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Regular open sets in fuzzifying topology redefined

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Fuzzifying topology;
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Abstract In 2000 [1], Zahran introduced the concept of regular open sets in fuzzifying topology. In 2004 [2], Sayed and Zahran, gave an example to illustrate that the statements:

(1) $\models A \in R_\tau \rightarrow A \in \tau$ (Lemma 2.2 [1]); and
(2) $\models (A \in R_\tau \wedge B \in R_\tau) \rightarrow A \cap B \in R_\tau$ (Theorem 2.4 [1]),

are incorrect. In the present paper we redefine this concept to make these statements correct. Furthermore, by making use of our definition of regular open sets, the concepts of almost continuity and δ-continuity are introduced and studied in fuzzifying topology.

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1. Introduction

In classical and fuzzy topology, Almost continuity, δ-continuity [3–5] have been defined and their properties have been obtained.

In 1991 [6], Ying used the semantics of fuzzy logic to propose a topology whose logical fundament is fuzzy. Proceeding in this direction many papers have been written [1,7–9]. The concept of regular open set in fuzzifying topology was given in 2000 [1] by Zahran. In 2004 [2], Sayed and Zahran illustrate by a counterexample that the statements:

(1) $\tau(A) \geqslant R_\tau(A)$ (Lemma 2.2 [1]); and
(2) $(R_\tau(A) \wedge R_\tau(B)) \leqslant R_\tau(A \cap B)$ (Theorem 2.4 [1]),

are incorrect. In the present paper we redefine the concept of regular open sets in fuzzifying topology to make these statements correct. Furthermore by making use of this concept we introduce and study the almost continuity and δ-continuity in fuzzifying topology.

For the definition of a fuzzifying topology and some of its basic concepts used in this paper we refer to [6,8,9]. For the definitions of the family of semi-open sets and the family of semi-closed sets in fuzzifying topology we refer to [7].

However we recall here some of the basic concepts used in this paper.

Definition 1.1. Let (X, τ) be a fuzzifying topological space. Then

(1) The family of all closed sets in X is denoted by F_τ or F if there is no confusion and defined as: $F_\tau(A) = \tau(X - A)\ \forall A \in 2^X$, where $X - A$ is the complement of A.
(2) The neighborhood system of x at a subset A of X is denoted by $\phi_{(\tau,x)}(A)$, and defined as:

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$$\phi_{(\tau,x)}(A) = \bigvee_{x\in B\subseteq A} \tau(B) \quad \forall\ A \in 2^X.$$

(3) The closure (resp. interior) of A is denoted by $cl_\tau(A)$, (resp. $int_\tau(A)$,) and defined as:

$$cl_\tau(A)(x) = 1 - \phi_{(\tau,x)}(X - A)(resp.\ int_\tau(A)(x) = \phi_{(\tau,x)}(A))\ \forall\ A \in 2^X, \quad \forall\ x \in X.$$

(4) Let $f \in I^X$, where $I = [0, 1]$. Then
  (a) The closure of f is denoted by $\widetilde{cl}_\tau(f)$, and defined as:

$$\widetilde{cl}_\tau(f)(x) = \bigvee_{\alpha\in[0,1]} (f(x) \wedge cl_\tau(f_\alpha))(x) \quad \forall\ x \in X;\ and$$

  (b) The interior of f is denoted by $\widetilde{int}_\tau(f)$, and defined as:

$$\widetilde{int}_\tau(f) = 1 - \widetilde{cl}_\tau(1 - f).$$

(5) The family of semi-open sets is denoted by $S\tau$, and defined as:

$$S\tau(A) = \bigwedge_{x\in A} \widetilde{cl}_\tau(int_\tau(A))(x) \quad \forall\ A \in 2^X.$$

(6) The family of semi-closed sets is denoted by SF, and defined as:

$$SF(A) = S\tau(X - A) \quad \forall\ A \in 2^X.$$

(7) The degree of the convergence of a net S in X to $x \in X$ is denoted by $S\triangleright_\tau x$, and defined as:

$$S\triangleright_\tau x = \bigwedge_{S\not\subset A} (1 - \phi_{(\tau,x)}(A))$$

$\forall\ S \in N(X), \forall\ x \in X$, where $S \subset A$ means S almost in A and $N(X)$ denoted the set of all nets in X.

Definition 1.2. Let $f, g \in I^X$. The fuzzy inclusion of f in g is denoted by $[[f, g[[$, and defined as:

$$[[f, g[[= \bigwedge_{x\in X} (f(x) \to g(x)).$$

Note, that "$\to$" is defined by: $\alpha \to \beta = \min(1, 1 - \alpha + \beta)\ \alpha,\ \beta \in I$.

2. Regular open sets and δ-open sets

Definition 2.1. Let (X, τ) be a fuzzifying topological space. Then

(1) The family of all regular open sets is denoted by $R\tau \in I^{(2^X)}$ and defined as follows:

$$R\tau(A) = \tau(A) \wedge SF(A).$$

(2) The family of all regular closed sets is denoted by $RF \in I^{(2^X)}$ and defined as follows:

$$RF(A) = R\tau(X - A) \quad \forall\ A \in 2^X.$$

Theorem 2.1. *Let (X, τ) be a fuzzifying topological space. Then*

(1) (a) $R\tau(X) = 1, R\tau(\phi) = 1$;
  (b) $R\tau(A \cap B) \geqslant R\tau(A) \wedge R\tau(B)$;
  (c) $\tau(A) \geqslant R\tau(A), SF(A) \geqslant R\tau(A)$;

(2) (a) $RF(X) = 1, RF(\phi) = 1$;
  (b) $RF(A \cup B) \geqslant RF(A) \wedge RF(B)$;
  (c) $F(A) \geqslant RF(A), S\tau(A) \geqslant RF(A)$;
  (d) $RF(A) = F(A) \wedge S\tau(A)$.

Proof. We just prove (1) (b). From Theorem 3.2 (1) (b) [7], we have

$$R\tau(A \cap B) = \tau(A \cap B) \wedge SF(A \cap B) \geqslant \tau(A) \wedge \tau(B) \wedge SF(A) \wedge SF(B) = R\tau(A) \wedge R\tau(B).$$

The other statements are clear. □

In 2004 [2], Sayed and Zahran illustrate by the following example that the statements:

(1) $\tau(A) \geqslant R\tau(A)$ (Lemma 2.2 [1]); and
(2) $(R\tau(A) \wedge R\tau(B)) \leqslant R\tau(A \cap B)$ (Theorem 2.4 [1]), are incorrect.

Example 2.1. Let $X = \{a, b, c\}$ and τ be a fuzzifying topology on X defined as $\tau(X) = \tau(\emptyset) = \tau(\{a\}) = \tau(\{a, c\}) = 1$, $\tau(\{b\}) = \tau(\{a, b\}) = 0$ and $\tau(\{c\}) = \tau(\{b, c\}) = \frac{1}{8}$.

Sayed and Zahran have obtained the regular openness degree of every $A \in 2^X$ according to the definition of regular open in form $R\tau(A) = (A \equiv int_\tau(cl_\tau(A)))$ as follows:

$R\tau(X) = R\tau(\emptyset) = 1$, $R\tau(\{a\}) = R\tau(\{c\}) = R\tau(\{a, b\}) = R\tau(\{b, c\}) = \frac{1}{8}$ and $R\tau(\{b\}) = R\tau(\{a, c\}) = 0$. Therefore, as we see $R\tau(\{a, b\}) > \tau(\{a, b\})$ and $R\tau(\{a, b\} \cap \{b, c\}) < R\tau(\{a, b\}) \cap R\tau(\{b, c\})$.

Now, we obtain the regular openness degree of every $A \in 2^X$ according to the definition of regular open in form $R\tau(A) = \tau(A) \wedge SF(A)$ as follows:

Example 2.2. Let $X = \{a, b, c\}$ and τ be a fuzzifying topology on X that defined in Example 2.1. So, $SF(X) = SF(\emptyset) = SF(\{b\}) = 1$, $SF(\{a\}) = SF(\{a, b\}) = SF(\{a, c\}) = 0$ and $SF(\{c\}) = SF(\{b, c\}) = \frac{7}{8}$. Therefore, as we see $R\tau(X) = R\tau(\emptyset) = 1$, $R\tau(\{a\}) = R\tau(\{b\}) = R\tau(\{a, b\}) = R\tau(\{a, c\}) = 0$ and $R\tau(\{c\}) = R\tau(\{b, c\}) = \frac{1}{8}$. Thus $R\tau(A) \leqslant \tau(A)$ for every $A \in 2^X$ and $R\tau(A) \cap R\tau(B) \leqslant R\tau(A \cap B)$ for every $A, B \in 2^X$.

Definition 2.2. Let (X, τ) be a fuzzifying topological space and let $x \in X$. The δ-neighborhood system of x is denoted by $\delta\phi_{(\tau,x)} \in I^{(2^X)}$ and defined as follows:

$$\delta\phi_{(\tau,x)}(A) = \bigvee_{x\in B\subseteq A} R\tau(B) \quad \forall\ A \in 2^X.$$

Theorem 2.2. *Let (X,τ) be a fuzzifying topological space. The mapping $\delta\phi_{(\tau,.)} : X \rightharpoonup I^{(2^X)}$ has the following properties:*

(1) $\delta\phi_{(\tau,x)}(X) = 1$ and $\delta\phi_{(\tau,x)}(\phi) = 0$;
(2) If $A \subseteq B$, then $\delta\phi_{(\tau,x)}(A) \leqslant \delta\phi_{(\tau,x)}(B)$;
(3) $\delta\phi_{(\tau,x)}(A) = 0$ whenever $x \notin A$;
(4) $\delta\phi_{(\tau,x)}(A \cap B) \geqslant \delta\phi_{(\tau,x)}(A) \wedge \delta\phi_{(\tau,x)}(B)$;
(5) $\delta\phi_{(\tau,x)}(A) \leqslant \bigvee_{C\subseteq A}(\delta\phi_{(\tau,x)}(C) \wedge \bigwedge_{y\in C}\delta\phi_{(\tau,y)}(C))$.

Conversely, if a mapping $\delta\phi_{(\tau,.)}$ satisfies (4), (5), then τ_δ is a fuzzifying topology which is defined as follows:

$$\tau_\delta(A) = \bigwedge_{x\in A}\delta\phi_{(\tau,x)}(A) \quad \forall\ A \in 2^X.$$

Proof. It is similar to the proof of Theorem 3.2 [6]. □

Remark 2.1. τ_δ is called the family of δ-open sets in (X,τ).

Theorem 2.3. *Let (X,τ) be a fuzzifying topological space. Then, $R\tau$ is a base for τ_δ.*

Proof. First, let $A \in 2^X$. Then $\tau_\delta(A) = \bigwedge_{x\in A}\delta\phi_{(\tau,x)} = \bigwedge_{x\in B}\bigvee_{x\in B\subseteq A}R\tau(B) \geqslant R\tau(A)$. Second,

$$\begin{aligned}\phi_{(\tau_\delta,x)}(A) &= \bigvee_{x\in H\subseteq A}\tau_\delta(H) = \bigvee_{x\in H\subseteq A}\bigwedge_{y\in H}\delta\phi_{(\tau,y)}(H)\\ &\leqslant \bigvee_{x\in H\subseteq A}\delta\phi_{(\tau,x)}(H) \leqslant \bigvee_{x\in H\subseteq A}\bigvee_{x\in B\subseteq H}R\tau(B)\\ &\leqslant \bigvee_{H\subseteq A}\bigvee_{x\in B\subseteq H}R\tau(B) \leqslant \bigvee_{x\in B\subseteq A}R\tau(B).\end{aligned}$$

Then from Theorem 4.1 [7], $R\tau$ is a base for τ_δ. □

Theorem 2.4. *Let (X,τ) be a fuzzifying topological space. Then $\forall\ A \in 2^X, \forall\ x \in X$,*

$$\delta\phi_{(\tau,x)}(A) = \phi_{(\tau_\delta,x)}(A).$$

Proof. From Theorem 2.3, we have $\phi_{(\tau_\delta,x)}(A) \leqslant \delta\phi_{(\tau,x)}(A)$ $\forall\ A \in 2^X$, $\forall\ x \in X$. Now,

$$\delta\phi_{(\tau,x)}(A) = \bigvee_{x\in B\subseteq A}R\tau(B) \leqslant \bigvee_{x\in B\subseteq A}\tau_\delta(B) = \phi_{(\tau_\delta,x)}(A). \qquad \square$$

3. Almost continuity and δ-continuity in fuzzifying topology

Definition 3.1. Let (X,τ) and (Y,σ) be two fuzzifying topological spaces. The unary fuzzy predicates $AC, \delta C \in I^{(Y^X)}$ are called fuzzy almost continuity and fuzzy δ-continuity, respectively, and defined as follows:

(1) $AC(f) = \bigwedge_{B\in 2^Y}(R\sigma(B)\longrightarrow\tau(f^{-1}(B)))$;
(2) $\delta C(f) = \bigwedge_{B\in 2^Y}(R\sigma(B)\longrightarrow\tau_\delta(f^{-1}(B)))$.

Theorem 3.1. *Let (X,τ) and (Y,σ) be two fuzzifying topological spaces. For any $f \in Y^X$, we set*

(1) $AC_1(f) = \bigwedge_{B\in 2^Y}(RF_\sigma(B)\longrightarrow F_\tau(f^{-1}(B))$;
(2) $AC_2(f) = \bigwedge_{x\in X}\bigwedge_{U\in 2^Y}(\delta\phi_{(\sigma,f(x))}(U))\longrightarrow(\delta\phi_{(\tau,x)}(f^{-1}(U))$;
(3) $AC_3(f) = \bigwedge_{x\in X}\bigwedge_{U\in 2^Y}(\delta\phi_{(\sigma,f(x))}(U))\longrightarrow\bigvee_{V\subseteq f^{-1}(U)}(\delta\phi_{(\tau,x)}(V))$;
(4) $AC_4(f) = \bigwedge_{x\in X}\bigwedge_{S\in N(X)}(S\rhd_\tau x\longrightarrow(f\circ S)\rhd_{\sigma_\delta}f(x))$;
(5) $AC_5(f) = \bigwedge_{A\in 2^X}[[f(cl_\tau(A)), cl_{\sigma_\delta}(f(A))[[$;
(6) $AC_6(f) = \bigwedge_{B\in 2^Y}[[cl_\tau(f^{-1}(B)), f^{-1}(cl_{\sigma_\delta}(B))[[$;
(7) $AC_7(f) = \bigwedge_{A\in 2^Y}[[f^{-1}(int_{\sigma_\delta}(A)), int_\tau(f^{-1}(A))[[$.

Then $AC(f) = AC_i(f), i = 1, 2, 3, 4, 5, 6, 7$.

Proof.

(1) We want to prove that $AC(f) = AC_1(f)$. Now,

$$\begin{aligned}AC_1(f) &= \bigwedge_{B\in 2^Y}(RF_\sigma(B))\longrightarrow F_\tau(f^{-1}(B))\\ &= \bigwedge_{B\in 2^Y}\min(1, 1 - RF_\sigma(B) + F_\tau(f^{-1}(B)))\\ &= \bigwedge_{B\in 2^Y}\min(1, 1 - R\sigma(Y - B) + \tau(X - f^{-1}(B)))\\ &= \bigwedge_{B\in 2^Y}\min(1, 1 - R\sigma(Y - B) + \tau(f^{-1}(Y - B)))\\ &= \bigwedge_{U\in 2^Y}\min(1, 1 - R\sigma(U) + \tau(f^{-1}(U))) = AC(f).\end{aligned}$$

(2) To prove that $AC(f) = AC_2(f)$. First, we prove $AC(f) \leqslant AC_2(f)$. Suppose that $\delta\phi_{(\sigma,f(x))}(U) \leqslant \phi_{(\tau,x)}(f^{-1}(U))$. Then we obtain that

$$\min(1, 1 - \delta\phi_{(\sigma,f(x))}(U) + \phi_{(\tau,x)}(f^{-1}(U))) = 1.$$

Therefore, the result holds. Now, suppose that $\delta\phi_{(\sigma,f(x))}(U) > \phi_{(\tau,x)}(f^{-1}(U))$. We prove that

$$\min(1, 1 - \delta\phi_{(\sigma,f(x))}(U) + \phi_{(\tau,x)}(f^{-1}(U))) \geqslant AC(f).$$

If $f(x) \in A \subseteq U$, then $x \in f^{-1}(A) \subseteq f^{-1}(U)$. So

$$\begin{aligned}\delta\phi_{(\sigma,f(x))}(U) - \phi_{(\tau,x)}(f^{-1}(U)) &= \bigvee_{f(x)\in A\subseteq U}R\sigma(A) - \bigvee_{x\in B\subseteq f^{-1}(U)}\tau(B)\\ &\leqslant \bigvee_{f(x)\in A\subseteq U}R\sigma(A) - \bigvee_{f(x)\in A\subseteq U}\tau(f^{-1}(A))\\ &\leqslant \bigvee_{f(x)\in A\subseteq U}(R\sigma(A) - \tau(f^{-1}(A))).\end{aligned}$$

Then

$$1 - \delta\phi_{(\sigma,f(x))}(U) + \phi_{(\tau,x)}(f^{-1}(U)) \geqslant \bigwedge_{f(x)\in A\subseteq U}(1 - R\sigma(A) + \tau(f^{-1}(A))).$$

So

$$\begin{aligned}&\min(1, 1 - \delta\phi_{(\sigma,f(x))}(U) + \phi_{(\tau,x)}(f^{-1}(U)))\\ &\geqslant \bigwedge_{f(x)\in A\subseteq U}\min(1, 1 - R\sigma(A) + \tau(f^{-1}(A)))\\ &\geqslant \bigwedge_{V\in 2^Y}\min(1, 1 - R\sigma(V) + \tau(f^{-1}(V))) = AC(f).\end{aligned}$$

Hence,

$$\bigwedge_{x\in X}\bigwedge_{U\in 2^Y}\min(1, 1 - \delta\phi_{(\sigma,f(x))}(U) + \phi_{(\tau,x)}(f^{-1}(U))) \geqslant AC(f).$$

Second, we prove $AC(f) \geqslant AC_2(f)$.

(3) We will prove that $AC_2(f) = AC_3(f)$. From Theorem 3.2 (3) [6] we have

$$\bigvee_{V\in 2^X,\, f(V)\subseteq U} \phi_{(\tau,x)}(V) = \bigvee_{V\in 2^X, V\subseteq f^{-1}(U)} \phi_{(\tau,x)}(V) = \phi_{(\tau,x)}(f^{-1}(U)).$$

Then,

$$\begin{aligned} AC_3(f) &= \bigwedge_{x\in X}\bigwedge_{U\in 2^Y} \min(1, 1-\delta\phi_{(\sigma,f(x))}(U) + \bigvee_{V\in 2^X, f(V)\subseteq U} \phi_{(\tau,x)}(V)) \\ &= \bigwedge_{x\in X}\bigwedge_{U\in 2^Y} \min(1, 1-\delta\phi_{(\sigma,f(x))}(U) + \phi_{(\tau,x)}(f^{-1}(U)) \\ &= AC_2(f)). \end{aligned}$$

(4) We prove that $AC_2(f) \leqslant AC_4(f)$, it suffices to show that for any $x\in X$ and $S\in N(X)$,

$$\min(1, 1 - S\rhd_\tau x + f\circ S\rhd_{\sigma_\delta} f(x)) \geqslant AC_2(f),$$

In fact, if $S\rhd_\tau x \leqslant f\circ S\rhd_{\sigma_\delta} f(x)$, it is obvious. Assume $S\rhd_\tau x > f\circ S\rhd_{\sigma_\delta} f(x)$. Since $f\circ S\not\subseteq B$ implies $S\not\subseteq f^{-1}(B)$, then

$$\begin{aligned} &S\rhd_\tau x - f\circ S\rhd_{\sigma_\delta} f(x) \\ &\quad= \bigwedge_{A\in 2^X,\ S\not\subseteq A} (1-\phi_{(\tau,x)}(A)) - \bigwedge_{B\in 2^Y,\ f\circ S\not\subseteq B} (1-\delta\phi_{(\sigma,f(x))}(B)) \\ &\quad\leqslant \bigwedge_{B\in 2^Y,\ f\circ S\not\subseteq B} (1-\phi_{(\tau,x)}(f^{-1}(B))) - \bigwedge_{B\in 2^Y,\ f\circ S\not\subseteq B} (1-\delta\phi_{(\sigma,f(x))}(B)) \\ &\quad\leqslant \bigvee_{B\in 2^Y,\ f\circ S\not\subseteq B} (\delta\phi_{(\sigma,f(x))}(B) - \phi_{(\tau,x)}(f^{-1}(B))), \end{aligned}$$

So,

$$\begin{aligned} &\min(1, 1 - S\rhd_\tau x + f\circ S\rhd_{\sigma_\delta} f(x)) \\ &\quad\geqslant \bigwedge_{B\in 2^Y, f\circ S\not\subseteq B} \min(1, 1-\delta\phi_{(\sigma,f(x))}(B) + \phi_{(\tau,x)}(f^{-1}(B))) \\ &\quad\geqslant \bigwedge_{x\in X}\bigwedge_{U\in 2^Y} \min(1, 1-\delta\phi_{(\sigma,f(x))}(U) + \phi_{(\tau,x)}(f^{-1}(U))) \\ &\quad= AC_2(f). \qquad \square \end{aligned}$$

(5) We will prove that $AC_4(f) \leqslant AC_5(f)$. Since $AC_5(f) = \bigwedge_{A\in 2^X}[[f(cl_\tau(A)), cl_{\sigma_\delta}(f(A))\ [[= \bigwedge_{A\in 2^X}\bigwedge_{y\in Y} \min(1, 1 - f(cl_\tau(A)) + cl_{\sigma_\delta}(f(A)))$, then the result holds if we proved that for every $A\in 2^X$ and every $y\in Y$, $\min(1,\ 1 - f(cl_\tau(A)) + cl_{\sigma_\delta}(f(A))) \geqslant AC_4(f)$. If $f(cl_\tau\ (A)) \leqslant cl_{\sigma_\delta}(f(A))$, the result holds. Now, suppose that $f(cl_\tau(A)) > cl_{\sigma_\delta}(f(A))$. Then from Theorem 6.1 (2) [6], we have

$$\begin{aligned} f(cl_\tau(A)) - cl_{\sigma_\delta}(f(A)) &= \bigvee_{f(x)=y} cl_\tau(A) - cl_{\sigma_\delta}(f(A)) \\ &= \bigvee_{f(x)=y}\bigvee_{S\subseteq A} S\rhd_\tau x - \bigvee_{T\subseteq f(A)} T\rhd_{\sigma_\delta} y \\ &\leqslant \bigvee_{f(x)=y}\bigvee_{S\subseteq A} S\rhd_\tau x - \bigvee_{f(S)\subseteq f(A)} f\circ S\rhd_{\sigma_\delta} y \\ &\leqslant \bigvee_{f(x)=y}\bigvee_{S\subseteq A} S\rhd_\tau x - \bigvee_{f(x)=y}\bigvee_{S\subseteq A} f\circ S\rhd_{\sigma_\delta} y \\ &\leqslant \bigvee_{f(x)=y}\bigvee_{S\subseteq A} (S\rhd_\tau x - f\circ S\rhd_{\sigma_\delta} y). \end{aligned}$$

Therefore,

$$\begin{aligned} &\min(1, 1 - f(cl_\tau(A)) + cl_{\sigma_\delta}(f(A))) \\ &\quad\geqslant \bigwedge_{f(x)=y}\bigwedge_{S\subseteq A} \min(1, 1 - S\rhd_\tau x + f\circ S\rhd_{\sigma_\delta} f(x)) \\ &\quad\geqslant \bigwedge_{x\in X}\bigwedge_{S\in N(X)} \min(1, 1 - S\rhd_\tau x + f\circ S\rhd_{\sigma_\delta} f(x)) \geqslant AC_4(f). \end{aligned}$$

(6) We will prove that $AC_5(f) \leqslant AC_6(f)$. Now, for any $B\subseteq Y$ one can deduce that

$$[[cl_\tau(f^{-1}(B)), f^{-1}(f(cl_\tau(f^{-1}(B))))[[= 1, [[cl_{\sigma_\delta}(f(f^{-1}(B))), cl_{\sigma_\delta}(B)[[= 1$$

and $[[f^{-1}(cl_{\sigma_\delta}(f(f^{-1}(B)))), f^{-1}(cl_{\sigma_\delta}(B))[[= 1$.
So, from Lemma 1.2 (2) [9] we have

$$\begin{aligned} [[cl_\tau(f^{-1}(B)), f^{-1}(cl_{\sigma_\delta}(B))[[&\geqslant [[f^{-1}(f(cl_\tau(f^{-1}(B)))), f^{-1}(cl_{\sigma_\delta}(B))[[\\ &\geqslant [[f^{-1}(f(cl_\tau(f^{-1}(B)))), f^{-1}(cl_{\sigma_\delta}(f(f^{-1}(B))))[[\\ &\geqslant [[f(cl_\tau(f^{-1}(B))), cl_{\sigma_\delta}(f(f^{-1}(B)))[[, \end{aligned}$$

Therefore,

$$\begin{aligned} AC_6(f) &= \bigwedge_{B\in 2^Y} [[cl_\tau(f^{-1}(B)), f^{-1}(cl_{\sigma_\delta}(B))[[\\ &\geqslant \bigwedge_{B\in 2^Y} [[f(cl_\tau(f^{-1}(B))), cl_{\sigma_\delta}(f(f^{-1}(B)))[[\\ &\geqslant \bigwedge_{A\in 2^X} [[f(cl_\tau(A)), cl_{\sigma_\delta}(f(A))[[= AC_5(f). \end{aligned}$$

(7) We prove that $AC_6(f) = AC_2(f)$.

$$\begin{aligned} AC_6(f) &= \bigwedge_{B\in 2^Y} [[cl_\tau(f^{-1}(B)), f^{-1}(cl_{\sigma_\delta}(B))[[\\ &= \bigwedge_{B\in 2^Y}\bigwedge_{x\in X} \min(1, 1-(1-\phi_{(\tau,x)}(X - f^{-1}(B))) \\ &\quad + (1-\delta\phi_{(\sigma,f(x))}(Y-B))) \\ &= \bigwedge_{B\in 2^Y}\bigwedge_{x\in X} \min(1, 1-\delta\phi_{(\sigma,f(x))}(Y-B) \\ &\quad + \phi_{(\tau,x)}(f^{-1}(Y-B))) \\ &= \bigwedge_{U\in 2^Y}\bigwedge_{x\in X} \min(1, 1-\delta\phi_{(\sigma,f(x))}(U) + \phi_{(\tau,x)}(f^{-1}(U))) \\ &= AC_2(f). \end{aligned}$$

(8) We prove that $AC_7(f) = AC_2(f)$.

$$\begin{aligned} AC_7(f) &= \bigwedge_{A\in 2^Y}\bigwedge_{x\in X} \min(1, 1 - int_{\sigma_\delta}(A)(f(x)) + int_\tau(f^{-1}(A))(x)) \\ &= \bigwedge_{A\in 2^Y}\bigwedge_{x\in X} \min(1, 1-\delta\phi_{(\sigma,f(x))}(A) + \phi_{(\tau,x)}(f^{-1}(A))) \\ &= AC_2(f). \qquad \square \end{aligned}$$

Theorem 3.2. *For any $f\in Y^X$, $C(f) \leqslant AC(f)$, where $C(f)$ is the fuzzy continuity of f.*

Proof. It follows from Theorem 2.1 (1) (c). $\square$

Theorem 3.3. *Let (X,τ), (Y,σ) and (Z,η) be three fuzzifying topological spaces. Then for any $f\in Y^X$ and for any $g\in Z^Y$ we have*

(1) $C(f) \leqslant (AC(g) \longrightarrow AC(g\circ f))$;
(2) $AC(g) \leqslant (C(f) \longrightarrow AC(g\circ f))$.

Proof.

(1) If $AC(g) \leqslant AC(g\circ f)$, the result holds, if $AC(g) > AC(g\circ f)$, then

$$\begin{aligned} AC(g) - AC(g\circ f)] &= \bigwedge_{v\in 2^Z} \min(1, 1 - R\xi(v) + \sigma(g^{-1}(v))) \\ &\quad - \bigwedge_{v\in 2^Z} \min(1, 1 - R\xi(v) + \tau((g\circ f)^{-1}(v))) \\ &\leqslant \bigvee_{v\in 2^Z} (\sigma(g^{-1}(v)) - \tau((g\circ f)^{-1}(v))) \\ &= \bigvee_{v\in 2^Z} (\sigma(g^{-1}(v)) - \tau(f^{-1}(g^{-1}(v)))) \\ &\leqslant \bigvee_{u\in 2^Y} (\sigma(u) - \tau(f^{-1}(u))) \end{aligned}$$

Therefore,

$$\begin{aligned} AC(g) \longrightarrow AC(g\circ f) &= \min(1, 1 - AC(g) + AC(g\circ f)) \\ &\geqslant \bigwedge_{u\in 2^Y} \min(1, 1 - \sigma(u) + \tau(f^{-1}(u))) = C(f). \end{aligned}$$

(2)
$$\begin{aligned} &AC(g) \longrightarrow (C(f) \longrightarrow AC(g\circ f)) \\ &= AC(g) \longrightarrow \neg(C(f) \odot \neg(AC(g\circ f))) \\ &= \neg(AC(g) \odot \neg\neg(C(f) \odot \neg(AC(g\circ f)))) \\ &= \neg(AC(g) \odot C(f) \odot \neg(AC(g\circ f))) \\ &= \neg(C(f) \odot AC(g) \odot \neg(AC(g\circ f))) \\ &= \neg(C(f) \odot \neg\neg(AC(g) \odot \neg(AC(g\circ f)))) \\ &= C(f) \longrightarrow \neg(AC(g) \odot \neg(AC(g\circ f))) \\ &= C(f) \longrightarrow (AC(g) \longrightarrow AC(g\circ f)). \qquad \square \end{aligned}$$

Theorem 3.4. *Let (X,τ) and (Y,σ) be two fuzzifying topological spaces. For any $f\in Y^X$, we set*

(1) $\delta C_1(f) = \bigwedge_{B\in 2^Y} (RF_\sigma(B)) \longrightarrow F_{\tau_\delta}(f^{-1}(B))$;
(2) $\delta C_2(f) = \bigwedge_{B\in X} \bigwedge_{B\in 2^Y} (\delta\phi_{(\sigma,f(x))}(B)) \longrightarrow (\delta\phi_{(\tau,x)}(f^{-1}(B)))$;
(3) $\delta C_3(f) = \bigwedge_{B\in X} \bigwedge_{B\in 2^Y} (\delta\phi_{(\sigma,f(x))}(B)) \longrightarrow \bigvee_{V\subseteq f^{-1}(B)} (\delta\phi_{(\tau,x)}(V))$;
(4) $\delta C_4(f) = \bigwedge_{x\in X} \bigwedge_{S\in N(X)} (S \rhd_{\tau_\delta} x \longrightarrow (f\circ S) \rhd_{\sigma_\delta} f(x)$;
(5) $\delta C_5(f) = \bigwedge_{B\in 2^Y} [[cl_{\tau_\delta}(f^{-1}(B)), f^{-1}(cl_{\sigma_\delta}(B))[[$;
(6) $\delta C_6(f) = \bigwedge_{A\in 2^X} [[f(cl_{\tau_\delta}(A)), cl_{\sigma_\delta}(f(A))[[$;
(7) $\delta C_7(f) = \bigwedge_{A\in 2^Y} [[f^{-1}(int_{\sigma_\delta}(A)), int_\tau(f^{-1}(A))[[$.

Then $\delta C(f) = \delta C_i(f), i = 1,2,3,4,5,6,7$.

Proof. It is similar to the proof of Theorem 3.1. $\square$

Theorem 3.5. *Let (X,τ), (Y,σ) and (Z,η) be three fuzzifying topological spaces. Then for any $f\in Y^X$ and for any $g\in Z^Y$ we have*

(1) $\delta C(f) \leqslant (\delta C(g) \longrightarrow \delta C(g\circ f))$;
(2) $\delta C(g) \leqslant (\delta C(f) \longrightarrow \delta C(g\circ f))$.

Proof. It is similar to the proof of Theorem 3.3. $\square$

4. Conclusion

The concept of regular open set in fuzzifying topology was given in 2000 [1] by Zahran. In 2004 [2], Sayed and Zahran illustrate by a counterexample that the statements:

(1) $\tau(A) \geqslant R_\tau(A)$ (Lemma 2.2 [1]); and
(2) $(R_\tau(A) \wedge R_\tau(B)) \leqslant R_\tau(A\cap B)$ (Theorem 2.4 [1]),

are incorrect. So, we note, if we extend the equivalent definition of regular open sets in general topology to fuzzifying topology these statements will be correct. Furthermore, as application of this concept we introduce and study the almost continuity and δ-continuity in fuzzifying topology.

In future, we hope to study this work in the framework of L-fuzzifying topological spaces "where L is a complete residuated lattice".

In the end, we would like to point that Definition 2.1 above and the definition of regular open by Zahran [1], are equivalent in general topology but it are Independent in fuzzifying topology.

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