Journal of the Egyptian Mathematical Society (2014) 22, 481–483

Egyptian Mathematical Society

Journal of the Egyptian Mathematical Society

www.etms-eg.org
www.elsevier.com/locate/joems

SHORT COMMUNICATION

An alternative approach to "fixed point theorems for occasionally weakly compatible mappings"

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Received 22 November 2013; accepted 23 December 2013
Available online 31 January 2014

KEYWORDS

Occasionally weakly compatible maps;
Common fixed point;
Conditionally absorbing maps

Abstract The aim of this short communication is to provide an alternative of Bisht and Pant result (2013) [1, Theorem 1.2] in the context of framing proper setting for the application of occasionally weakly compatible mappings.

2000 MATHEMATICS SUBJECT CLASSIFICATION: 47H09; 47H10; 54H25

1. Introduction and preliminaries

In a recent work, Bisht and Pant [1], pointed out that the concept of occasionally weakly compatible (in short o.w.c.) employed in [2] does not provide a proper setting for the study of existence of common fixed point in respect of contractive condition or unique coincidence point, since in the presence of contractive condition (or unique coincidence point) the assumption of o.w.c. and the existence of a unique common fixed point coincide (this fact has also been illustrated in [3–6]). However, the situation is a bit different in case of multiple coincidence points or multiple fixed points. In this connection the authors in [1] suggested the possible approach (Theorem 1.2) to remedy the situation and improve the results contained in [2].

The main purpose of this note is to provide an alternative of Bisht and Pant result (see Theorem 1.2 in [1]) by introducing a new class of mappings called conditionally absorbing. Consequently, a host of recent results contained in the literature of non-unique common fixed point theory are significantly improved.

Before proceeding further, we recall some relevant concepts and results.

Definition 1.1 (*Jungck and Rhoades [2]*). Let X be a nonempty set. A symmetric on X is a mapping $r : X \times X \to [0, \infty)$ such that

(i) $r(x, y) = 0$ iff $x = y$,
(ii) $r(x, y) = r(y, x)\ \forall x, y \in X$.

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[1] The author thanks for the support of CSIR, Govt. of India, grant number 25(0215)/13/EMR-II.
Peer review under responsibility of Egyptian Mathematical Society.

Production and hosting by Elsevier

1110-256X
http://dx.doi.org/10.1016/j.joems.2013.12.017

Definition 1.2 (*Jungck [7]*). Two self-mappings f and g on a nonempty set X are said to be weakly compatible if they commute on the set of coincidence points that is, if $fx = gx$ for x in X, then $fgx = gfx$.

Recall that a point $x \in X$ is called a coincidence point of the pair (f, g) if $fx = gx (= w)$, the point w is then called a point of coincidence for (f, g).

Definition 1.3 (*Al-Thagafi and Shahzad [8]*). Two self-mappings f and g on a nonempty set X are said to be occasionally weakly compatible (in short o.w.c.) if there exists a point x in X which is a coincidence point of f and g at which f and g commute.

Theorem 1.1 (*Bisht and Pant [1]*). *Let (X, r) be a symmetric space with symmetric r and f and g are occasionally weakly compatible self-mappings of X satisfying*

$$r(fx, f^2x) \neq \max\{r(fx, gfx), r(f^2x, gfx), r(f^2x, g^2x)\}, \tag{1}$$

whenever the right hand side is nonzero. Then f and g have a common fixed point.

Remark 1.1 [1]. Theorem 1.1 also remains true if we replace condition (1) by any of the following.

(i) $r(fx, f^2x) \neq r(gx, g^2x)$,
(ii) $r(fx, f^2x) \neq \max\{r(gx, gfx), r(fx, gx), r(f^2x, gfx), r(fx, gfx), r(gx, f^2x)\}$,
(iii) $r(gx, g^2x) \neq \max\{r(fx, fgx), r(gx, fx), r(g^2x, fgx), r(gx, fgx), r(fx, g^2x)\}$,
(iv) $r(x, fx) \neq \max\{r(x, gx), r(fx, gx)\}$,
(v) $r(x, gx) \neq \max\{r(x, fx), r(gx, fx)\}$, and
(vi) $r(fx, f^2x) \neq r(fx, g^2x) + r(g^2x, fgx) + r(fgx, f^2x)$,

whenever the right hand side is nonzero.

2. Main results

We begin with the following example.

Example 2.1. Let $X = [2, 20]$ equipped with the symmetric $r(x, y) = (x - y)^2$. Define $f, g : X \to X$ as follows

$$fx = \begin{cases} 6 & \text{if } 2 \leqslant x < 6 \\ & \text{or } x > 6, \\ \frac{13}{2} & \text{if } x = 6, \end{cases} \qquad gx = \begin{cases} 5 & \text{if } 2 \leqslant x \leqslant 5, \\ \frac{x+7}{2} & \text{if } 5 < x \leqslant 6, \\ 10 & \text{if } 6 < x < 13/2 \\ & \text{or } x > 13/2, \\ 6 & \text{if } x = 13/2. \end{cases}$$

Then, it can be verified that (1) the pair (f, g) is weakly compatible and hence o.w.c., (2) at $x = 6$, the pair do not satisfy the conditions; (1) and (i)–(vi) mentioned earlier, (3) also, at $x = 6$, the pair fails to satisfy the condition $fx = fgx = ggx$; which actually required in the proof of Theorem 1.1. Note that the esteemed pair has no common fixed point.

The above example motivates us to define the following.

Definition 2.1. Two self-mappings f and g on a nonempty set X are called conditionally absorbing if $fx = fgx$ and $gx = gfx$ hold on a nonempty subset of the set of coincidence points whenever the set of their coincidences is nonempty.

Example 2.2. Let $X = [2, 20]$ and define $f, g : X \to X$ as follows

$$fx = \begin{cases} x + 2 & \text{if } x \in (2, 3], \\ 5 & \text{otherwise}, \end{cases} \qquad gx = \begin{cases} 2x & \text{if } x \in (2, 3], \\ 20 & \text{otherwise}. \end{cases}$$

Here the pair (f, g) has no coincidence point therefore it is not o.w.c. but it is vacuously conditionally absorbing.

Example 2.3. Let $X = [0, 1]$ and define $f, g : X \to X$ as follows

$$fx = 1 - x \quad \text{and} \quad gx = (1 - x)^2 \quad \forall x \in X.$$

Then, the pair (f, g) is o.w.c. but not conditionally absorbing. Note that the esteemed pair has two ($x = 0$ and $x = 1$) coincidence points but not common fixed point.

In view of the above examples (Examples 2.1, 2.2 and 2.3), we observe that in the event of no common fixed points, the notion of o.w.c. and conditionally absorbing are very different. However, in the context of existence of common fixed points conditionally absorbing is stronger than o.w.c. To illustrate this fact we prove the following result which provide an alternative of Theorem 1.1 in the context of framing proper setting for the application of o.w.c. mappings.

Proposition 2.1. *If a pair of self-mappings f and g on a nonempty set X are o.w.c. and conditionally absorbing, then f and g have a common fixed point.*

Proof. Since f and g are o.w.c., there exists a point u in X such that $fu = gu$ and $fgu = gfu$. Now, the conditionally absorbing property gives us $gu = gfu$ and $fu = fgu$. Thus in all, we get $fu = gu = gfu = fgu$, i.e. fu is a common fixed point of f and g. □

Example 2.4. Let $X = [2, 23]$ and define $f, g : X \to X$ as follows

$$fx = \begin{cases} 2 & \text{if } x \in \{2\} \cup (5, 7) \cup (7, 10) \cup (10, 11) \cup (11, 12) \\ & \cup (12, 13) \cup (13, 21) \cup (21, 23), \\ \frac{x+5}{2} & \text{if } 2 < x \leqslant 5, \\ 7 & \text{if } x = 7, 23, \\ 11 & \text{if } x = 10, 11, 13, \\ 11.5 & \text{if } x = 12, \\ 10 & \text{if } x = 21, \end{cases}$$

$$gx = \begin{cases} 2 & \text{if } x \in \{2\} \cup [7, 10) \cup (10, 11) \cup (11, 12) \\ & \cup (12, 13) \cup (13, 21) \cup (21, 22) \cup (22, 23), \\ 6 & \text{if } 2 < x \leqslant 5, \\ \frac{x+1}{3} & \text{if } x \in (5, 7), \\ 11 & \text{if } x = 10, 11, 13, 22, \\ 11.6 & \text{if } x = 12, \\ 10 & \text{if } x = 21, \\ 7 & \text{if } x = 23. \end{cases}$$

In this example, the pair (f, g) has two common fixed points namely, $x = 2$ and $x = 11$. Indeed, it is easy to verify that the pair (f, g) is o.w.c. and conditionally absorbing but not weakly compatible. Also, at $x = 21$, the involved maps do not satisfy any of the conditions (1) and (i)–(vi) mentioned earlier. Here it is worth noting that none of the relevant theorems contained in Imdad and Soliman [9], Pant and Pant [10], Karapinar et al. [11], Pant and Bisht [12] and Bisht and Shahzad [13] can be used in the context of this example.

Remark 2.1. It is important to note that the Proposition 2.1 do not impose symmetric or metric structure on X whereas Theorem 1.1 involves such structure.

Acknowledgement

The authors would like to express their sincere thanks to the anonymous referee(s) for their valuable suggestions and comments. The second author is thankful to S.V. National Institute of Technology, Surat, India for awarding Senior Research Fellow.

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