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SHORT COMMUNICATION

Normalizing extensions of homogeneous semilocal rings and related rings

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Abstract The aim of this paper is to study the transfer of homogeneous semilocality, FGFP and W_n properties from a ring R to a normalizing extension S and vice versa.

1. Introduction

Throughout let S be a finite normalizing extension of a ring R, that is S is finitely generated as an R-module by elements $x_1, x_2, \ldots, x_n$ of S with $Rx_i = x_iR$ for $i = 1, 2, \ldots, n$, $J(R)$ and $J(S)$ denote the Jacobson radicals of R and S, respectively. A ring R is said to be semilocal if $R/J(R)$ is an artinian ring and R is said to be homogeneous semilocal if $R/J(R)$ is a simple artinian ring.

Several authors studied the transfer of algebraic properties from S to R and from R to S. Resco [8] proved that if S is right artinian, semiprimary or perfect, then so is R. Moreover Lee [5] proved that if S is a local ring then R is a local ring. In [3] the authors studied necessary and sufficient conditions on R which forces the homogeneous semilocality of the group ring $R[G]$ or the crossed product $R * G$ and vice versa. In this paper, we shall focus on the transfer of homogeneous semilocality, or more general conditions such as W_n or FGFP rings from S to R (see Sections 3 and 4 below).

The given Examples 2.4 and 2.5 in this paper show that the homogeneous semilocality cannot be transferred directly either from S to R or from R to S. However, we will show that if R or S satisfies some weaker properties such as normal homogeneous semilocal, W_n or FGFP then those properties can be transferred from S to R (see Sections 3 and 4 below).

2. Homogeneous semilocal rings

We need the following two results.

Lemma 2.1. *Let S be a finite normalizing extension of R. Then $S/J(S)$ is a finite normalizing extension of $R/J(R)$ whose normal generators are $\bar{x}_i = x_i + J(S)$.*

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Proof. For any $\bar{s} = s + J(S) \in S/J(S) = \overline{S}; \bar{s} = \sum_{i=1}^{n} x_i r_i + J(S) = \sum_{i=1}^{n}(x_i + J(S))(r_i + J(R)) = \sum_{i=1}^{n} \bar{x}_i \bar{r}_i$. Thus $\bar{x}_i = x_i + J(S)$ is a normal basis for $\overline{S} = S/J(S)$ over $R/J(R)$. □

Theorem 2.2 [8]. *Let S be a finite normalizing extension of R. Then S is artinian if and only if R is artinian.*

Theorem 2.3. *Let S be a finite normalizing extension of R. Then S is a semilocal ring if and only if R is a semilocal ring.*

Proof. S is a semilocal ring $\Longleftrightarrow \overline{S} = S/J(S)$ is an artinian ring $\Longleftrightarrow \overline{R} = R/J(R)$ is an artinian ring (by 2.1 and 2.2) $\Longleftrightarrow$ R is a semilocal ring. □

The following two examples show that the transfer of homogeneous semilocality between R and S does not hold in either.

Example 2.4. Let $A = Z^{(p)} = \{\frac{m}{n} : m, n \in \mathbb{Z}, p \nmid n\}$ and let $R = D_n(Z^p)$, the diagonal $n \times n$ matrices over A. Let $s_{ij} = e_{ij}$ be the unit matrices. Then it is clear that $s_{ij}R = Rs_{ij}$ and $S = \sum_{i,j=1}^{n} s_{ij}R = \sum_{i,j=1}^{n} Rs_{ij} = M_n(A)$, the ring of $n \times n$ matrices over A. Hence S is a finite normalizing extension of R with n^2 generators. Since A is a local ring with unique maximal ideal $m = J(A) = pZ^{(p)}$, then $K = A/m \cong \mathbb{Z} mod p$ is a field. Therefore $R/J(R) \cong D_n(K) \cong K \oplus K \oplus \cdots \oplus K$ is an artinian ring which is not simple. However $S/J(S) \cong M_n(K)$, which is simple artinian ring. Thus S is homogeneous semilocal and R is just semilocal.

Example 2.5. Let $R = \mathbb{Z} mod 3$ be the field of integers mod 3 and $G = \langle a, b | a^2 = b^2 = 1, ab = ba \rangle$ the 4-Klein group and let $S = R[G]$ be the group algebra of G over $\mathbb{Z} mod 3$. Since $char R = 3 \nmid |G| = 4$, we know that $J(R[G]) = 0$ by Maschkes theorem and $S/J(S) \simeq S = R[G]$. In spite of being artinian S is not simple as it has the augmentation ideal $\Delta(R[G]) = \{0, 1 + a + b, 1 + a + ab, \ldots\}$.

However, if S is homogeneous semilocal, then R satisfies an intermediate property between homogeneous semilocality and semilocality (see Theorem 2.9 below).

Throughout the following definitions, S is a fixed finite normalizing extension of a ring R with normal generators $x_1, \ldots, x_n$.

Definition 2.6. An ideal I in R is called S-normal ideal if $Ix_i = x_i I$ for each $i = 1, \ldots, n$.

Definition 2.7. The ring R is called S-normal simple if R has no nonzero proper S-normal ideal.

Definition 2.8. A ring R is called normal homogeneous semilocal if $\overline{R} = R/J(R)$ is artinian and is $\overline{S} = S/J(S)$-normal simple ring.

Theorem 2.9. *Let S be a finite normalizing extension of R. If S is homogeneous semilocal, then R is S-normal homogeneous semilocal.*

Proof. Since S is a homogeneous semilocal ring, then $\overline{S} = S/J(G)$ is simple artinian ring and by Theorem 2.2 $\overline{R} = R/J(R)$ is artinian ring. To show that $\overline{R}$ has no nonzero normal $\overline{S}$-ideal, assume that $\overline{I}$ is a nonzero normal $\overline{S}$-ideal in $\overline{R}$. Since by Lemma 2.1 $\overline{S}$ is a finite normalizing extension of $\overline{R}$, then $\overline{S} = \sum \bar{x}_i \overline{R} = \sum \overline{R} \bar{x}_i$, moreover since $\bar{x}_i \overline{R} = \overline{R} \bar{x}_i$ and $\overline{I} = \overline{RI} = \overline{IR}$. Then $\overline{SI} = \sum \bar{x}_i \overline{RI} = \sum \overline{R} \bar{x}_i \overline{I} = \sum \overline{RI} \bar{x}_i = \sum \overline{IR} \bar{x}_i = \overline{IS}$. If $\overline{SI} = \overline{S}$, then $\overline{I}$ contains an element $\bar{u}$ such that $\bar{s}\bar{u} = \bar{1}$, where $s \in S$. By Proposition 3.3, the element $\bar{u} \in \overline{I}$ is invertible in $\overline{R}$, which contradicts that $\overline{I}$ is a nonzero proper ideal in $\overline{R}$. Therefore $\overline{IS}$ is a nonzero two sided ideal in $\overline{S}$ which contradicts the simplicity of $\overline{S}$. Hence R is S-normal homogeneous semilocal. □

3. W_n-rings

In [12] Woods introduced the notion of W_n-ring to study the transfer of semiperfectness between the ring R and the group ring $R[G]$ and in [6] Okninski used the notation of W_n-ring to show that the class of W_n-rings contains the class of semilocal rings.

Definition 3.1. A ring R is said to be W_n-ring if for any $r \in R$ there exists an integer $i, 1 \leqslant i \leqslant n$, such that $1 - f_i(r)$ is invertible in R; where

$$f_1(r) = r, \quad f_i(r) = f_{i-1}(r)(1 - f_{i-1}(r)).$$

The next theorem gives a new characterization for homogeneous semilocal rings.

Theorem 3.2. *A ring R is homogeneous semilocal if and only if R is a W_n-ring for some n and has a unique maximal two sided ideal.*

Proof. Assume that R is homogeneous semilocal ring, then it follows from [2] that R has a unique maximal ideal $m = J(R)$. Since R is a homogeneous semilocal ring, then R is semilocal and hence by [6, Lemma 3.1] R satisfies W_n fore some n.

Assume that R satisfies W_n and has a unique maximal ideal m. Let P be any primitive ideal then R/P is a primitive ring satisfying W_n. Thus by [6, Lemma 1] $R/P \cong M_t(D)(t \times t$ matrices over a division ring $D)$ is simple artinian. Thus P is a maximal ideal and so every primitive ideal equals m. Therefore $m = J(R)$ and $R/J(R) = R/P$ is simple artinian. Hence R is homogeneous semilocal. □

In spite of the non-transfer of homogeneous semilocality from S to R and vice versa, we can transfer the W_n structure from S to R using the next result of Resco.

Proposition 3.3 [8]. *Let S be a finite normalizing extension of R. Then an element $u \in R$ is invertible in S if and only if u is invertible in R.*

Thus we can show the following:

Theorem 3.4. *If S is a W_n-ring, then so is R.*

Proof. Since S is a W_n-ring for any $s \in S$ there exists $i, 1 \leqslant i \leqslant n$, such that $1 - f_i(s)$ is invertible in S. Then for any $r \in R \subseteq S$ there exists $i, 1 \leqslant i \leqslant n$, such that $1 - f_i(r)$ is invertible in S. By (Proposition 3.3) $1 - f_i(r)$ is invertible in R for some $1 \leqslant i \leqslant n$. Hence R is W_n-ring. □

4. FGFP rings

Throughout this section all modules are unitary right modules. A well known theorem due to Bass [1] states that a ring R is left perfect if and only if every flat R-module is projective. Therefore, it is natural to study the rings R, for which every finitely generated flat R-module is projective. Such rings are called FGFP rings. In [10] Sakhaev and Chirkov proved that a semilocal ring with a unique primitive ideal is an FGFP ring.

In [7] Puninski and Rothmaler proved that the FGFP property is closed under Morita equivalence, finite direct sum and subrings. Jondrup [4] proved that FGFP property is interchanged between a ring R and the group ring $R[G]$, also between R and the ring of power series $R[[x]]$. Right notherian rings, right Ore domains with right Krull dimension and semiperfect rings are FGFP rings. The latter two rings are even left and right FGFP rings.

If a sequence $A_1, A_2, A_3, \ldots$ of $M_n(R)$ the $n \times n$ matrices over R such that $A_{i+1}A_i = A_i$ for every i, eventually consists of idempotent generating the same principal right ideal in $M_n(R)$, then we say in this case that the sequence converges [7]. Following Sakhaev [9] an ideal I of a ring R is called weakly commutative if there exists $m \geqslant 2$ such that for all $a_1, a_2, \ldots, a_m \in I$ there exists a non identity permutation $\sigma \in S_m$ such that $a_1, a_2, \ldots, a_m \in Ra_{\sigma(1)}, a_{\sigma(2)}, \ldots, a_{\sigma(m)}$. A ring R is called M_nCP if every right cyclic $M_n(R)$-module is projective.

Theorem 4.1 [9]. *Let R is a semilocal ring. then*

(1) *R is FGFP if and only if R is M_nCP for some $n > 0$.*
(2) *If $J(R)$ is weakly commutative, then R is FGFP.*
(3) *If R is FGFP then every sequence $A_1, A_2, A_3, \ldots$ of $M_n(R)$ converges.*

Theorem 4.2 [7]. *If for every $n > 0$, one of the following conditions are satisfied (i) $M_n(R)$ satisfies a.c.c an right annihilators of elements or (ii) $M_n(R)$ satisfies d.c.c on left annihilators of elements, then R is FGFP.*

Theorem 4.3 [11]. *An R-module T is flat if and only if $S \otimes_R T$ is a flat S-module.*

Proposition 4.4. *Let R be a homogeneous semilocal ring. Then R is an FGFP ring.*

Proof. If R is homogeneous semilocal, then R is W_n-ring having a unique maximal ideal $\mathscr{M}$ by Theorem 3.2. Using the same argument as in the proof of Theorem 3.2 we deduce that every primitive ideal is maximal. Thus R has a unique primitive ideal $\mathscr{M}$. Since R is semilocal, then by [10] R is FGFP. □

In the rest of this paper S is a finite normalizing extension of R.

Proposition 4.5. *If S is FGFP, then so is R.*

Proof. Follows directly by [7, Lemma 3.1]. □

In Example 2.4 we showed that the homogeneous semilocality of S does not imply the homogeneous semilocality of R, however the homogeneous semilocality of S guarantees that at least R is FGFP, M_nCP and W_n-ring.

Corollary 4.6. *Let S be a homogeneous semilocal ring. Then:*

(i) *R is FGFP ring,*
(ii) *R is M_nCP,*
(iii) *every sequence $A_1, A_2, A_3, \ldots$ of $M_n(R)$ converges,*
(iv) *R is W_n-ring.*

Proof.

(i) Since S is homogeneous semilocal, then by Proposition 4.4 S is FGFP. Thus R is FGFP by Proposition 4.5.
(ii) Since S is homogeneous semilocal, then R is semilocal by Theorem 2.3. Thus by Theorem 4.1, R is M_nCP
(iii) From (ii) R is semilocal and by Theorem 4.1 (3) the result follows.
(iv) Follows directly from Theorems 3.2 and 3.4. □

Proposition 4.7. *Let S be a flat R-module, and R is a FGFP ring. Then S is a FGFP ring.*

Proof. Since S is finitely generated flat R-module and there is a ring monomorphism from R to S, then the ring S is FGFP by [4, Proposition 1.2]. □

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