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REVIEW PAPER

On operator ideals using weighted Cesàro sequence space

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Abstract Let $s = (s_n)$ be a sequence of s-numbers in the sense of Pietsch. In this paper we have introduced a class $\mathcal{A}^{(s)}_{p,q}$ of s-type $ces(p,q)$ operators by using weighted Cesàro sequence space for $1 < p < \infty$. It is shown that the class $\mathcal{A}^{(s)}_{p,q}$ forms a quasi-Banach operator ideal. Moreover, the inclusion relations among the operator ideals as well as the inclusion relations among their duals are established. Finally, we have proved that the class $\mathcal{A}^{(a)}_{p,q}$ of approximation type $ces(p,q)$ operators is small.

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Contents

1. Introduction

Due to the immense applications in spectral theory, geometry of Banach spaces, theory of eigenvalue distributions etc., the theory of operator ideals occupies a special importance in functional analysis. Many useful operator ideals have been defined by using sequence of s-numbers. In 1963, Pietsch [1] introduced the approximation numbers of a bounded linear operator in Banach spaces. Subsequently, different s-numbers, namely Kolmogorov numbers, Gel'fand numbers, etc. are introduced to the Banach space setting. For the unifications of different s-numbers, Pietsch [2] defined an axiomatic theory of s-numbers in Banach spaces.

For $1 < p < \infty$, the Cesàro sequence space ces_p [3,4] is defined as

$$ces_p = \left\{ x = (x_k) \in w : \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{k=1}^{n} |x_k| \right)^p < \infty \right\},$$

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where w is the set of all real or complex sequences. The space ces_p is complete with respect to the norm $\|x\| = \left(\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}|x_k|\right)^p\right)^{\frac{1}{p}}$. It is easy to verify that if $1 < p \leqslant r < \infty$, then $ces_p \subseteq ces_r$. For more on the Cesàro sequence space and Cesàro operator one can refer [5–8].

Pietsch [1] defined an operator $T \in \mathcal{L}(E,F)$ to be l^p type operator if $\sum_{n=1}^{\infty}(a_n(T))^p$ is finite for $0 < p < \infty$, where $(a_n(T))$ is the sequence of approximation numbers of the bounded linear operator T. Later on Constantin [9] generalized the class of l_p type operators to the class of $ces-p$ type operators by using the Cesàro sequence spaces, where an operator $T \in \mathcal{L}(E,F)$ is called $ces-p$ type if $\sum_{n=1}^{\infty}\left(\frac{1}{n}\sum_{k=1}^{n}a_k(T)\right)^p$ is finite, $1 < p < \infty$. For $A-p$ type operators and Stolz mappings one can see [10,11].

The purpose of this paper is to study s-type $ces(p,q)$ operators using weighted Cesàro sequence space. The s-type $ces(p,q)$ operators are more general than the $ces-p$ type operators. We show that the class $\mathcal{A}^{(s)}_{p,q}$ of s-type $ces(p,q)$ operators is a quasi-Banach operator ideal. Moreover, the inclusion relations among the operator ideals as well as the inclusion relations among their duals are established. Finally, we also prove that the class $\mathcal{A}^{(a)}_{p,q}$ of approximation type $ces(p,q)$ operators is small.

2. Preliminaries

Throughout this paper we denote E, F as the real or complex Banach spaces and $\mathcal{L}(E,F)$ as the space of all bounded linear operators from E to F. Let $\mathcal{L}$ be the class of all bounded linear operators between arbitrary Banach spaces. We denote E' as the dual of E and x' is the continuous linear functional on E. $\mathbb{N}$ and $\mathbb{R}^+$ stand for the set of all natural numbers and the set of all nonnegative real numbers respectively. Let $x' \in E'$ and $y \in F$. Then the map $x' \otimes y : E \to F$ is defined by $(x' \otimes y)(x) = x'(x)y, x \in E$.

We now state few results which will be used in the sequel. Before it, we recall some basic definitions and terminologies of **s-numbers of operators** and **operator ideals**.

Definition 2.1. A finite rank operator is a bounded linear operator whose dimension of the range space is finite.

Definition 2.2 (*[12,13]*). A map $s = (s_n) : \mathcal{L} \to \mathbb{R}^+$ assigning to every operator $T \in \mathcal{L}$ a nonnegative scalar sequence $(s_n(T))_{n\in\mathbb{N}}$ is called an s-number sequence if the following conditions are satisfied:

- $(S1)$ monotonicity: $\|T\| = s_1(T) \geqslant s_2(T) \geqslant \cdots \geqslant 0$, for $T \in \mathcal{L}(E,F)$
- $(S2)$ additivity: $s_{m+n-1}(S+T) \leqslant s_m(S) + s_n(T)$, for $S, T \in \mathcal{L}(E,F)$, $m, n \in \mathbb{N}$
- $(S3)$ ideal property: $s_n(RST) \leqslant \|R\| s_n(S) \|T\|$, for some $R \in \mathcal{L}(F, F_0)$, $S \in \mathcal{L}(E,F)$ and $T \in \mathcal{L}(E_0, E)$, where E_0, F_0 are arbitrary Banach spaces
- $(S4)$ rank property: If $rank(T) \leqslant n$ then $s_n(T) = 0$
- $(S5)$ norming property: $s_n(I : l_2^n \to l_2^n) = 1$, where I denotes the identity operator on the n-dimensional Hilbert space l_2^n.

We call $s_n(T)$ the nth s-number of the operator T. For results on s-number sequence, refer [2,14–16].

We now give some examples of s-number sequences of a bounded linear operator.

Let $T \in \mathcal{L}(E,F)$ and $n \in \mathbb{N}$.

The nth approximation number, denoted by $a_n(T)$, is defined as

$a_n(T) = \inf\{\|T - L\| : \quad L \in \mathcal{L}(E,F), \mathrm{rank(L)} < \mathrm{n}\}$.

The nth Gel'fand number, denoted by $c_n(T)$, is defined as

$c_n(T) = \inf\{\|TJ_M\| : \quad M \subset E, \mathrm{codim(M)} < \mathrm{n}\}$, where $J_M : M \to E$ be the natural embedding from subspace M of E into E.

The nth Kolmogorov number, denoted by $d_n(T)$, is defined as

$d_n(T) = \inf\{\|Q_N(T)\| : \quad N \subset F, \mathrm{dim(N)} < \mathrm{n}\}$, where $Q_N : E \to E/N$ be the quotient map from E onto E/N.

The nth Weyl number, denoted by $x_n(T)$, is defined as

$x_n(T) = \inf\{a_n(TA) : \|A\| \leqslant 1, \text{ where } A : \ell_2 \to E\}$, where $a_n(TA)$ is an nth approximation number of the operator TA.

The nth Chang number, denoted by $y_n(T)$, is defined as

$y_n(T) = \inf\{a_n(BT) : \|B\| \leqslant 1, \text{ where } B : F \to \ell_2\}$, where $a_n(BT)$ is an nth approximation number of the operator BT.

The nth Hilbert number, denoted by $h_n(T)$, is defined as

$h_n(T) = \sup\{a_n(BTA) : \|B\| \leqslant 1, \|A\| \leqslant 1, \text{ where } B : F \to \ell_2 \text{ and } A : \ell_2 \to E\}$.

Remark 2.1 [16]. If T is compact and is defined on a Hilbert space, then all the s-numbers coincide with the singular values of T i.e. the eigenvalues of $|T|$, where $|T| = (T^*T)^{\frac{1}{2}}$.

Proposition 2.1 [16, p. 115]. *Let $T \in \mathcal{L}(E,F)$. Then*

$h_n(T) \leqslant x_n(T) \leqslant c_n(T) \leqslant a_n(T)$ *and* $h_n(T) \leqslant y_n(T) \leqslant d_n(T) \leqslant a_n(T)$.

Definition 2.3 [16, p. 90]. An s-number sequence $s = (s_n)$ is called injective if, given any metric injection $J \in \mathcal{L}(F, F_0)$, $s_n(T) = s_n(JT)$ for all $T \in \mathcal{L}(E,F)$.

Definition 2.4 [16, p. 95]. An s-number sequence $s = (s_n)$ is called surjective if, given any metric surjection $Q \in \mathcal{L}(E_0, E), s_n(T) = s_n(TQ)$ for all $T \in \mathcal{L}(E,F)$.

Proposition 2.2 [16, pp. 90–94]. *The Gel'fand numbers and the Weyl numbers are injective.*

Proposition 2.3 [16, p. 95]. *The Kolmogorov numbers and the Chang numbers are surjective.*

The following lemma is required to prove our theorems.

Lemma 2.1 [2]. *Let $S, T \in \mathcal{L}(E,F)$. Then* $|s_n(T) - s_n(S)| \leqslant \|T - S\|$ for $n = 1, 2, \ldots$.

Definition 2.5 (*Dual s-numbers [2]*). For each s-number sequence $\mathbf{s} = (\mathbf{s_n})$, a dual s-number sequence $\mathbf{s^D} = (\mathbf{s_n^D})$ is defined by

$s_n^D(T) = s_n(T')$ for all $T \in \mathcal{L}$, where T' is the dual of T.

Definition 2.6 [14, p. 152]. An s-number sequence is called symmetric if $s_n(T) \geqslant s_n(T')$ for all $T \in \mathcal{L}$. If $s_n(T) = s_n(T')$ then the s-number sequence is said to be completely symmetric.

Theorem 2.1 [14, p. 152]. *The approximation numbers are symmetric i.e.* $a_n(T') \leqslant a_n(T)$ *for* $T \in \mathcal{L}$.

Theorem 2.2 [16, p. 95]. *Let* $T \in \mathcal{L}$. *Then*

$$c_n(T) = d_n(T') \quad \text{and} \quad c_n(T') \leqslant d_n(T).$$

In addition, if T *is a compact operator, then* $c_n(T') = d_n(T)$.

Theorem 2.3 [16, p. 96]. *Let* $T \in \mathcal{L}$. *Then*

$$x_n(T) = y_n(T') \quad \text{and} \quad y_n(T) = x_n(T').$$

Theorem 2.4 [16, p. 97]. *The Hilbert numbers are completely symmetric i.e.* $h_n(T) = h_n(T')$ *for all* $T \in \mathcal{L}$.

Definition 2.7 ([14,17]). Let $\mathcal{L}$ be the class of all bounded linear operators between arbitrary Banach spaces and $\mathcal{L}(E,F)$ be the set of all such operators from E to F. A sub collection $\mathcal{M}$ of $\mathcal{L}$ is said to be an ideal if each component $\mathcal{M}(E,F) = \mathcal{M} \bigcap \mathcal{L}(E,F)$ satisfies the following conditions:

($OI1$) if $x\prime \in E\prime, y \in F$ then $x\prime \otimes y \in \mathcal{M}(E,F)$
($OI2$) if $S, T \in \mathcal{M}(E,F)$ then $S + T \in \mathcal{M}(E,F)$
($OI3$) if $S \in \mathcal{M}(E,F), T \in \mathcal{L}(E_0,E)$ and $R \in \mathcal{L}(F,F_0)$ then $RST \in \mathcal{M}(E_0,F_0)$.

Definition 2.8 ([14,17]). A function $\alpha : \mathcal{M} \to \mathbb{R}^+$ is said to be a quasi-norm on the ideal $\mathcal{M}$ if the following conditions hold:

($QON1$) if $x\prime \in E\prime, y \in F$ then $\alpha(x\prime \otimes y) = \|x'\|\|y\|$
($QON2$) if $S, T \in \mathcal{M}(E,F)$ then there exists a constant $C \geqslant 1$ such that $\alpha(S+T) \leqslant C(\alpha(S) + \alpha(T))$
($QON3$) if $S \in \mathcal{M}(E,F), T \in \mathcal{L}(E_0,E)$ and $R \in \mathcal{L}(F,F_0)$, then $\alpha(RST) \leqslant \|R\|\alpha(S)\|T\|$.

In particular if $C = 1$ then α becomes a norm on the operator ideal $\mathcal{M}$.

An ideal $\mathcal{M}$ with a quasi-norm α, denoted by $[\mathcal{M}, \alpha]$ is said to be a quasi-Banach operator ideal if each component $\mathcal{M}(E,F)$ is complete under the quasi-norm α. A quasi-normed operator ideal $[\mathcal{M}, \alpha]$ is called injective if for every operator $T \in \mathcal{L}(E,F)$ and a metric injection $J \in \mathcal{L}(F,F_0)$, $JT \in \mathcal{M}(E,F_0)$ we have $T \in \mathcal{M}(E,F)$ and $\alpha(JT) = \alpha(T)$. Moreover, a quasi-normed operator ideal $[\mathcal{M}, \alpha]$ is called surjective if for every operator $T \in \mathcal{L}(E,F)$ and a metric surjection $Q \in \mathcal{L}(E_0,E), TQ \in \mathcal{M}(E_0,F)$ we have $T \in \mathcal{M}(E,F)$ and $\alpha(TQ) = \alpha(T)$. Thus injectivity and surjectivity are dual concepts. For its various properties, please refer to [14,17–19].

Definition 2.9 ([14,17]). For every operator ideal $\mathcal{M}$, the dual operator ideal denoted by $\mathcal{M}'$ is defined as

$$\mathcal{M}'(E,F) = \{T \in \mathcal{L}(E,F) : \quad T' \in \mathcal{M}(F',E')\},$$

where T' is the dual of T and E' and F' are the duals of E and F respectively.

Definition 2.10 [14, p. 68]. *An operator ideal* $\mathcal{M}$ *is called symmetric if* $\mathcal{M} \subset \mathcal{M}'$ *and is called completely symmetric if* $\mathcal{M} = \mathcal{M}'$.

3. *s*-type $ces(p,q)$ operators

Let $q = (q_k)$ be a bounded sequence of positive real numbers. Define $Q_n = \sum_{k=1}^n q_k, n \in \mathbb{N}$. Then the weighted Cesàro sequence space $ces(p,q), 1 < p < \infty$ is defined by

$$ces(p,q) = \left\{ x \in w : \sum_{n=1}^{\infty} \left(\frac{1}{Q_n} \sum_{k=1}^{n} |q_k x_k| \right)^p < \infty \right\}.$$

In particular $q_k = 1$ for all k, then the sequence space $ces(p,q)$ reduces to ces_p. We call an operator $T \in \mathcal{L}(E,F)$ is of s-type $ces(p,q)$ if

$$\sum_{n=1}^{\infty} \left(\frac{1}{Q_n} \sum_{k=1}^{n} q_k s_k(T) \right)^p < \infty, \qquad 1 < p < \infty.$$

We denote by $\mathcal{A}_{p,q}^{(s)}$ class of all s-type $ces(p,q)$ operators between any two Banach spaces.

Let $q = (q_k)$ be a bounded sequence of positive numbers such that

$$q_{2k-1} + q_{2k} \leqslant M q_k \qquad \text{for all } k = 1, 2, \ldots, \tag{3.1}$$

where $M > 0$ is independent of k.

Then we have.

Theorem 3.1. *Let* $q = (q_k)$ *be a bounded sequence of positive numbers satisfying* (3.1). *Let* $1 < p < \infty$. *If* $\sum_{n=1}^{\infty} \left(\frac{1}{Q_n}\right)^p < \infty$, *then the class* $\mathcal{A}_{p,q}^{(s)}$ *is an operator ideal.*

Proof. In order to show $\mathcal{A}_{p,q}^{(s)}$ be an operator ideal, we prove the conditions ($OI1$) to ($OI3$).

Let E and F be any two Banach spaces. Let $x' \in E', \ y \in F$. Then $x' \otimes y$ is a rank one operator. So

$$s_n(x' \otimes y) = 0, \text{ for all } n \geqslant 2.$$

We have

$$\begin{aligned} \sum_{n=1}^{\infty} \left(\frac{1}{Q_n} \sum_{k=1}^{n} q_k s_k(x' \otimes y) \right)^p &= \sum_{n=1}^{\infty} \left(\frac{1}{Q_n} q_1 s_1(x' \otimes y) \right)^p \\ &= (q_1 \|x' \otimes y\|)^p \left(\sum_{n=1}^{\infty} \left(\frac{1}{Q_n} \right)^p \right) < \infty. \end{aligned}$$

Thus $x' \otimes y \in \mathcal{A}_{p,q}^{(s)}(E \to F)$; hence ($OI1$) is proved.

Let $S, T \in \mathcal{A}_{p,q}^{(s)}(E \to F)$. Since s-number is nonnegative and nonincreasing

$$\begin{aligned} \sum_{k=1}^{n} q_k s_k(T+S) &\leqslant \sum_{k=1}^{n} q_{2k-1} s_{2k-1}(T+S) + \sum_{k=1}^{n} q_{2k} s_{2k}(T+S) \\ &\leqslant \sum_{k=1}^{n} (q_{2k-1} + q_{2k}) s_{2k-1}(T+S) \\ &\leqslant M \left(\sum_{k=1}^{n} q_k s_k(T) + \sum_{k=1}^{n} q_k s_k(S) \right). \end{aligned} \tag{3.2}$$

Using Minkowski inequality for $1 < p < \infty$, we have from (3.2)

$$\left(\sum_{n=1}^{\infty}\left(\frac{1}{Q_n}\sum_{k=1}^{n}q_k s_k(T+S)\right)^p\right)^{\frac{1}{p}}$$
$$\leqslant M\left(\sum_{n=1}^{\infty}\left(\frac{1}{Q_n}\sum_{k=1}^{n}q_k s_k(T)+\frac{1}{Q_n}\sum_{k=1}^{n}q_k s_k(S)\right)^p\right)^{\frac{1}{p}}$$
$$\leqslant M\left[\left(\sum_{n=1}^{\infty}\left(\frac{1}{Q_n}\sum_{k=1}^{n}q_k s_k(T)\right)^p\right)^{\frac{1}{p}}+\left(\sum_{n=1}^{\infty}\left(\frac{1}{Q_n}\sum_{k=1}^{n}q_k s_k(S)\right)^p\right)^{\frac{1}{p}}\right]$$
$$< \infty.$$

Thus $S+T \in \mathcal{A}^{(s)}_{p,q}(E\to F)$; hence $(OI2)$ is proved.

Let $T \in \mathcal{L}(E_0,E), R\in\mathcal{L}(F,F_0)$ and $S\in\mathcal{A}^{(s)}_{p,q}(E\to F)$.

Using the property $(S3)$ in the Definition 2.2, we have

$$\left(\sum_{n=1}^{\infty}\left(\frac{1}{Q_n}\sum_{k=1}^{n}q_k s_k(RST)\right)^p\right)^{\frac{1}{p}}$$
$$\leqslant \|R\|\|T\|\left(\sum_{n=1}^{\infty}\left(\frac{1}{Q_n}\sum_{k=1}^{n}q_k s_k(S)\right)^p\right)^{\frac{1}{p}} < \infty.$$

Thus $RST \in \mathcal{A}^{(s)}_{p,q}(E_0\to F_0)$ and therefore $(OI3)$ is proved.

Hence the class $\mathcal{A}^{(s)}_{p,q}$ is an operator ideal. □

Remark 3.1. It is observed that the set $\mathcal{A}^{(s)}_{p,q}(E\to F)$ of s-type $ces(p,q)$ operators from E to F is a linear space. In particular if we take s-number sequence as the sequence of approximation numbers and $q_k=1$, then the set $\mathcal{A}^{(a)}_{p,q}(E\to F)$ coincides with the set of $ces-p$ type operators from E to F introduced by Constantin [9].

Proposition 3.1. *For $1<p\leqslant r<\infty$, we have $\mathcal{A}^{(s)}_{p,q}\subseteq\mathcal{A}^{(s)}_{r,q}$.*

Proof. The result follows from the inclusion $ces(p;q)\subseteq ces(r;q)$ for $1<p\leqslant r<\infty$. □

Let $\mathcal{A}^{(s)}_{p,q}$ be an operator ideal. Define $\beta^{(s)}_{p,q}:\mathcal{A}^{(s)}_{p,q}\to\mathbb{R}^+$ for $1<p<\infty$ by

$$\beta^{(s)}_{p,q}(T)=\left(\sum_{n=1}^{\infty}\left(\frac{1}{Q_n}\sum_{k=1}^{n}q_k s_k(T)\right)^p\right)^{\frac{1}{p}},$$

where $T\in\mathcal{A}^{(s)}_{p,q}$.

Theorem 3.2. *Let $q=(q_k)$ be a bounded sequence of positive numbers satisfying (3.1). Let $1<p<\infty$. If $\sum_{n=1}^{\infty}\left(\frac{1}{Q_n}\right)^p<\infty$, then the function $\hat{\beta}^{(s)}_{p,q}$ is a quasi-norm on the operator ideal $\mathcal{A}^{(s)}_{p,q}$, where*

$$\hat{\beta}^{(s)}_{p,q}(T)=\frac{\beta^{(s)}_{p,q}(T)}{\left(\sum_{n=1}^{\infty}\left(\frac{q_1}{Q_n}\right)^p\right)^{\frac{1}{p}}}.$$

Proof. Let E and F be two Banach spaces and $\mathcal{A}^{(s)}_{p,q}(E\to F)$ be any one of the components of $\mathcal{A}^{(s)}_{p,q}$.

Since $x'\otimes y: E\to F$ is a rank one operator, $s_n(x'\otimes y)=0$ for all $n\geqslant 2$.

Therefore,

$$\beta^{(s)}_{p,q}(x'\otimes y)=\left(\sum_{n=1}^{\infty}\left(\frac{1}{Q_n}\sum_{k=1}^{n}q_k s_k(x'\otimes y)\right)^p\right)^{\frac{1}{p}}$$
$$=\|x'\otimes y\|\left(\sum_{n=1}^{\infty}\left(\frac{q_1}{Q_n}\right)^p\right)^{\frac{1}{p}}.$$

Again $\|x'\otimes y\|=\sup_{\|x\|=1}\|(x'\otimes y)(x)\|=(\sup_{\|x\|=1}|x'(x)|)\|y\|=\|x'\|\|y\|$.

Therefore

$\hat{\beta}^{(s)}_{p,q}(x'\otimes y)=\|x'\|\|y\|$.

Suppose that $S,T\in\mathcal{A}^{(s)}_{p,q}(E\to F)$, then

$$\begin{aligned}\beta^{(s)}_{p,q}(S+T)&=\left(\sum_{n=1}^{\infty}\left(\frac{1}{Q_n}\sum_{k=1}^{n}q_k s_k(S+T)\right)^p\right)^{\frac{1}{p}}\\&\leqslant M\left[\left(\sum_{n=1}^{\infty}\left(\frac{1}{Q_n}\sum_{k=1}^{n}q_k s_k(S)\right)^p\right)^{\frac{1}{p}}\right.\\&\quad\left.+\left(\sum_{n=1}^{\infty}\left(\frac{1}{Q_n}\sum_{k=1}^{n}q_k s_k(T)\right)^p\right)^{\frac{1}{p}}\right]\\&\leqslant M\left(\beta^{(s)}_{p,q}(S)+\beta^{(s)}_{p,q}(T)\right).\end{aligned}$$

Thus $\hat{\beta}^{(s)}_{p,q}(S+T)\leqslant M\left(\hat{\beta}^{(s)}_{p,q}(S)+\hat{\beta}^{(s)}_{p,q}(T)\right)$.

Finally, let $S\in\mathcal{A}^{(s)}_{p,q}(E\to F)$, $R\in\mathcal{L}(F,F_0)$ and $T\in\mathcal{L}(E_0,E)$. Then

$$\begin{aligned}\beta^{(s)}_{p}(RST)&=\left(\sum_{n=1}^{\infty}\left(\frac{1}{Q_n}\sum_{k=1}^{n}q_k s_k(RST)\right)^p\right)^{\frac{1}{p}}\\&\leqslant\|R\|\|T\|\left(\sum_{n=1}^{\infty}\left(\frac{1}{Q_n}\sum_{k=1}^{n}q_k s_k(S)\right)^p\right)^{\frac{1}{p}}\\&\leqslant\|R\|\beta^{(s)}_{p,q}(S)\|T\|.\end{aligned}$$

Thus

$\hat{\beta}^{(s)}_{p,q}(RST)\leqslant\|R\|\hat{\beta}^{(s)}_{p,q}(S)\|T\|$.

Hence $\hat{\beta}^{(s)}_{p,q}$ is a quasi-norm on the operator ideal $\mathcal{A}^{(s)}_{p,q}$. □

Theorem 3.3. *The operator ideal $\mathcal{A}^{(s)}_{p,q}$ is complete under the quasi-norm $\hat{\beta}^{(s)}_{p,q}$ i.e. $[\mathcal{A}^{(s)}_{p,q},\hat{\beta}^{(s)}_{p,q}]$ is a quasi-Banach operator ideal for $1<p<\infty$.*

Proof. Let $1<p<\infty$. To prove $\mathcal{A}^{(s)}_{p,q}$ is a quasi-Banach operator ideal, it is enough to prove that each component $\mathcal{A}^{(s)}_{p,q}(E\to F)$ of $\mathcal{A}^{(s)}_{p,q}$ is complete under the quasi-norm $\hat{\beta}^{(s)}_{p,q}$.

We have

$$\begin{aligned}\beta^{(s)}_{p,q}(T)&=\left(\sum_{n=1}^{\infty}\left(\frac{1}{Q_n}\sum_{k=1}^{n}q_k s_k(T)\right)^p\right)^{\frac{1}{p}}\\&\geqslant\left(\sum_{n=1}^{\infty}\left(\frac{q_1}{Q_n}s_1(T)\right)^p\right)^{\frac{1}{p}}\\&=\|T\|\left(\sum_{n=1}^{\infty}\left(\frac{q_1}{Q_n}\right)^p\right)^{\frac{1}{p}},\end{aligned}$$

$$\Rightarrow \|T\| \leqslant \hat{\beta}^{(s)}_{p,q}(T) \quad \text{for } T \in \mathcal{A}^{(s)}_{p,q}(E \to F). \tag{3.3}$$

Let (T_m) be a Cauchy sequence in $\mathcal{A}^{(s)}_{p,q}(E \to F)$. Then $\forall \epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that

$$\hat{\beta}^{(s)}_{p,q}(T_m - T_l) < \epsilon, \quad \forall m, l \geqslant n_0. \tag{3.4}$$

Now from (3.3),
$\|T_m - T_l\| \leqslant \hat{\beta}^{(s)}_p(T_m - T_l)$.

Using (3.4), we have

$$\|T_m - T_l\| \leqslant \hat{\beta}^{(s)}_{p,q}(T_m - T_l) < \epsilon \quad \forall m, l \geqslant n_0.$$

Hence (T_m) is a Cauchy sequence in $\mathcal{L}(E,F)$. As F is a Banach space, $\mathcal{L}(E,F)$ is also a Banach space. Therefore $T_m \to T$ as $m \to \infty$ in $\mathcal{L}(E,F)$. We shall now show that $T_m \to T$ as $m \to \infty$ in $\mathcal{A}^{(s)}_{p,q}(E \to F)$.

Using Lemma 2.1., we have

$$|s_n(T_l - T_m) - s_n(T - T_m)| \leqslant \|T_l - T\|.$$

On taking $l \to \infty$, we have

$$s_n(T_l - T_m) \to s_n(T - T_m). \tag{3.5}$$

From (3.4), we get

$$\left(\sum_{n=1}^{\infty}\left(\frac{1}{Q_n}\sum_{k=1}^{n} q_k s_k(T_l - T_m)\right)^p\right)^{\frac{1}{p}} < \epsilon\left(\sum_{n=1}^{\infty}\left(\frac{q_1}{Q_n}\right)^p\right)^{\frac{1}{p}}, \quad \forall m, l \geqslant n_0.$$

Using (3.5), it can be shown that as $l \to \infty$ (keeping $m \geqslant n_0$ fixed)

$$\left(\sum_{n=1}^{\infty}\left(\frac{1}{Q_n}\sum_{k=1}^{n} q_k s_k(T - T_m)\right)^p\right)^{\frac{1}{p}} \leqslant \epsilon\left(\sum_{n=1}^{\infty}\left(\frac{q_1}{Q_n}\right)^p\right)^{\frac{1}{p}}$$

$$\Rightarrow \hat{\beta}^{(s)}_p(T - T_m) \leqslant \epsilon \quad \forall m \geqslant n_0.$$

This means that $T_m \to T$ under the quasi-norm $\hat{\beta}^{(s)}_{p,q}$.

Next we show that $T \in \mathcal{A}^{(s)}_{p,q}(E \to F)$. Now

$$\begin{aligned}\sum_{k=1}^{n} q_k s_k(T) &\leqslant \sum_{k=1}^{n} q_{2k-1} s_{2k-1}(T) + \sum_{k=1}^{n} q_{2k} s_{2k}(T)\\ &\leqslant \sum_{k=1}^{n}(q_{2k-1} + q_{2k}) s_{2k-1}(T)\\ &\leqslant M\left(\sum_{k=1}^{n} q_k s_k(T - T_m) + \sum_{k=1}^{n} q_k s_k(T_m)\right).\end{aligned}$$

Therefore

$$\begin{aligned}&\left(\sum_{n=1}^{\infty}\left(\frac{1}{Q_n}\sum_{k=1}^{n} q_k s_k(T)\right)^p\right)^{\frac{1}{p}}\\ &\leqslant M\left[\left(\sum_{n=1}^{\infty}\left(\frac{1}{Q_n}\sum_{k=1}^{n} q_k s_k(T - T_m)\right)^p\right)^{\frac{1}{p}}\right.\\ &\left.+\left(\sum_{n=1}^{\infty}\left(\frac{1}{Q_n}\sum_{k=1}^{n} q_k s_k(T_m)\right)^p\right)^{\frac{1}{p}}\right] < \infty,\end{aligned}$$

since $\hat{\beta}^{(s)}_{p,q}(T - T_m) \to 0$ as $m \to \infty$ and for each m, $T_m \in \mathcal{A}^{(s)}_{p,q}(E \to F)$.

Hence $T \in \mathcal{A}^{(s)}_{p,q}(E \to F)$. This completes the proof. □

We now study some properties of the quasi-Banach operator ideal $\mathcal{A}^{(s)}_{p,q}$ for $1 < p < \infty$.

Theorem 3.4. *If s-number sequence is injective, then the quasi-Banach operator ideal* $[\mathcal{A}^{(s)}_{p,q}, \hat{\beta}^{(s)}_{p,q}]$ *is injective for* $1 < p < \infty$.

Proof. Let $1 < p < \infty$. Let $T \in \mathcal{L}(E,F)$ and $J \in \mathcal{L}(F,F_0)$ be any metric injection. Suppose that $JT \in \mathcal{A}^{(s)}_{p,q}(E \to F_0)$. Then

$$\sum_{n=1}^{\infty}\left(\frac{1}{Q_n}\sum_{k=1}^{n} q_k s_k(JT)\right)^p < \infty.$$

Since $s = (s_n)$ is an injective s-number sequence, we have $s_n(T) = s_n(JT)$, for all $T \in \mathcal{L}(E,F), n = 1,2,\ldots$. Hence

$$\sum_{n=1}^{\infty}\left(\frac{1}{Q_n}\sum_{k=1}^{n} q_k s_k(T)\right)^p = \sum_{n=1}^{\infty}\left(\frac{1}{Q_n}\sum_{k=1}^{n} q_k s_k(JT)\right)^p < \infty.$$

Thus $T \in \mathcal{A}^{(s)}_{p,q}(E \to F)$ and clearly $\hat{\beta}^{(s)}_p(JT) = \hat{\beta}^{(s)}_{p,q}(T)$ holds.

Hence the operator ideal $[\mathcal{A}^{(s)}_{p,q}, \hat{\beta}^{(s)}_{p,q}]$ is injective. □

Remark 3.2. The quasi-Banach operator ideal $[\mathcal{A}^{(c)}_{p,q}, \hat{\beta}^{(c)}_{p,q}]$ formed by Gel'fand numbers $c = (c_n)$ and the quasi-Banach operator ideal $[\mathcal{A}^{(x)}_{p,q}, \hat{\beta}^{(x)}_{p,q}]$ formed by Weyl numbers $x = (x_n)$ are injective quasi-Banach operator ideals.

Theorem 3.5. *If s-number sequence is surjective, then the quasi-Banach operator ideal* $[\mathcal{A}^{(s)}_{p,q}, \hat{\beta}^{(s)}_{p,q}]$ *is surjective for* $1 < p < \infty$.

Proof. Let $1 < p < \infty$. Let $T \in \mathcal{L}(E,F)$ and $Q \in \mathcal{L}(E_0,E)$ be any metric surjection. Suppose that $TQ \in \mathcal{A}^{(s)}_{p,q}(E_0 \to F)$. Then

$$\sum_{n=1}^{\infty}\left(\frac{1}{Q_n}\sum_{k=1}^{n} q_k s_k(TQ)\right)^p < \infty.$$

Since $s = (s_n)$ is a surjective s-number sequence, we have $s_n(T) = s_n(TQ)$, for all $T \in \mathcal{L}(E,F)$ and $n = 1,2,\ldots$. Hence

$$\sum_{n=1}^{\infty}\left(\frac{1}{Q_n}\sum_{k=1}^{n} q_k s_k(T)\right)^p = \sum_{n=1}^{\infty}\left(\frac{1}{Q_n}\sum_{k=1}^{n} q_k s_k(TQ)\right)^p < \infty.$$

Thus $T \in \mathcal{A}^{(s)}_{p,q}(E \to F)$ and also $\hat{\beta}^{(s)}_{p,q}(TQ) = \hat{\beta}^{(s)}_{p,q}(T)$.

Hence the operator ideal $[\mathcal{A}^{(s)}_{p,q}, \hat{\beta}^{(s)}_{p,q}]$ is surjective. □

Remark 3.3. The quasi-Banach operator ideal $[\mathcal{A}^{(d)}_{p,q}, \hat{\beta}^{(d)}_{p,q}]$ formed by Kolmogorov numbers $d = (d_n)$ and the quasi-Banach operator ideal $[\mathcal{A}^{(y)}_{p,q}, \hat{\beta}^{(y)}_{p,q}]$ formed by Chang numbers $y = (y_n)$ are surjective quasi-Banach operator ideals.

Let us consider $[\mathcal{A}^{(a)}_{p,q}, \hat{\beta}^{(a)}_{p,q}]$ and $[\mathcal{A}^{(h)}_{p,q}, \hat{\beta}^{(h)}_{p,q}]$ be the quasi-Banach operator ideals corresponding to the approximation numbers $a = (a_n)$ and the Hilbert numbers $h = (h_n)$ respectively. Let $\mathcal{A}^{(a)}_{p,q}$, $\mathcal{A}^{(c)}_{p,q}$, $\mathcal{A}^{(d)}_{p,q}$, $\mathcal{A}^{(x)}_{p,q}$, $\mathcal{A}^{(y)}_{p,q}$ and $\mathcal{A}^{(h)}_{p,q}$ be the class of approximation type, the class of Gel'fand type, the class of Kolmogorov type, the class of Weyl type, the class of Chang type and the class of Hilbert type $ces(p,q)$ operators respectively. Then we have the following inclusion relations among the operator ideals.

Theorem 3.6. *Let $1 < p < \infty$. Then*

(I) $\mathcal{A}^{(a)}_{p,q} \subseteq \mathcal{A}^{(c)}_{p,q} \subseteq \mathcal{A}^{(x)}_{p,q} \subseteq \mathcal{A}^{(h)}_{p,q}$ and
(II) $\mathcal{A}^{(a)}_{p,q} \subseteq \mathcal{A}^{(d)}_{p,q} \subseteq \mathcal{A}^{(y)}_{p,q} \subseteq \mathcal{A}^{(h)}_{p,q}$.

Proof. Let $1 < p < \infty$. Suppose that $T \in \mathcal{A}^{(a)}_{p,q}$. Then

$$\sum_{n=1}^{\infty}\left(\frac{1}{Q_n}\sum_{k=1}^{n} q_k a_k(T)\right)^p < \infty.$$

From Proposition 2.1., we have

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\frac{1}{Q_n}\sum_{k=1}^{n} q_k h_k(T)\right)^p &\leqslant \sum_{n=1}^{\infty}\left(\frac{1}{Q_n}\sum_{k=1}^{n} q_k x_k(T)\right)^p \\ &\leqslant \sum_{n=1}^{\infty}\left(\frac{1}{Q_n}\sum_{k=1}^{n} q_k c_k(T)\right)^p \\ &\leqslant \sum_{n=1}^{\infty}\left(\frac{1}{Q_n}\sum_{k=1}^{n} q_k a_k(T)\right)^p.\end{aligned}$$

This proves (I).

(II) The proof of this part can be established following the technique used in part (I) above □

We now state the dual of the operator ideals formed by different s-number sequences.

Theorem 3.7. *The operator ideal $\mathcal{A}^{(a)}_{p,q}$ is symmetric and the operator ideal $\mathcal{A}^{(h)}_{p,q}$ is completely symmetric for $1 < p < \infty$.*

Proof. Since $a_n(T') \leqslant a_n(T)$ and $h_n(T') = h_n(T)$, for all $T \in \mathcal{L}(E,F)$, we have $\mathcal{A}^{(a)}_{p,q} \subseteq (\mathcal{A}^{(a)}_{p,q})'$ and $\mathcal{A}^{(h)}_{p,q} = (\mathcal{A}^{(h)}_{p,q})'$. □

Theorem 3.8. *Let $1 < p < \infty$. Then $\mathcal{A}^{(c)}_{p,q} = (\mathcal{A}^{(d)}_{p,q})'$ and $\mathcal{A}^{(d)}_{p,q} \subseteq (\mathcal{A}^{(c)}_{p,q})'$. In addition, if T belongs to the class of compact operators, then $\mathcal{A}^{(d)}_{p,q} = (\mathcal{A}^{(c)}_{p,q})'$.*

Proof. The proof follows from the Theorem 2.2. □

Theorem 3.9. *Let $1 < p < \infty$. Then $\mathcal{A}^{(x)}_{p,q} = (\mathcal{A}^{(y)}_{p,q})'$ and $\mathcal{A}^{(y)}_{p,q} = (\mathcal{A}^{(x)}_{p,q})'$.*

Proof. The proof follows from the Theorem 2.3. □

4. Small operator ideal

This section deals with the notion of small ideal of operators. In [20], Pietsch proved that the ideal $S^{(a)}_p$ of approximation type l_p operators is small for $0 < p < \infty$.

Definition 4.1. [20] An operator ideal $\mathcal{M}$ is said to be small if $\mathcal{M}(E,F) = \mathcal{L}(E,F)$ implies that at least one of the Banach spaces E and F is of finite dimension.

Then we have.

Theorem 4.1. *The quasi-Banach ideal $\mathcal{A}^{(a)}_{p,q}$ of approximation type $ces(p,q)$ operators is small for $1 < p < \infty$.*

Proof. Let $\alpha = \left(\sum_{n=1}^{\infty}\left(\frac{q_1}{Q_n}\right)^p\right)^{\frac{1}{p}} < \infty, 1 < p < \infty.$ Then $[\mathcal{A}^{(a)}_{p,q}, \hat{\beta}^{(a)}_{p,q}]$ is a quasi-Banach operator ideal, where $\hat{\beta}^{(a)}_{p,q}(T) = \frac{1}{\alpha}\left(\sum_{n=1}^{\infty}\left(\frac{1}{Q_n}\sum_{k=1}^{n} q_k a_k(T)\right)^p\right)^{\frac{1}{p}}$. Let E, F be any two Banach spaces. Suppose that $\mathcal{A}^{(a)}_{p,q}(E \to F) = \mathcal{L}(E,F)$. Then there exists a constant $C > 0$ such that $\hat{\beta}^{(a)}_{p,q}(T) \leqslant C\|T\|$ for all $T \in \mathcal{L}(E,F)$. Assume that E and F both are infinite dimensional Banach spaces. Then by Dvoretzky's theorem [20] for $m = 1, 2, \ldots$, we have quotient spaces E/N_m and subspaces M_m of F which can be mapped onto l^m_2 by isomorphisms X_m and A_m such that $\|X_m\|\|X_m^{-1}\| \leqslant 2$ and $\|A_m\|\|A_m^{-1}\| \leqslant 2$. Consider I_m be the identity map on l^m_2, Q_m be the quotient map from E onto E/N_m and J_m be the natural embedding map from M_m into F. Let a_n, d_n and u_n be approximation numbers, Kolmogorov numbers and Bernstein numbers [2] respectively. Then

$$\begin{aligned}1 = u_n(I_m) &= u_n(A_m A_m^{-1} I_m X_m X_m^{-1}) \\ &\leqslant \|A_m\| u_n(A_m^{-1} I_m X_m)\|X_m^{-1}\| \\ &= \|A_m\| u_n(J_m A_m^{-1} I_m X_m)\|X_m^{-1}\| \\ &\leqslant \|A_m\| d_n(J_m A_m^{-1} I_m X_m)\|X_m^{-1}\| \\ &= \|A_m\| d_n(J_m A_m^{-1} I_m X_m Q_m)\|X_m^{-1}\| \\ &\leqslant \|A_m\| a_n(J_m A_m^{-1} I_m X_m Q_m)\|X_m^{-1}\| \quad \text{for } n = 1, 2, \ldots, m.\end{aligned}$$

Now

$$\begin{aligned}&\sum_{k=1}^{n} q_k \leqslant \sum_{k=1}^{n} q_k \|A_m\| a_k(J_m A_m^{-1} I_m X_m Q_m)\|X_m^{-1}\| \\ \Rightarrow &\frac{1}{Q_n}\sum_{k=1}^{n} q_k \leqslant \|A_m\|\left(\frac{1}{Q_n}\sum_{k=1}^{n} q_k a_k(J_m A_m^{-1} I_m X_m Q_m)\right)\|X_m^{-1}\| \\ &\Rightarrow 1 \leqslant (\|A_m\|\|X_m^{-1}\|)^p\left(\frac{1}{Q_n}\sum_{k=1}^{n} q_k a_k(J_m A_m^{-1} I_m X_m Q_m)\right)^p\end{aligned}$$

$$\text{Therefore } \left(\sum_{n=1}^{m}(1)\right)^{\frac{1}{p}} \leqslant (\|A_m\|\|X_m^{-1}\|)\left(\sum_{n=1}^{m}\left(\frac{1}{Q_n}\sum_{k=1}^{n} q_k a_k(J_m A_m^{-1} I_m X_m Q_m)\right)^p\right)^{\frac{1}{p}}$$

$$\begin{aligned}&\Rightarrow \frac{1}{\alpha} m^{\frac{1}{p}} \leqslant \|A_m\|\|X_m^{-1}\|\frac{1}{\alpha}\left(\sum_{n=1}^{m}\left(\frac{1}{Q_n}\sum_{k=1}^{n} q_k a_k(J_m A_m^{-1} I_m X_m Q_m)\right)^p\right)^{\frac{1}{p}} \\ &\Rightarrow \frac{1}{\alpha} m^{\frac{1}{p}} \leqslant \|A_m\|\|X_m^{-1}\|\hat{\beta}^{(a)}_{p,q}(J_m A_m^{-1} I_m X_m Q_m) \\ &\quad \leqslant C\|A_m\|\|X_m^{-1}\|\|J_m A_m^{-1} I_m X_m Q_m\| \\ &\quad \leqslant C\|A_m\|\|X_m^{-1}\|\|J_m A_m^{-1}\|\|I_m\|\|X_m Q_m\| \\ &\quad = C\|A_m\|\|X_m^{-1}\|\|A_m^{-1}\|\|X_m\| \\ &\quad \leqslant 4C.\end{aligned}$$

This is a contradiction as m is any arbitrary number. Thus E and F both cannot be infinite dimensional when $\mathcal{A}^{(a)}_{p,q}(E \to F) = \mathcal{L}(E,F)$. This completes the proof. □

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