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SHORT COMMUNICATION

Unitarily invariant norm inequalities for operators

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Abstract We present several norm inequalities for Hilbert space operators. In particular, we prove that if $A_1, A_2, \ldots, A_n \in \mathbb{B}(\mathscr{H})$, then

$$|||A_1A_2^* + A_2A_3^* + \cdots + A_nA_1^*||| \leqslant \left|\left|\left|\sum_{i=1}^{n} A_iA_i^*\right|\right|\right|$$

for all unitarily invariant norms.

We also show that if A_1, A_2, A_3, A_4 are projections in $\mathbb{B}(\mathscr{H})$, then

$$\left|\left|\left|\left(\sum_{i=1}^{4}(-1)^{i+1}A_i\right) \oplus 0 \oplus 0 \oplus 0\right|\right|\right| \leqslant |||(A_1 + |A_3A_1|) \oplus (A_2 + |A_4A_2|) \oplus (A_3 + |A_1A_3|) \oplus (A_4 + |A_2A_4|)|||$$

for any unitarily invariant norm.

1. Introduction and preliminaries

Let $\mathbb{B}(\mathscr{H})$ stand for the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathscr{H}$ with inner product $\langle \cdot, \cdot \rangle$ and let I denote the identity operator. For $A \in \mathbb{B}(\mathscr{H})$, let $\|A\| = \sup\{\|Ax\| : \|x\| = 1\}$ denote the usual operator norm of A and $|A| = (A^*A)^{1/2}$ be the absolute value of A. For $1 \leqslant p < \infty$, the Schatten p-norm of a compact operator A is defined by $\|A\|_p = (\mathrm{tr}|A|^p)^{1/p}$, where tr is the usual trace functional. If A and B are operators in $\mathbb{B}(\mathscr{H})$ we use $A \oplus B$ to denote the 2×2 operator matrix $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$, regarded as an operator on $\mathscr{H} \oplus \mathscr{H}$. One can show that

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$$\|A \oplus B\| = \max(\|A\|, \|B\|) \tag{1.1}$$

$$\|A \oplus B\|_p = \left(\|A\|_p^p + \|B\|_p^p\right)^{1/p} \tag{1.2}$$

An operator $A \in \mathbb{B}(\mathscr{H})$ is positive and write $A \geqslant 0$ if $\langle A(x), x\rangle \geqslant 0$ for all $x \in \mathscr{H}$. We say $A \leqslant B$ whenever $B - A \geqslant 0$.

We consider the wide class of unitarily invariant norms $|||\cdot|||$. Each of these norms is defined on an ideal in $\mathbb{B}(\mathscr{H})$ and it will be implicitly understood that when we talk about $|||T|||$, then the operator T belongs to the norm ideal associated with $|||\cdot|||$. Each unitarily invariant norm $|||\cdot|||$ is characterized by the invariance property $|||UTV||| = |||T|||$ for all operators T in the norm ideal associated with $|||\cdot|||$ and for all unitary operators U and V in $\mathbb{B}(\mathscr{H})$. The following are easily deduced by utilizing the basic properties of unitarily invariant norms

$$|||A \oplus A^*||| = |||A \oplus A|||, \tag{1.3}$$

$$|||A \oplus B||| = \left|\left|\left|\begin{bmatrix} 0 & A \\ B & 0 \end{bmatrix}\right|\right|\right| \tag{1.4}$$

$$|||AA^*||| = |||A^*A||| \tag{1.5}$$

for all operators $A, B \in \mathbb{B}(\mathscr{H})$. For the general theory of unitarily invariant norms, we refer the reader to Bhatia and Simon [1,12].

It follows from the Fan dominance principle (see [7] and [11]) that the following three inequalities for all unitarily invariant norms are equivalence:

$$|||A||| \leqslant |||B|||, \tag{1.6}$$

$$|||A \oplus 0||| \leqslant |||B \oplus 0|||, \tag{1.7}$$

$$|||A \oplus A||| \leqslant |||B \oplus B|||. \tag{1.8}$$

It has been shown by Kittaneh [9] that if A_1, A_2, B_1, B_2, X, and Y are operators in $\mathbb{B}(\mathscr{H})$, then

$$2|||(A_1XA_2^* + B_1YB_2^*) \oplus 0||| \leqslant \left|\left|\left|\begin{bmatrix} A_1^*A_1X + XA_2^*A_2 & A_1^*B_1Y + XA_2^*B_2 \\ B_1^*A_1X + YB_2^*A_2 & B_1^*B_1Y + YB_2^*B_2 \end{bmatrix}\right|\right|\right| \tag{1.9}$$

for all unitarily invariant norms.

It has been shown by Bhatia and Kittaneh [2] that if A and B are operators in $\mathbb{B}(\mathscr{H})$, then

$$|||A^*B + B^*A||| \leqslant |||A^*A + B^*B|||, \tag{1.10}$$

for all unitarily invariant norms.

Kittaneh [10] proved that if A and B are positive operators in $\mathbb{B}(\mathscr{H})$, then

$$|||(A+B) \oplus 0||| \leqslant \left|\left|\left|\left(A + \left|B^{1/2}A^{1/2}\right|\right) \oplus \left(B + \left|A^{1/2}B^{1/2}\right|\right)\right|\right|\right| \tag{1.11}$$

for any unitarily invariant norm.

It was shown by Fong [4] that if $A \in M_n(C)$, then

$$\|AA^* - A^*A\| \leqslant \|A\|^2, \tag{1.12}$$

and it was shown by Kittaneh [8] that

$$\|AA^* + A^*A\| \leqslant \|A^2\| + \|A\|^2. \tag{1.13}$$

In this paper we establish some norm and norm inequalities. We generalize inequalities (1.9) and (1.10) and present a norm inequality analogue to (1.11). Based on our main result, we provide new proofs of inequalities (1.12) and (1.13).

2. Main results

To achieve our main result we need the following lemma.

Lemma 2.1 (3, Theorem 1). *If A, B and X are operators in $\mathbb{B}(\mathscr{H})$, then*

$$2|||AXB^*||| \leqslant |||A^*AX + XB^*B||| \tag{2.1}$$

for any unitarily invariant norm.

The main result below is an extension of [9, Theorem 2.2]. We will prove it by an approach different from [9, Theorem 2.2].

Theorem 2.2. *Let $A_i, B_i, X_i \in \mathbb{B}(\mathscr{H})$ for $i, j = 1, 2 \ldots, n$. Then*

$$2|||\sum_{i=1}^{n}(A_iX_iB_i^*) \oplus 0 \oplus \cdots \oplus 0||| \leqslant \left|\left|\left|\begin{bmatrix} A_1^*A_1X_1 + X_1B_1^*B_1 & A_1^*A_2X_2 + X_1B_1^*B_2 & \cdots & A_1^*A_nX_n + X_1B_1^*B_n \\ A_2^*A_1X_1 + X_2B_2^*B_1 & A_2^*A_2X_2 + X_2B_2^*B_2 & \cdots & A_2^*A_nX_n + X_2B_2^*B_n \\ \vdots & \vdots & \ddots & \vdots \\ A_n^*A_1X_1 + X_nB_n^*B_1 & A_n^*A_2X_2 + X_nB_n^*B_2 & \cdots & A_n^*A_nX_n + X_nB_n^*B_n \end{bmatrix}\right|\right|\right|$$

for all unitarily invariant norms.

Proof. Consider the following operators on $\oplus_{i=1}^{n}\mathscr{H}$

$$A = \begin{bmatrix} A_1 & A_2 & \cdots & A_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

$$B = \begin{bmatrix} B_1 & B_2 & \cdots & B_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

and

$$X = \begin{bmatrix} X_1 & 0 & \cdots & 0 \\ 0 & X_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & X_n \end{bmatrix}.$$

Then

$$
\begin{aligned}
&2|||\sum_{i=1}^{n}(A_iX_iB_i^*)\oplus 0\oplus\cdots\oplus 0|||\\
&=2\left|\left|\left|\begin{bmatrix} A_1 & A_2 & \cdots & A_n\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 0\end{bmatrix}\begin{bmatrix} X_1 & 0 & \cdots & 0\\ 0 & X_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & X_n\end{bmatrix}\begin{bmatrix} B_1^* & 0 & \cdots & 0\\ B_2^* & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ B_n^* & 0 & \cdots & 0\end{bmatrix}\right|\right|\right|\\
&=2|||AXB^*|||\\
&\leqslant|||A^*AX+XB^*B||| \qquad \text{(by (2.1))}\\
&=\left|\left|\left|\begin{bmatrix} A_1^* & 0 & \cdots & 0\\ A_2^* & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ A_n^* & 0 & \cdots & 0\end{bmatrix}\begin{bmatrix} A_1 & A_2 & \cdots & A_n\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 0\end{bmatrix}\begin{bmatrix} X_1 & 0 & \cdots & 0\\ 0 & X_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & X_n\end{bmatrix}\right.\right.\right.\\
&\quad+\left.\left.\left.\begin{bmatrix} X_1 & 0 & \cdots & 0\\ 0 & X_2 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & X_n\end{bmatrix}\begin{bmatrix} B_1^* & 0 & \cdots & 0\\ B_2^* & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ B_n^* & 0 & \cdots & 0\end{bmatrix}\begin{bmatrix} B_1 & B_2 & \cdots & B_n\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 0\end{bmatrix}\right|\right|\right|\\
&=\left|\left|\left|\begin{bmatrix} A_1^*A_1X_1+X_1B_1^*B_1 & A_1^*A_2X_2+X_1B_1^*B_2 & \cdots & A_1^*A_nX_n+X_1B_1^*B_n\\ A_2^*A_1X_1+X_2B_2^*B_1 & A_2^*A_2X_2+X_2B_2^*B_2 & \cdots & A_2^*A_nX_n+X_2B_2^*B_n\\ \vdots & \vdots & \ddots & \vdots\\ A_n^*A_1X_1+X_nB_n^*B_1 & A_n^*A_2X_2+X_nB_n^*B_2 & \cdots & A_n^*A_nX_n+X_nB_n^*B_n\end{bmatrix}\right|\right|\right|.
\end{aligned}
$$

□

Corollary 2.3. Let $A_1, A_2, \ldots, A_n \in \mathbb{B}(\mathscr{H})$. Then

$$|||A_1A_2^*+A_2A_3^*+\cdots+A_nA_1^*|||\leqslant\left|\left|\left|\sum_{i=1}^{n}A_iA_i^*\right|\right|\right|,$$

for all unitarily invariant norms. In particular,

$$\|A_1A_2^*+A_2A_3^*+\cdots+A_nA_1^*\|_p\leqslant\left\|\sum_{i=1}^{n}A_iA_i^*\right\|_p \quad \textit{for}\ \ 1\leqslant p\leqslant\infty.$$

Proof. Letting $B_i = A_{i+1}$ for $i = 1, 2\ldots, n-1$ and $B_n = A_1$ and $X_i = I$ in Theorem 2.2, we get

$$
\begin{aligned}
&2|||A_1A_2^*+A_2A_3^*+\cdots+A_nA_1^*\oplus 0\cdots\oplus 0|||\\
&\leqslant\left|\left|\left|\begin{bmatrix} A_1^*A_1+A_2^*A_2 & A_1^*A_2+A_2^*A_3 & \cdots & A_1^*A_n+A_2^*A_1\\ A_2^*A_1+A_3^*A_2 & A_2^*A_2+A_3^*A_3 & \cdots & A_2^*A_n+A_3^*A_1\\ \vdots & \vdots & \ddots & \vdots\\ A_n^*A_1+A_1^*A_2 & A_n^*A_2+A_1^*A_3 & \cdots & A_n^*A_n+A_1^*A_1\end{bmatrix}\right|\right|\right|\\
&\leqslant\left|\left|\left|\begin{bmatrix} A_1^*A_1 & A_1^*A_2 & \cdots & A_1^*A_n\\ A_2^*A_1 & A_2^*A_2 & \cdots & A_2^*A_n\\ \vdots & \vdots & \ddots & \vdots\\ A_n^*A_1 & A_n^*A_2 & \cdots & A_n^*A_n\end{bmatrix}\right|\right|\right|+\left|\left|\left|\begin{bmatrix} A_2^*A_2 & A_2^*A_3 & \cdots & A_2^*A_1\\ A_3^*A_2 & A_3^*A_3 & \cdots & A_3^*A_1\\ \vdots & \vdots & \ddots & \vdots\\ A_1^*A_2 & A_1^*A_3 & \cdots & A_1^*A_1\end{bmatrix}\right|\right|\right|\\
&=\left|\left|\left|\begin{bmatrix} A_1^* & 0 & \cdots & 0\\ A_2^* & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ A_n^* & 0 & \cdots & 0\end{bmatrix}\begin{bmatrix} A_1 & A_2 & \cdots & A_n\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 0\end{bmatrix}\right|\right|\right|\\
&\quad+\left|\left|\left|\begin{bmatrix} A_2^* & 0 & \cdots & 0\\ A_3^* & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ A_1^* & 0 & \cdots & 0\end{bmatrix}\begin{bmatrix} A_2 & A_3 & \cdots & A_1\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 0\end{bmatrix}\right|\right|\right|\\
&=\left|\left|\left|\begin{bmatrix} A_1 & A_2 & \cdots & A_n\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 0\end{bmatrix}\begin{bmatrix} A_1^* & 0 & \cdots & 0\\ A_2^* & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ A_n^* & 0 & \cdots & 0\end{bmatrix}\right|\right|\right|\\
&\quad+\left|\left|\left|\begin{bmatrix} A_2 & A_3 & \cdots & A_1\\ 0 & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \cdots & 0\end{bmatrix}\begin{bmatrix} A_2^* & 0 & \cdots & 0\\ A_3^* & 0 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ A_1^* & 0 & \cdots & 0\end{bmatrix}\right|\right|\right| \text{(by (1.5))}\\
&=2\left|\left|\left|\left(\sum_{i=1}^{n}A_iA_i^*\right)\oplus 0\cdots\oplus 0\right|\right|\right|
\end{aligned}
$$

By the equivalence of inequalities (1.6) and (1.7) we have

$$|||A_1A_2^*+A_2A_3^*+\cdots+A_nA_1^*|||\leqslant\left|\left|\left|\sum_{i=1}^{n}A_iA_i^*\right|\right|\right|$$

for all unitarily invariant norms. □

To establish the next result we need the following Lemma. The lemma is a basic triangle inequality comparing, in unitarily invariant norms, the sum of two normal operators to the sum of their absolute values.

Lemma 2.4. [5] *If A and B are normal operators in $\mathbb{B}(\mathscr{H})$, then*

$$|||A+B|||\leqslant|||\ |A|+|B|\ |||. \tag{2.2}$$

for all unitarily invariant norms.

Corollary 2.5. Let A_1, A_2, A_3, A_4 be projections in $\mathbb{B}(\mathscr{H})$. Then

$$
\begin{aligned}
&\left|\left|\left|\left(\sum_{i=1}^{4}(-1)^{i+1}A_i\right)\oplus 0\oplus 0\oplus 0\right|\right|\right|\\
&\leqslant|||(A_1+|A_3A_1|)\oplus(A_2+|A_4A_2|)\oplus(A_3+|A_1A_3|)\oplus(A_4+|A_2A_4|)||| \qquad (2.3)
\end{aligned}
$$

for all unitarily invariant norms. In particular,

$$
\begin{aligned}
&\left\|\sum_{i=1}^{4}(-1)^{i+1}A_i\right\|\\
&\leqslant\max\{\|A_1+|A_3A_1|\|,\|A_2+|A_4A_2|\|,\|A_3+|A_1A_3|\|,\|A_4+|A_2A_4|\|\}
\end{aligned}
$$

and

$$
\begin{aligned}
\left\|\sum_{i=1}^{4}(-1)^{i+1}A_i\right\|_p &\leqslant\Big(\|A_1+|A_3A_1|\|_p^p+\|A_2+|A_4A_2|\|_p^p+\|A_3+|A_1A_3|\|_p^p\\
&\quad+\|A_4+|A_2A_4|\|_p^p\Big)^{1/p}(1\leqslant p<\infty).
\end{aligned}
$$

Proof. Letting $n = 4$, and replacing both A_i and B_i by A_i, and putting $X_i = (-1)^{i+1}I$ for $i = 1, 2, 3, 4$ in Theorem 2.2, we get

$$
\begin{aligned}
&\left|\left|\left|\begin{bmatrix} \sum_{i=1}^{4}(-1)^{i+1}A_i & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}\right|\right|\right|\\
&\leqslant\left|\left|\left|\begin{bmatrix} A_1 & 0 & A_1A_3 & 0\\ 0 & -A_2 & 0 & -A_2A_4\\ A_3A_1 & 0 & A_3 & 0\\ 0 & -A_4A_2 & 0 & -A_4\end{bmatrix}\right|\right|\right|\\
&=\left|\left|\left|\begin{bmatrix} A_1 & 0 & A_1A_3 & 0\\ 0 & A_2 & 0 & A_2A_4\\ A_3A_1 & 0 & A_3 & 0\\ 0 & A_4A_2 & 0 & A_4\end{bmatrix}\right|\right|\right| \quad \text{(by the unitary invariance of the norm)}\\
&=\left|\left|\left|\begin{bmatrix} A_1 & 0 & 0 & 0\\ 0 & A_2 & 0 & 0\\ 0 & 0 & A_3 & 0\\ 0 & 0 & 0 & A_4\end{bmatrix}+\begin{bmatrix} 0 & 0 & A_1A_3 & 0\\ 0 & 0 & 0 & A_2A_4\\ A_3A_1 & 0 & 0 & 0\\ 0 & A_4A_2 & 0 & 0\end{bmatrix}\right|\right|\right|
\end{aligned}
$$

$$\leqslant \left|\left|\left|\begin{bmatrix} A_1 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 \\ 0 & 0 & A_3 & 0 \\ 0 & 0 & 0 & A_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 & A_1A_3 & 0 \\ 0 & 0 & 0 & A_2A_4 \\ A_3A_1 & 0 & 0 & 0 \\ 0 & A_4A_2 & 0 & 0 \end{bmatrix}\right|\right|\right| \quad (\text{by}\,(2.2))$$

$$= \left|\left|\left|\begin{bmatrix} A_1 & 0 & 0 & 0 \\ 0 & A_2 & 0 & 0 \\ 0 & 0 & A_3 & 0 \\ 0 & 0 & 0 & A_4 \end{bmatrix} + \begin{bmatrix} |A_3A_1| & 0 & 0 & 0 \\ 0 & |A_4A_2| & 0 & 0 \\ 0 & 0 & |A_1A_3| & 0 \\ 0 & 0 & 0 & |A_2A_4| \end{bmatrix}\right|\right|\right|$$

$$= \left|\left|\left|\begin{bmatrix} A_1+|A_3A_1| & 0 & 0 & 0 \\ 0 & A_2+|A_4A_2| & 0 & 0 \\ 0 & 0 & A_3+|A_1A_3| & 0 \\ 0 & 0 & 0 & A_4+|A_2A_4| \end{bmatrix}\right|\right|\right|.$$

This proves inequality (2.3).

The rest inequalities follow from (2.3), (1.1) and (1.2). □

Corollary 2.6. Let $A_1, A_2, \ldots, A_n$ be positive operators in $\mathbb{B}(\mathscr{H})$. Then

$$\|A_1 + A_2 + \cdots + A_n\| \leqslant \max\{\|A_i + (n-1)\|A_i\|\| : i = 1, 2, \ldots, n\}. \tag{2.4}$$

Proof. First we show that

$$\begin{bmatrix} A_1 & A_1^{1/2}A_2^{1/2} & \cdots & A_1^{1/2}A_n^{1/2} \\ A_2^{1/2}A_1^{1/2} & A_2 & \cdots & A_2^{1/2}A_n^{1/2} \\ \vdots & \vdots & \ddots & \vdots \\ A_n^{1/2}A_1^{1/2} & A_n^{1/2}A_2^{1/2} & \cdots & A_n \end{bmatrix}$$
$$\leqslant \begin{bmatrix} A_1 + (n-1)\|A_1\| & 0 & \cdots & 0 \\ 0 & A_2 + (n-1)\|A_2\| & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n + (n-1)\|A_n\| \end{bmatrix}. \tag{2.5}$$

It is enough to show that

$$C = \begin{bmatrix} (n-1)A_1 & -A_1^{1/2}A_2^{1/2} & \cdots & -A_1^{1/2}A_n^{1/2} \\ -A_2^{1/2}A_1^{1/2} & (n-1)A_2 & \cdots & -A_2^{1/2}A_n^{1/2} \\ \vdots & \vdots & \ddots & \vdots \\ -A_n^{1/2}A_1^{1/2} & -A_n^{1/2}A_2^{1/2} & \cdots & (n-1)A_n \end{bmatrix} \geqslant 0.$$

To see this, we note that

$$nC = \begin{bmatrix} (n-1)A_1^{1/2} & -A_1^{1/2} & \cdots & -A_1^{1/2} \\ -A_2^{1/2} & (n-1)A_2^{1/2} & \cdots & -A_2^{1/2} \\ \vdots & \vdots & \ddots & \vdots \\ -A_n^{1/2} & -A_n^{1/2} & \cdots & (n-1)A_n^{1/2} \end{bmatrix} \begin{bmatrix} (n-1)A_1^{1/2} & -A_2^{1/2} & \cdots & -A_n^{1/2} \\ -A_1^{1/2} & (n-1)A_2^{1/2} & \cdots & -A_n^{1/2} \\ \vdots & \vdots & \ddots & \vdots \\ -A_1^{1/2} & -A_2^{1/2} & \cdots & (n-1)A_n^{1/2} \end{bmatrix} \geqslant 0.$$

Next, by letting $X_i = I$ and replacing both A_i and B_i by $A_i^{1/2}$ in Theorem 2.2, we obtain

$$\|A_1 + A_2 + \cdots + A_n\| \leqslant \left\|\begin{bmatrix} A_1 & A_1^{1/2}A_2^{1/2} & \cdots & A_1^{1/2}A_n^{1/2} \\ A_2^{1/2}A_1^{1/2} & A_2 & \cdots & A_2^{1/2}A_n^{1/2} \\ \vdots & \vdots & \ddots & \vdots \\ A_n^{1/2}A_1^{1/2} & A_n^{1/2}A_2^{1/2} & \cdots & A_n \end{bmatrix}\right\|$$
$$\leqslant \left\|\begin{bmatrix} A_1 + (n-1)\|A_1\| & 0 & \cdots & 0 \\ 0 & A_2 + (n-1)\|A_2\| & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & A_n + (n-1)\|A_n\| \end{bmatrix}\right\| \quad (\text{by}(2.5))$$

Hence

$$\|A_1 + A_2 + \cdots + A_n\| \leqslant \max\{\|A_i + (n-1)\|A_i\|\| : i = 1, 2, \ldots, n\}. \qquad \square$$

Corollary 2.7. Let A and B be normal operators in $\mathbb{B}(\mathscr{H})$. Then
$\|A + B\| \leqslant \max\{\||A| + \|A\|\|, \||B| + \|B\|\|\}$.

Proof. Letting $n = 2, A_1 = |A|, A_2 = |B|$ in (2.4), we obtain

$$\begin{aligned} \|A + B\| &\leqslant \||A| + |B|\| \qquad (\text{by}(2.2)) \\ &\leqslant \max\{\||A| + \|A\|\|, \||B| + \|B\|\|\}. \quad \square \end{aligned}$$

To establish the next result we need the following lemma.

Lemma 2.8. [6, Theorem 1.1] *If A, B, C and D are operators in $\mathbb{B}(\mathscr{H})$, then*

$$\left\|\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right\| \leqslant \left\|\begin{bmatrix} \|A\| & \|B\| \\ \|C\| & \|D\| \end{bmatrix}\right\|. \tag{2.6}$$

Corollary 2.9. Let $A \in \mathbb{B}(\mathscr{H})$. Then

$$\|AA^* + A^*A\| \leqslant \|A^2\| + \|A\|^2 \tag{2.7}$$

and

$$\|AA^* - A^*A\| \leqslant \|A\|^2. \tag{2.8}$$

Proof. Letting $n = 2, A_1 = B_1 = A, A_2 = B_2 = A^*$ and $X_1 = X_2 = I$ in Theorem 2.2 we get

$$\begin{aligned} \|AA^* + A^*A\| &\leqslant \left\|\begin{bmatrix} A^*A & A^{*2} \\ A^2 & AA^* \end{bmatrix}\right\| \\ &\leqslant \left\|\begin{bmatrix} \|A^*A\| & \|A^{*2}\| \\ \|A^2\| & \|AA^*\| \end{bmatrix}\right\| (\text{by}\,(2.6)) \end{aligned}$$

Since

$$\begin{bmatrix} \|A^*A\| & \|A^{*2}\| \\ \|A^2\| & \|AA^*\| \end{bmatrix} = \begin{bmatrix} \|A\|^2 & \|A^2\| \\ \|A^2\| & \|A\|^2 \end{bmatrix}$$

is self-adjoint, the usual operator norm of this matrix is equal to its spectral radius, so

$$\left\|\begin{bmatrix} \|A\|^2 & \|A^2\| \\ \|A^2\| & \|A\|^2 \end{bmatrix}\right\| = \|A^2\| + \|A\|^2.$$

This proves inequality (2.7). To prove inequality (2.8), putting $n = 2, A_1 = B_1 = A, A_2 = B_2 = A^*$ and $X_1 = I = -X_2$ in Theorem 2.2, to get

$$\begin{aligned} \|AA^* - A^*A\| &\leqslant \left\|\begin{bmatrix} A^*A & 0 \\ 0 & AA^* \end{bmatrix}\right\| \\ &\leqslant \left\|\begin{bmatrix} \|A^*A\| & 0 \\ 0 & \|AA^*\| \end{bmatrix}\right\| (\text{by}\,(2.6)) \\ &= \|A\|^2. \qquad \square \end{aligned}$$

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