is complete. □

Taking $\lambda = 0$ in Theorem 3, we get the following corollary

Corollary 4 *A function f of the form (1) in the class $S_{\zeta}^*(A, B)$ if and only if*

$$\frac{1}{z} \left[f(z) * \frac{z - \zeta Cz^2}{(1 - z)(1 - \zeta z)} \right] \neq 0 (z \in \mathbb{U})$$

where $C = C_{\theta} = \frac{e^{-i\theta} + A}{A - B}$, $\theta \in [0, 2\pi)$.

Remark 5

Letting $A = b(1 + m) - m$ and $B = -m$ with $m = 1 - \frac{1}{M}$ and $M > \frac{1}{2}$ in Corollary 4 we get the result obtained by Aouf and Seoudy [25, Theorem 2.1].

Taking $\lambda = 1$ in Theorem 3, we get the following corollary

Corollary 6 *A function f of the form (1) is in the class $C_{\zeta}(A, B)$ if and only if*

$$\frac{1}{z} \left[f(z) * \frac{z + \zeta [1 - (1 + \zeta)C]z^2}{(1 - z)(1 - \zeta z)(1 - \zeta^2 z)} \right] \neq 0 \quad (z \in \mathbb{U})$$

where $C = C_\theta = \frac{e^{-i\theta} + A}{A - B}$, $\theta \in [0, 2\pi)$.

Remark 7

Letting $A = b(1 + m) - m$ and $B = -m$ with $m = 1 - \frac{1}{M}$ and $M > \frac{1}{2}$ in Corollary 6 we get the result obtained by Aouf and Seoudy [25, Theorem 2.4].

Remark 8

- 1 Letting $\zeta \rightarrow 1^{-1}$, $A = [b(1 + m) - m]$ and $B = -m$ with $m = 1 - \frac{1}{M}$ and $M > \frac{1}{2}$ in Theorem 3, we obtain the result obtained by Lashin [16].
- 2 Letting $\zeta \rightarrow 1^{-1}$, $A = [b(1 + m) - m]$ and $B = -m$ with $m = 1 - \frac{1}{M}$ and $M > \frac{1}{2}$ in Corollaries 4 and 6, respectively, we obtain the results obtained by El-Ashwah [26, Theorem 2.1 and Theorem 2.4].
- 3 Taking $\zeta \rightarrow 1^{-1}$, $A = 1 - 2\alpha$, $B = -1$ and $e^{i\theta} = x(|x| = 1)$ in Corollaries 4 and 6, we obtain the results obtained by Silverman et al. [23, Theorems 1,2].
- 4 Taking $\zeta \rightarrow 1^{-1}$ and $e^{i\theta} = x(|x| = 1)$ in Corollaries 4 and 6, respectively, we obtain the results obtained by Padmanabhan and Ganesan [27, Theorem 1,2].

Theorem 9 A necessary and sufficient condition for the function f of the form (1) to be in the class $S_{\lambda, \zeta}^*(n, A, B)$ is

$$1 - \sum_{k=2}^{\infty} [k]_{\zeta}^n \frac{\zeta^{[k-1]_{\zeta}} (e^{-i\theta} + B) - (A - B)}{(A - B)} [(1 - \lambda) + \lambda [k]_{\zeta}] a_k z^{k-1} \neq 0 \tag{16}$$

for all $\theta \in [0, 2\pi)$ and $z \in U$.

Proof From Theorem 3, we have $f(z) \in S_{\lambda, \zeta}^*(n, A, B)$ if and only if

$$\frac{1}{z} \left[S_{\zeta}^n f(z) * \left\{ (1 - \lambda) \frac{z - \zeta Cz^2}{(1 - z)(1 - \zeta z)} + \lambda \frac{z + \zeta(1 - (1 + \zeta)C)z^2}{(1 - z)(1 - \zeta z)(1 - \zeta^2 z)} \right\} \right] \neq 0 (z \in U) \tag{17}$$

where $C = C_\theta = \frac{e^{-i\theta} + A}{A - B}$, $\theta \in [0, 2\pi)$. Now, we can easily deduce that

$$\begin{aligned} & (1 - \lambda) \frac{z - \zeta Cz^2}{(1 - z)(1 - \zeta z)} + \lambda \frac{z + \zeta(1 - (1 + \zeta)C)z^2}{(1 - z)(1 - \zeta z)(1 - \zeta^2 z)} \\ &= (1 - \lambda) \left[\frac{Cz}{1 - z} + \frac{(1 - C)z}{(1 - z)(1 - \zeta z)} \right] \\ &+ \frac{\lambda}{\zeta} \left[\frac{(1 + \zeta)(1 - C)z}{(1 - z)(1 - \zeta z)(1 - \zeta^2 z)} - \frac{[1 - (1 + \zeta)C]z}{(1 - z)(1 - \zeta z)} \right]. \end{aligned} \tag{18}$$

Using Proposition 2 and the relation $\frac{z}{(1 - z)} = z + \sum_{k=2}^{\infty} z^k$, (18) can be written as

$$z + \sum_{k=2}^{\infty} \left\{ (1 - \lambda) [(1 - C)([k]_{\zeta} - 1) + 1] + \frac{\lambda}{\zeta} [k]_{\zeta} [(1 - C)([k + 1]_{\zeta} - (1 + \zeta)) + \zeta] \right\} z^k. \tag{19}$$

Setting $[k]_\zeta = \frac{1-\zeta^k}{1-\zeta}$ and $[k+1]_\zeta = \frac{1-\zeta^{k+1}}{1-\zeta}$, (19) gives

$$z + \sum_{k=2}^{\infty} [\zeta(1-C)[k-1]_\zeta + 1][(1-\lambda) + \lambda[k]_\zeta]z^k$$

which is equivalent to

$$z - \sum_{k=2}^{\infty} \frac{\zeta[k-1]_\zeta (e^{-i\theta} + B) - (A-B)}{(A-B)} [(1-\lambda) + \lambda[k]_\zeta]z^k. \tag{20}$$

Thus (17) together with (20) lead to (16). This completes the proof of Theorem 9. \square

Theorem 10 *If the function f defined by (1) satisfies the inequality*

$$\sum_{k=2}^{\infty} [[k-1]_\zeta(1+B) + (A-B)][(1-\lambda) + \lambda[k]_\zeta][k]_\zeta^n |a_k| \leq (A-B) \tag{21}$$

then $f(z) \in S_{\lambda,\zeta}^*(n, A, B)$.

Proof Since

$$\left| \frac{\zeta[k-1]_\zeta (e^{-i\theta} + B) - (A-B)}{(A-B)} \right| \leq \frac{[k-1]_\zeta(1+B) + (A-B)}{(A-B)}$$

then

$$\begin{aligned} & \left| 1 - \sum_{k=2}^{\infty} [k]_\zeta^n \frac{\zeta[k-1]_\zeta (e^{-i\theta} + B) - (A-B)}{(A-B)} [(1-\lambda) + \lambda[k]_\zeta] a_k z^{k-1} \right| \\ & \geq 1 - \sum_{k=2}^{\infty} \left| [k]_\zeta^n \frac{\zeta[k-1]_\zeta (e^{-i\theta} + B) + (A-B)}{(A-B)} \right| [(1-\lambda) + \lambda[k]_\zeta] |a_k| |z|^{k-1} \\ & \geq 1 - \sum_{k=2}^{\infty} \frac{[k-1]_\zeta(1+B) + (A-B)}{(A-B)} [(1-\lambda) + \lambda[k]_\zeta][k]_\zeta^n |a_k| > 0 \quad (z \in \mathbb{U}). \end{aligned}$$

Thus (16) holds, which ends the proof. \square

Theorem 11 $S_{\lambda,\zeta}^*(n+1, A, B) \subset S_{\lambda,\zeta}^*(n, A, B)$.

Proof Since $f(z) \in S_{\lambda,\zeta}^*(n+1, A, B)$, It follows from Theorem 9, that

$$1 - \sum_{k=2}^{\infty} [k]_\zeta^{n+1} \frac{\zeta[k-1]_\zeta (e^{-i\theta} + B) + (A-B)}{(A-B)} [(1-\lambda) + \lambda[k]_\zeta] a_k z^{k-1} \neq 0 \tag{22}$$

(22) can be written as

$$\left[1 + \sum_{k=2}^{\infty} [k]_{\zeta} z^{k-1} \right] * \left[1 - \sum_{k=2}^{\infty} [k]_{\zeta}^n \frac{\zeta^{[k-1]_{\zeta} (e^{-i\theta} + B) + (A-B)}}{(A-B)} [(1 - \lambda) + \lambda [k]_{\zeta}] a_k z^{k-1} \right] \neq 0. \tag{23}$$

(23) gives

$$1 - \sum_{k=2}^{\infty} [k]_{\zeta}^n \frac{\zeta^{[k-1]_{\zeta} (e^{-i\theta} + B) + (A-B)}}{(A-B)} [(1 - \lambda) + \lambda [k]_{\zeta}] a_k z^{k-1} \neq 0$$

which means that $f(z) \in S_{\lambda, \zeta}^*(n, A, B)$. This completes the proof of Theorem 11. □

Conclusions

q- Derivatives and q-integrals play an important and significant role in the study of quantum groups and q-deformed super-algebras. Recently, q-calculus has attracted the attention of many researchers in the field of geometric function theory. In this paper, we have used q-calculus to define and study some new sub-classes of analytic functions. Using the convolution technique some interesting properties of these new classes have been derived. Also, Coefficient conditions and inclusion properties of functions in these classes are found. Some special cases have been discussed as applications of our main results.

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Competing interests

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