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ORIGINAL ARTICLE

# Approximate solution to a singular, plane mixed boundary-value problem for Laplace's equation in a curved rectangle

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**Abstract** In this paper, we use a hybrid method based on a variant of Trefftz's method (TM), in combination with the usual Boundary Collocation Method (BCM) to find the approximate solution to a singular, two-dimensional mixed boundary-value problem for Laplace's equation in a rectangular sheet with one curved side.

After expressing the solution as a finite linear combination of harmonic trial functions, the usual BCM is used to enforce the boundary condition on the curved side, while a variant of TM is applied to the three remaining sides. The singularity at one corner of the rectangle is treated via the enrichment of the expansion with a specially built harmonic function which has a singularity at one corner.

The procedure ultimately produces a rectangular set of linear algebraic equations, which is solved by QR factorization method.

Numerical results are presented and discussed, in order to assess the efficiency of the proposed method.



## 1. Introduction

The semi-analytical methods for solving boundary-value problems of partial differential equations have acquired increasing popularity in the past few years, because of their efficiency and flexibility in dealing with complicated geometries, especially with the advent of fast computers. These methods are in essence approximation methods, aiming at finding relatively simple analytical formulae to approximate the solution of the problem and, at the same time, to retain the main features of the exact solution. Semi-analytical methods have evolved in two main streams: (i) To choose an expression for the approximate solution that exactly satisfies the differential (field) equation from the outset. The task then reduces to satisfying the boundary conditions approximately using well-known techniques (Galerkin-type Method, Boundary Collocation Method, Least Squares Method [1]. Trefftz's method belongs to

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this category [2,3]. (ii) To use a boundary representation of the solution, by means of which the unknown functions of the problem are expressed in terms of a set of boundary integrals and algebraic equations involving some unknowns. This is usually carried out using the Green's function technique.

BCM is one of the simplest numerical methods for solution of two-dimensional boundary-value problems for Laplace's equation [4–12] which belongs to the family of boundary procedures. The BCM was first used by Barta in 1937 to analyze thin square clamped plates under uniformly distributed or concentrated loads. The advantages of BCM resides in the high accuracy of the numerical solutions, and in the simplicity in computer programming. The basic feature of the method is the expansion of the solution as a linear combination of a complete set of functions, called the trial functions, satisfying the differential equation. In the present case these are harmonic functions. The enforcement of the boundary conditions at a certain number of boundary points provides a generally overdetermined system of linear algebraic equations for the coefficients of the expansion. When the number of boundary points at which the boundary conditions are satisfied is equal to the number of degrees of freedom (i.e. to the number of expansion coefficients), the method is said to be "straightforward". In this case, the system of linear algebraic equations is square. Two sets of trial functions have been frequently used in solving two-dimensional boundary-value problems for Laplace's equation: The set of polar harmonics and the set of fundamental solutions of Laplace's equation, which are trial functions in the form of logarithms of the distance from a general field point in the domain of solution to a certain set of points in the plane, lying outside the domain. Trefftz method (TM) was first proposed in 1926 According to TM, verification of the boundary condition produces a "boundary function" that assumes, in principle, zero values on some interval corresponding to the boundary. This function is then expanded in a Boundary Fourier Expansion in terms of a properly chosen set of orthogonal trigonometric functions. Equating to zero as many Fourier coefficients of this boundary function as needed by the level of approximation yields a rectangular system of linear algebraic equations for the coefficients of the expansion.

In this paper, we used both TM and BCM to find the steady-state temperature distribution in a rectangular sheet with one curved side, when the temperature is specified on two adjacent sides and the sheet is insulated along the other two sides. We shall use BCM on the curved side, where the geometry on the boundary is not simple; as for the other three sides, we shall use TM. The resulting systems of linear algebraic equations are solved using the QR-factorization technique.

We note that the performance of the two proposed methods largely depends on the choice of the set of trial functions. These could be constructed by inspection of the local behavior of the solution at the boundary points. Once this set has been appropriately chosen, the two methods will be easily adaptable to deal with more complicated differential operators, boundary geometries and boundary conditions in two and three dimensions.

## 2. Problem formulation and solution

Consider the mixed boundary-value problem for Laplace's equation in a curved rectangle region:

$$\Delta U = 0, \quad 0 < x < 1, \quad 0 < y < E, \tag{1}$$

with the boundary conditions

$$\begin{aligned} &\frac{\partial U}{\partial n} = 0, \quad y = g(x) = hx^2(1-x)^2, \quad 0 \leqslant x \leqslant 1, \\ &\frac{\partial U}{\partial \theta} = 0, \quad \theta = 0, \quad 0 \leqslant r \leqslant 1, \\ &U = -\frac{1}{2}r^2, \quad x = 1, \quad 0 \leqslant y \leqslant E \quad \text{and} \quad y = E, \quad 0 \leqslant x \leqslant 1, \end{aligned} \tag{2}$$

where $(r, \theta)$ are polar coordinates as on Fig. 1, $\boldsymbol{n}$ is the unit inward normal vector and $h$ is a constant. This problem models steady heat conduction in a curved rectangle with two insulated sides and specified temperature on the other two sides. It is required to study the influence of the curved side on the solution. The counterpart of this problem for a straight rectangle was treated in [13].

The approximate solution is taken in the form

$$U_a(r, \theta) = a_0 + \sum_{n=1}^{N} (a_n r^n \cos n\theta + b_n r^n \sin n\theta) + a_s \xi(x, y). \tag{3}$$

The analysis of the analytical solution as produced in [13] reveals that the mixed second derivative behaves like $\frac{4}{\pi}\ln(1-x)$ as one approaches the upper right corner (1, $E$) along the upper side of the boundary [14]. The other second order derivatives are regular at this point. A harmonic function possessing this type of behavior may be taken as [13]

$$\xi(x, y) = \frac{2}{\pi}\left(\rho^2 \ln \rho \sin 2\phi + \rho^2 \phi \cos 2\phi\right), \tag{4}$$

$(\rho, \phi)$ being polar coordinates centered at the upper right corner, with initial line taken along the upper side of the boundary, so that

$$\rho = [(1-x)^2 + (E-y)^2]^{\frac{1}{2}}, \quad \phi = \tan^{-1}\frac{E-y}{1-x}. \tag{5}$$

Boundary condition on the curved side:

$$\frac{\partial U}{\partial n} = 0, \quad y = g(x) = hx^2(1-x)^2, \quad 0 \leqslant x \leqslant 1, \tag{6}$$

the normal derivative of function $U$ is

$$\begin{aligned} &\frac{\partial U}{\partial n} = \nabla U \cdot \hat{\boldsymbol{n}}, \\ &\nabla U = \frac{\partial U}{\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial U}{\partial \theta}\hat{\boldsymbol{\theta}}, \\ &x = x(s), \quad y = y(s) \\ &\hat{\boldsymbol{n}} = -\dot{y}\hat{\boldsymbol{i}} + \dot{x}\hat{\boldsymbol{j}}, \end{aligned}$$

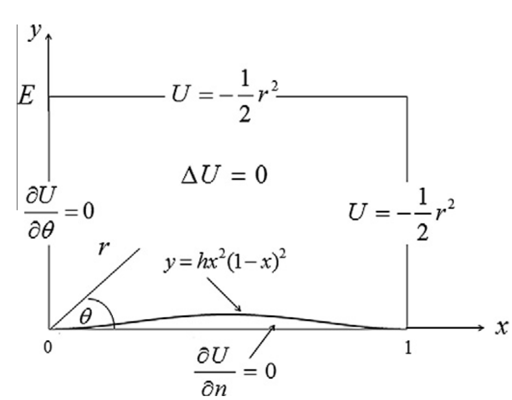


**Figure 1** The mixed boundary-value problem.

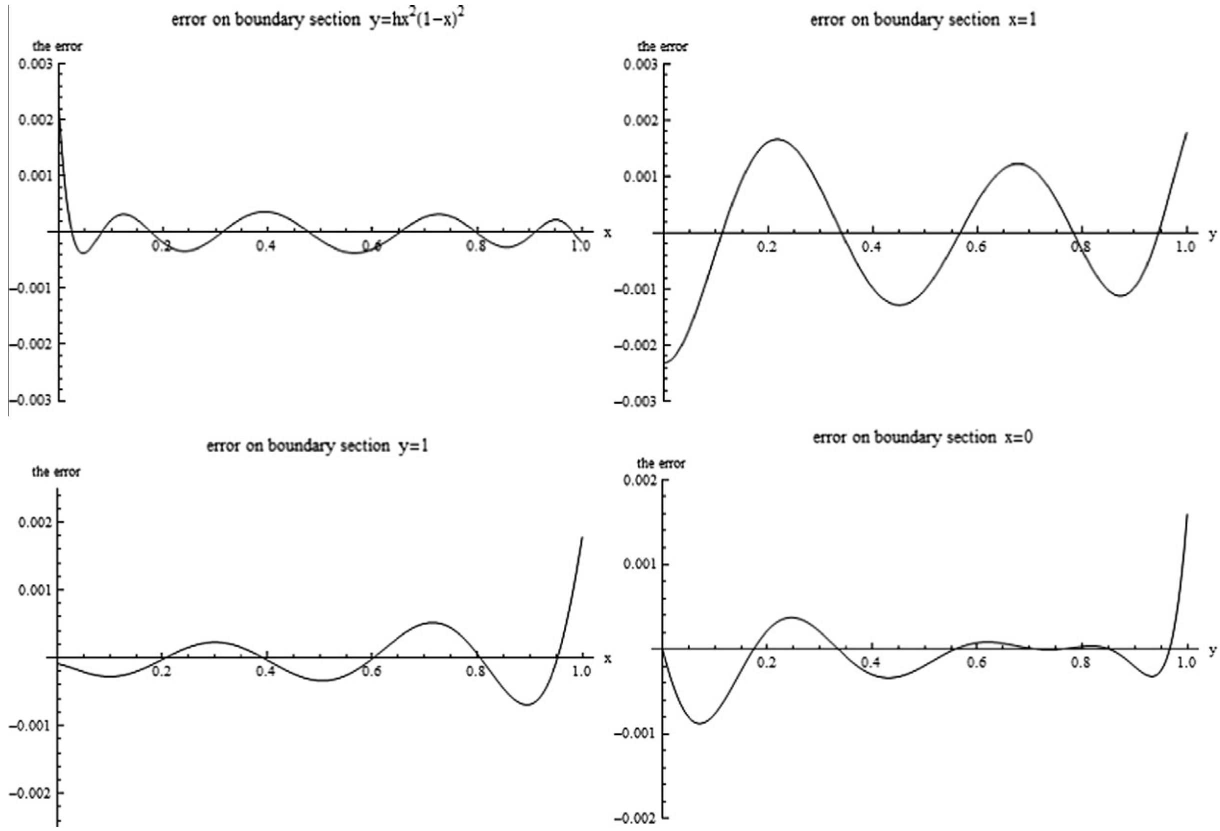


**Figure 2** The errors for $N = 13$ and $h = 1$.

$$\hat{\boldsymbol{i}} = \cos\theta\hat{\boldsymbol{r}} - \sin\theta\hat{\boldsymbol{\theta}}, \quad \hat{\boldsymbol{j}} = \sin\theta\hat{\boldsymbol{r}} + \cos\theta\hat{\boldsymbol{\theta}},$$

$$\hat{\boldsymbol{n}} = (\dot{x}\sin\theta - \dot{y}\cos\theta)\hat{\boldsymbol{r}} + (\dot{x}\cos\theta + \dot{y}\sin\theta)\hat{\boldsymbol{\theta}}.$$

$$\frac{\partial U}{\partial n} = (\dot{x}\sin\theta - \dot{y}\cos\theta)\frac{\partial U}{\partial r} + \frac{1}{r}(\dot{x}\cos\theta + \dot{y}\sin\theta)\frac{\partial U}{\partial\theta}, \tag{7}$$

where $\hat{\boldsymbol{i}}$ and $\hat{\boldsymbol{j}}$ are unit vectors in the directions of increase of $x$ and $y$ respectively, $\hat{\boldsymbol{r}}$ and $\hat{\boldsymbol{\theta}}$ are unit vectors in the directions of increase of $r$ and $\theta$ respectively, and $s$ is a parameter.

$$\dot{y} = \frac{\frac{dg}{dx}}{\sqrt{1 + \left(\frac{dg}{dx}\right)^2}}, \\ = \frac{2hx(1-x)(1-2x)}{\sqrt{1 + [2hx(1-x)(1-2x)]^2}}, \tag{8}$$

$$\dot{x} = \frac{1}{\sqrt{1 + [2hx(1-x)(1-2x)]^2}}.$$

$$\frac{\partial U_a}{\partial n} = \sum_{n=1}^{N} nr^{n-1}[a_n(\dot{x}\sin[(1-n)\theta] - \dot{y}\cos[(1-n)\theta]) \\ + b_n(\dot{x}\cos[(1-n)\theta] + \dot{y}\sin[(1-n)\theta]) \\ + a_s\left(-\dot{y}\ \frac{\partial\xi(x,y)}{\partial x} + \dot{x}\frac{\partial\xi(x,y)}{\partial y}\right) = f(x). \tag{9}$$

Choose $M(\geqslant N)$ collocation points, and enforce the function (3) to satisfy (6) at those points

$$\frac{\partial U_a}{\partial n} = f(x_m) = 0 \quad m = 1, 2, \ldots, M, \tag{10}$$

where $x_m$ are the collocation nodes, to get the system of linear algebraic equations

$$\sum_{n=1}^{N}[A_{mn}a_n + B_{mn}b_n] = C_m, \quad m = 1, 2, \ldots, M, \quad n = 1, 2, \ldots, N \tag{11}$$

with

$$A_{mn} = n\left(x_m^2 + hx_m^2(1-x_m)^2\right)^{\frac{n-1}{2}} \\ \times \left[\dot{x}_m \sin\left[(1-n)\tan^{-1}\left[\frac{hx_m^2(1-x_m)^2}{x_m}\right]\right] \\ - \dot{y}_m \cos\left[(1-n)\tan^{-1}\left[\frac{hx_m^2(1-x_m)^2}{x_m}\right]\right]\right], \tag{12}$$

and

$$B_{mn} = n\left(x_m^2 + hx_m^2(1-x_m)^2\right)^{\frac{n-1}{2}} \\ \times \left[\dot{x}_m \cos\left[(1-n)\tan^{-1}\left[\frac{hx_m^2(1-x_m)^2}{x_m}\right]\right] \\ + \dot{y}_m \sin\left[(1-n)\tan^{-1}\left[\frac{hx_m^2(1-x_m)^2}{x_m}\right]\right]\right], \tag{13}$$

and

$$C_m = a_s\left(-\dot{y}_m\frac{\partial\xi(x_m,y)}{\partial x}\bigg|_{y=hx_m^2(1-x_m)^2} + \dot{x}_m\frac{\partial\xi(x_m,y)}{\partial y}\bigg|_{y=hx_m^2(1-x_m)^2}\right). \tag{14}$$

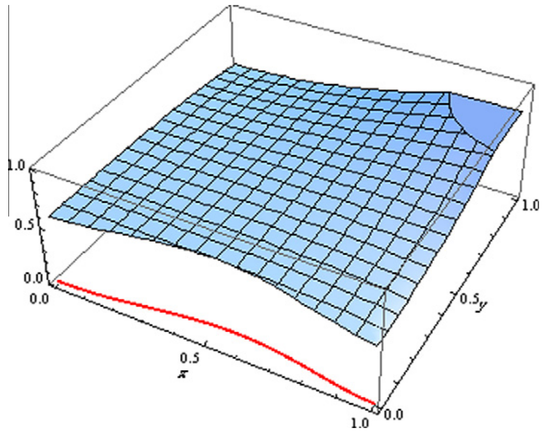


**Figure 3** $-U_a$ and $hx^2(1-x)^2$.

Satisfaction of the boundary condition

$$U = -\frac{1}{2}r^2, \quad x = 1, \quad 0 \leqslant y \leqslant E \quad \text{and} \quad y = E, \quad 0 \leqslant x \leqslant 1, \tag{15}$$

on the two sides of the curved rectangle yields

$$ER(\theta) = U_a + \frac{1}{2}r^2 \tag{16}$$

$$ER(\theta) \equiv a_0 + \sum_{n=1}^{N} r^n (a_n \cos n\theta + b_n \sin n\theta) + a_s \xi(r\cos\theta, r\sin\theta) + \frac{1}{2}r^2 = 0, \quad 0 \leqslant \theta \leqslant \frac{\pi}{2}, \tag{17}$$

where $r$ must now be considered as a function of $\theta$:

$$r(\theta) = \begin{cases} \sec\theta, & 0 \leqslant \theta \leqslant \frac{\pi}{4} \\ \csc\theta, & \frac{\pi}{4} \leqslant \theta \leqslant \frac{\pi}{2} \end{cases}. \tag{18}$$

Extending odd the function $ER(\theta)$ symmetrically to $\left[-\frac{\pi}{2}, 0\right]$, and equating to zero its first $M$ Fourier coefficients leads to the system of linear algebraic equations

$$\sum_{n=1}^{N} [A_{mn} a_n + B_{mn} b_n] = C_m, \quad m = 1, 2, \ldots, M, \quad n = 1, 2, \ldots, N \tag{19}$$

with

$$A_{m0} = \int_0^{\frac{\pi}{2}} \sin 2mt \; dt, = \frac{1}{2m}[1 - (-1)^m], \tag{20}$$

and

$$A_{mn} = \int_0^{\frac{\pi}{4}} (\sec\theta)^n \cos n\theta \sin 2mt \; dt + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\csc\theta)^n \times \cos n\theta \sin 2mt, \tag{21}$$

$$B_{mn} = \int_0^{\frac{\pi}{4}} (\sec\theta)^n \sin n\theta \sin 2mt \, dt + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\csc\theta)^n \sin n\theta \sin 2mt, \tag{22}$$

$$C_m = -\int_0^{\frac{\pi}{4}} \left[a_s \xi(1, \tan\theta) + \frac{1}{2}\sec^2\theta\right] \sin 2mt \; dt - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left[a_s \xi(\cot\theta, 1) + \frac{1}{2}\cos^2\theta\right] \sin 2mt \; dt. \tag{23}$$

Satisfaction of the boundary condition

$$\frac{\partial U}{\partial \theta} = 0, \quad 0 \leqslant y \leqslant E, \quad \theta = \frac{\pi}{2}, \tag{24}$$

yields

$$ER(y) = \sum_{n=1}^{N} n y^n \left(-a_n \sin n\frac{\pi}{2} + b_n \cos n\frac{\pi}{2}\right) - a_s y \left.\frac{\partial \xi(x, y)}{\partial x}\right|_{x=0} = 0, \quad 0 \leqslant y \leqslant E. \tag{25}$$

Extending function $ER(y)$ symmetrically to $[-E, 0]$, and equating to zero its first $M$ Fourier coefficients leads to the system of linear algebraic equations

$$\sum_{n=1}^{N} [A_{mn} a_n + B_{mn} b_n] = C_m, \quad m = 1, 2, \ldots, M, \quad n = 1, 2, \ldots, N, \tag{26}$$

with

$$A_{mn} = \int_0^E -n y^n \sin n\frac{\pi}{2} \sin\frac{m\pi y}{E} dy, \tag{27}$$

$$B_{mn} = \int_0^E n y^n \cos n\frac{\pi}{2} \sin\frac{m\pi y}{E}, dy, \tag{28}$$

$$C_m = \int_0^E a_s y \left.\frac{\partial \xi(x, y)}{\partial x}\right|_{x=0} \sin\frac{m\pi y}{E} dy. \tag{29}$$

## 3. Numerical results and discussion

For definiteness, we have taken a square area with side length equal to unity, i.e. $E = 1$. The value $h = 1$ was used, giving a maximum height equal to 0.06 for the curved part of the boundary. The form of the function $h(x)$ was chosen in other to preserve the right angles at the lower corners. For best results, the number of terms in the expansion of the approximate solution was 13, the number of collocation points used on the curved side was 50, the number of zeroed Fourier coefficients on the right vertical side and upper side taken together was 30, while this number was 16 for the left vertical side. An increase in the number of collocation points on the curved side may affect the errors on the other sides. The curves on Fig. 2 show the errors on the four straight sections of the boundary. The maximal error was about 0.0025, attained at the left end of the curved side. It was noted that this error increased with the increase of $h$ as expected. We notice that the errors are higher than in the case of a straight rectangle treated in [13] . The cause for this is that in the latter case the problem can be extended by symmetry across the axes and the expansion is thus simplified (see Fig. 3).

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