Journal of the Egyptian Mathematical Society (2014) **22**, 420–423

Egyptian Mathematical Society
Journal of the Egyptian Mathematical Society
www.etms-eg.org
www.elsevier.com/locate/joems

ORIGINAL ARTICLE

On some Toeplitz matrices and their inversions

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Received 5 August 2013; revised 30 September 2013; accepted 3 October 2013
Available online 14 November 2013

KEYWORDS

Difference operator $B(a[m])$;
Toeplitz matrix;
Fibonacci matrix;
Pascal matrix and weighted mean factorable difference matrix

Abstract In this article, using the difference operator $B(a[m])$, we introduce a lower triangular Toeplitz matrix T which includes several difference matrices such as $\Delta^{(1)}$, $\Delta^{(m)}$, $B(r,s)$, $B(r,s,t)$, and $B(\tilde{r},\tilde{s},\tilde{t},\tilde{u})$ in different special cases. For any $x \in w$ and $m \in \mathbb{N}_0 = \{0,1,2,\ldots\}$, the difference operator $B(a[m])$ is defined by $(B(a[m])x)_k = a_k(0)x_k + a_{k-1}(1)x_{k-1} + a_{k-2}(2)x_{k-2} + \cdots + a_{k-m}(m)x_{k-m}$, $(k \in \mathbb{N}_0)$ where $a[m] = \{a(0), a(1), \ldots, a(m)\}$ and $a(i) = (a_k(i))$ for $0 \leqslant i \leqslant m$ are convergent sequences of real numbers. We use the convention that any term with negative subscript is equal to zero. The main results of this article relate to the determination and applications of the inverse of the Toeplitz matrix T.

2010 MATHEMATICS SUBJECT CLASSIFICATION: 40C05; 47A10; 46A45

1. Introduction

Let w be the space all real valued sequences. We write $a[m]$ for any convergent sequence $a(i) = (a_k(i))$ of real numbers satisfying $a(i) \neq a(j)$, where $m \in \mathbb{N}_0$ and $0 \leqslant i, j \leqslant m$. Let $x = (x_k)$ be any sequence in w, then we define the generalized difference operator $B(a[m])$ as follows:

$$(B(a[m])x)_k = a_k(0)x_k + a_{k-1}(1)x_{k-1} + a_{k-2}(2)x_{k-2} + \cdots + a_{k-m}(m)x_{k-m}, \quad (k \in \mathbb{N}_0). \tag{1.1}$$

We assume throughout that any term with negative subscript is zero. It is natural that the difference operator given in Eq. (1.1), can be expressed as a lower triangular Toeplitz matrix $T = (b_{nk})$, where

$$(b_{nk}) = \begin{pmatrix} a_0(0) & 0 & 0 & \ldots & 0 & 0 & \ldots \\ a_0(1) & a_1(0) & 0 & \ldots & 0 & 0 & \ldots \\ a_0(2) & a_1(1) & a_2(0) & \ldots & 0 & 0 & \ldots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ a_0(m) & a_1(m-1) & a_2(m-2) & \ldots & a_m(0) & 0 & \ldots \\ 0 & a_1(m) & a_2(m-1) & \ldots & a_m(1) & a_{m+1}(0) & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

In particular, the difference operator $B(a[m])$ has the following generalizations:

(i) For $a(0) = e = (1, 1, 1, \ldots)$, $a(1) = -e$ and $a(i) = \theta = (0, 0, 0, \ldots)$ for $2 \leqslant i \leqslant m$, the difference matrix $B(a(m))$ reduces to $\Delta^{(1)}$ studied by Kızmaz [1].

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Peer review under responsibility of Egyptian Mathematical Society.

Production and hosting by Elsevier

1110-256X
http://dx.doi.org/10.1016/j.joems.2013.10.001

(ii) For $a(0) = e$, $a(1) = -2e$, $a(2) = e$ and $a(i) = \theta$ for $3 \leqslant i \leqslant m$, the difference matrix $B(a(m))$ reduces to Δ^2 studied by Dutta and Baliarsingh [2].

(iii) For $a(0) = re$, $a(1) = se$, $0 \neq r$, $s \in \mathbb{R}$ and $a(i) = \theta$ for $2 \leqslant i \leqslant m$, the difference matrix $B(a(m))$ reduces to $B(r, s)$ studied by Altay and Başar [3].

(iv) For $a(0) = re$, $a(1) = se$, $a(2) = te$, $0 \neq r, s, t \in \mathbb{R}$ and $a(i) = \theta$ for $3 \leqslant i \leqslant m$, the difference matrix $B(a(m))$ reduces to $B(r, s, t)$ studied by Furkan et al. [4].

(v) For $a(i) = \binom{m}{i}$ for $0 \leqslant i \leqslant m$ and $m = r$, the difference matrix $B(a(m))$ reduces to Δ^r studied by Dutta and Baliarsingh [5].

(vi) For $a(0) = re$, $a(1) = se$, $a(2) = te$, $a(3) = ue$, $0 \neq r, s, t, u \in \mathbb{R}$ and $a(i) = \theta$ for $4 \leqslant i \leqslant m$, the difference matrix $B(a(m))$ reduces to $B(\tilde{r}, \tilde{s}, \tilde{t}, \tilde{u})$ studied by Baliarsingh and Dutta [6].

For last several decades, many new theories and fundamental results have been introduced and studied by different authors contributing to the development of sequence spaces. Amongst all, one of the most interesting idea is the study of sequence spaces by using difference matrices. For example: Kızmaz [1] introduced the difference matrix Δ and studied the sequence spaces $X(\Delta)$, for $X = \ell_\infty$, c, c_0, Et and Çolak [7] generalized these results by introducing the generalized difference matrix $\Delta^m, (m \in \mathbb{N}_0)$ and Baliarsingh [8] studied the difference sequence spaces $\lambda(\Gamma, \Delta^\alpha, u)$ for $\lambda \in \{\ell_\infty, c_0, c\}$ by introducing the difference matrix $\Delta^\alpha, (\alpha \in \mathbb{R})$. The difference matrices $B(r, s)$, $B(r, s, t)$ and $B(\tilde{r}, \tilde{s}, \tilde{t}, \tilde{u})$ have been introduced and studied by Altay and Başar [3], Furkan et al. [4] and Baliarsingh and Dutta [6], respectively. Recently, using difference matrices, various sequence spaces have been defined and different results concerning their topological properties, matrix transformations, spectral properties and many more (see [8–28]) have been established. The main objective of this work is to define a generalized difference operator and unify most of the difference matrices defined earlier and establish certain results concerning its inverse.

2. Main results

The most general and effective application of the difference matrix $a[m]$ is to redefine some triangles and find their inverses. In the present section, we redefine some well known lower triangular matrices such as generalized Fibonacci, Pascal and weighted mean factorable difference matrices, and we obtain some results related to the linearity, boundedness and inverse of the difference matrix $B(a[m])$.

Let F_n, $(n \in \mathbb{N}_0)$ be the nth Fibonacci number which satisfies the recurrence relation $F_n = F_{n-1} + F_{n-2}$ with $F_0 = 0$, $F_1 = 1$. Then for any $s, t \in \mathbb{R}$, we define generalized lower triangular Fibonacci matrix $F(r, s)$ as follows:

$$(F(r,s))_{nk} = \begin{cases} r, & (k = n) \\ r + s, & (k = n-1) \\ F_{n-1}r + F_{n-2}s, & (0 \leqslant k \leqslant n-2) \\ 0, & (k > n) \end{cases}, \quad (n, k \in \mathbb{N}_0).$$

Clearly, for $r = 0$, $s = 1$, the matrix $F(r, s)$ reduces to the usual Fibonacci matrix F studied in [28,29]. The lower triangular Pascal matrix $P = (p_{nk})$ is defined by

$$p_{nk} = \begin{cases} \binom{n}{n-k}, & (0 \leqslant k \leqslant n) \\ 0, & (k > n) \end{cases}, \quad (n, k \in \mathbb{N}_0).$$

The well known weighted mean factorable difference matrix $G(u, v; \Delta) = (g^{\Delta}_{nk})$ is defined as follows:

$$g^{\Delta}_{nk} = \begin{cases} u_n v_n, & (k = n) \\ u_n(v_k - v_{k+1}), & (0 \leqslant k \leqslant n-1) \\ 0, & (k > n). \end{cases}, \quad (n, k \in \mathbb{N}_0),$$

where we write $\mathcal{U}$ for the set of all sequences $u = (u_n)$ such that $u_n \neq 0$ for all $n \in \mathbb{N}_0$, and $u, v \in \mathcal{U}$. Now, we state some important theorems.

Theorem 1. *The difference operator $B(a[m])$: $w \to w$ is a linear operator and satisfying*

$$\|B(a[m])\| = \sup_k(|a_k(0)| + |a_k(1)| + \cdots + |a_k(m)|).$$

Proof. The proof is trivial, so we omit it. □

Theorem 2. *If $a_k(0) \neq 0$ for all $k \in \mathbb{N}_0$, then the inverse of the difference operator $B(a[m])$ is given by a lower triangular Toeplitz matrix $C = (c_{nk})$ as follows:*

$$c_{nk} = \begin{cases} \frac{1}{a_n(0)}, & (k = n), \\ \frac{\frac{(-1)^{n-k}}{n}}{\prod_{j=k} a_j(0)} D^{(k)}_{n-k}(a[m]), & (0 \leqslant k \leqslant n-1), \\ 0, & (k > n), \end{cases} \quad (n, k \in \mathbb{N}_0),$$

where

$$D^{(k)}_n(a[m]) = \begin{vmatrix} a_k(1) & a_{k+1}(0) & 0 & \dots & 0 & 0 & \dots & 0 \\ a_k(2) & a_{k+1}(1) & a_{k+2}(0) & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_k(m) & a_{k+1}(m-1) & a_{k+2}(m-2) & \dots & a_{m-1}(1) & a_m(0) & \dots & 0 \\ 0 & a_{k+1}(m) & a_{k+2}(m-1) & \dots & a_{m-1}(2) & a_m(1) & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{n-m}(m) & \dots & \dots & \dots & a_{n-1}(1) \end{vmatrix}.$$

Proof. The proof is clear from the following examples. □

Examples

(i) The inverse of the difference matrix $\Delta^{(1)}$ is

$$((\Delta^{(1)})^{-1})_{nk} = \begin{cases} 1, & (0 \leqslant k \leqslant n) \\ 0, & (k > n) \end{cases}, \quad (n, k \in \mathbb{N}_0).$$

This follows from the fact that $a(0) = e$, $a(1) = -e$, $a(2) = a(3) \ldots = a(m) = \theta$ and $D^{(0)}_n(\Delta^{(1)}) = (-1)^n$.

(ii) The inverse of the difference matrix $\Delta^{(m)}, m \in \mathbb{N}_0$ is

$$((\Delta^{(m)})^{-1})_{nk} = \begin{cases} 1, & (k = n) \\ \frac{m(m+1)\ldots(m+n-k)}{(n-k)!}, & (0 \leqslant k \leqslant n-1) \\ 0, & (k > n) \end{cases}, \quad (n, k \in \mathbb{N}_0).$$

Note that for this case, $a(0) = e, a(1) = -me, a(2) = \frac{m(m-1)}{2!}e, \ldots, a(m) = (-1)^m e$ and $D_n^{(0)}(\Delta^{(m)}) = \frac{m(m+1)\ldots(m+n)}{n!}$ for all $n \geqslant 1$.

(iii) The inverse of the difference matrix $B(r,\ s)$, $(r \neq 0)$ is

$$((B(r,s))^{-1})_{nk} = \begin{cases} \frac{1}{r}, & (k=n) \\ (-1)^{n-k}\frac{s^{n-k}}{r^{n-k+1}}, & (0 \leqslant k \leqslant n-1), \\ 0, & (k>n) \end{cases} \quad (n,k \in \mathbb{N}_0).$$

In fact, here $a(0) = re$, $a(1) = se$, $a(2) = a(3)\ldots = a(m) = \theta$ and $D_n^{(0)}(B(r,s)) = s^n$ for all $n \geqslant 1$.

(iv) The inverse of the difference matrix $B(\tilde{r},\tilde{s})$, $(r_k \neq 0)$ for all $k \in \mathbb{N}_0$, is

$$((B(\tilde{r},\tilde{s}))^{-1})_{nk} = \begin{cases} \frac{1}{r_k}, & (k=n) \\ (-1)^{n-k}\frac{\prod_{j=k}^{n-k-1}s_j}{\prod_{j=k}^{n-k}r_j}, & (0 \leqslant k \leqslant n-1), \\ 0, & (k>n) \end{cases} \quad (n,k \in \mathbb{N}_0).$$

This follows from the fact that $a(0) = (r_k)$, $a(1) = (s_k)$, $a(2) = a(3)\ldots = a(m) = \theta$ and $D_n^{(k)}(B(\tilde{r},\tilde{s})) = \frac{\prod_{j=k}^{n-1}s_j}{\prod_{j=k}^{n}r_j}$ for all $n \geqslant 1$.

Theorem 3. *The inverse of the Fibonacci matrix $F(r,s)$ is given by*

$$(F(r,s)^{-1})_{nk} = \begin{cases} \frac{1}{r}, & (k=n) \\ -\frac{r+s}{r^2}, & (k=n-1) \\ (-1)^{n-k}\frac{(s^2+rs-r^2)s^{n-k-2}}{r^{n-k+1}}, & (0 \leqslant k \leqslant n-2) \\ 0, & (k>n) \end{cases}, \quad (n,k \in \mathbb{N}_0).$$

Proof. We prove the Theorem for the Fibonacci matrix of finite order n. By Theorem 2, we obtain that

$$(F(r,s)^{-1})_{nk} = \begin{cases} \frac{1}{r}, & (k=n), \\ \frac{(-1)^{n-k}}{r^{n+1}}D_{n-k}^{(0)}(F(r,s)), & (1 \leqslant k \leqslant n), \\ 0, & (k>n), \end{cases} \quad (n,k \in \mathbb{N}_0),$$

where

$$D_n^{(0)}(r,s) = \begin{vmatrix} r+s & r & \ldots & 0 \\ 2r+s & r+s & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ F_n r + F_{n-1}s & F_{n-1}r + F_{n-2}s & \ldots & r+s \end{vmatrix} (n \geqslant 1).$$

In fact, we use induction method for proving this theorem. For $n = 1$, it is obtained that $D_1^{(0)}(r,s) = r+s, (F(r,s)^{-1})_{1,0} = -\frac{(r+s)}{r^2}$, if $n = 2, D_2^{(0)}(r,s) = s^2+rs-r^2$ and $(F(r,s)^{-1})_{2,0} = \frac{s^2+rs-r^2}{r^3}$, and if $n = 3, D_3^{(0)}(r,s) = s(s^2+rs-r^2)$ and $(F(r,s)^{-1})_{3,0} = \frac{(s^2+rs-r^2)s}{r^4}$. This completes the basis step. As per the principle of mathematical induction, we have

$$D_{n+1}^{(0)}(r,s) = \begin{vmatrix} r+s & r & \ldots & 0 \\ 2r+s & r+s & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ F_{n+1}r + F_n s & F_n r + F_{n-1}s & \ldots & r+s \end{vmatrix} = (r+s)D_n^{(0)}(r,s) - rD_n^{(0)}(r,s) = s(D_n^{(0)}(r,s)) = (s^2+rs-r^2)s^{n-1}.$$

Therefore, we conclude that $(F(r,s)^{-1})_{n,k} = (-1)^{n-k}\frac{(s^2+rs-r^2)s^{n-k-2}}{r^{n-k+1}}$ $(n,k \geqslant 2)$. □

Theorem 4. *The inverse of the Pascal matrix $P = (p_{nk})$ is given by*

$$(P^{-1})_{nk} = \begin{cases} (-1)^{n-k}\binom{n}{n-k}, & (0 \leqslant k \leqslant n) \\ 0, & (k>n) \end{cases}, \quad (n,k \in \mathbb{N}_0).$$

Proof. The proof follows from the fact that

$$D_n^{(k)}(P) = \begin{vmatrix} \binom{k+1}{k} & 1 & 0 & \ldots & 0 \\ \binom{k+2}{k} & \binom{k+2}{k+1} & 1 & \ldots & 0 \\ \binom{k+3}{k} & \binom{k+3}{k+1} & \binom{k+3}{k+2} & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \binom{n}{k} & \binom{n}{k+1} & \binom{n}{k+2} & \ldots & n \end{vmatrix} = \binom{n}{k}.$$

In particular,

$$D_n^{(0)}(P) = \begin{vmatrix} 1 & 1 & 0 & \ldots & 0 \\ 1 & 2 & 1 & \ldots & 0 \\ 1 & 3 & 3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \binom{n}{1} & \binom{n}{2} & \ldots & n \end{vmatrix} = 1 \qquad \square$$

Theorem 5. *The inverse of the generalized weighted mean factorable difference matrix $G(u,\ v;\Delta)$ is given by*

$$(G(u,v;\Delta)^{-1})_{nk} = \begin{cases} \frac{1}{u_n v_n}, & (k=n) \\ (-1)^{2(n-k)-1}\frac{(v_k - v_{k+1})}{u_k v_k v_{k+1}}, & (0 \leqslant k \leqslant n-1), \\ 0, & (k>n) \end{cases} \quad (n,k \in \mathbb{N}_0).$$

Proof. The proof of this theorem is the direct consequence of Theorem 2. By using Theorem 2, one can calculate

$$(G(u,v;\Delta)^{-1})_{nk} = \begin{cases} \frac{1}{u_n v_n}, & (k=n) \\ \frac{\frac{(-1)^{n-k}}{n}D_{n-k}^{(k)}(G(u,v;\Delta))}{\prod_{j=k}^{n} u_j v_j}, & (0 \leqslant k \leqslant n-1) \\ 0, & (k>n) \end{cases}, \quad (n,k \in \mathbb{N}_0),$$

where

$$D_n^{(k)}(G(u,v;\Delta)) = (v_k - v_{k+1}) \begin{vmatrix} u_{k+1} & u_{k+1}v_{k+1} & 0 & \dots & 0 \\ u_{k+2} & u_{k+2}(v_{k+1} - v_{k+2}) & u_{k+2}v_{k+2} & \dots & 0 \\ u_{k+3} & u_{k+3}(v_{k+1} - v_{k+2}) & u_{k+3}(v_{k+2} - v_{k+3}) & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_n & u_n(v_{k+1} - v_{k+2}) & u_n(v_{k+2} - v_{k+3}) & \dots & u_n(v_{n-1} - v_n) \end{vmatrix}.$$

On further simplification, we obtain that

$$\begin{aligned} D_n^{(k)}(G(u,v;\Delta)) &= (v_k - v_{k+1})(v_{k+1} - v_{k+2})(v_{k+3} - v_{k+4})\dots(v_{n-1} - v_n)u_n \\ &\quad \times \left(u_{k+1} - \frac{u_{k+1}v_{k+1}}{v_{k+1} - v_{k+2}}\right)\left(u_{k+2} - \frac{u_{k+2}v_{k+2}}{v_{k+2} - v_{k+3}}\right)\dots\left(u_{n-1} - \frac{u_{n-1}v_{n-1}}{v_{n-1} - v_n}\right) \\ &= (v_k - v_{k+1})(-u_{k+1}v_{k+2})(-u_{k+2}v_{k+3})\dots(-u_{n-1}v_n)u_n \\ &= (-1)^{n-k-1}(v_k - v_{k+1})u_n \prod_{j=k+1}^{n-1} u_j v_{j+1} \end{aligned}$$

Therefore, for $0 \leqslant k \leqslant n-1$, the exact entries of $(G(u,\ v;\ \Delta)^{-1})_{nk}$ are as follows:

$$\begin{aligned} (G(u,v:\Delta)^{-1})_{nk} &= (-1)^{n-k+n-k-1}\frac{(v_k - v_{k+1})u_n\prod_{j=k+1}^{n-1}u_jv_{j+1}}{\prod_{j=k}^{n-1}u_jv_j} \\ &= (-1)^{2(n-k)-1}\frac{(v_k - v_{k+1})}{u_kv_kv_{k+1}}. \qquad \square \end{aligned}$$

3. Conclusion

The most important tool of studying sequence spaces via different operators are the determination of their topological structures, duals, matrix characterizations, compactness, and spectral properties etc. In fact, for an operator, all these investigations are quite easier and even possible by finding its inverse. The main purpose of this work is to unify most of lower triangular Toeplitz matrices and determine their inverses. As the results of the present article relate to the infinite dimensional matrices it is natural to implement these results for finite dimensional cases. As an application of Theorem 2, in our next study we design algorithm for inverse of any lower triangular Toeplitz matrix of finite dimension. Therefore, this study is more essential and effective for different computer oriented languages such as C, C++, Matlab etc.

Acknowledgment

The authors are grateful to the anonymous referees for careful checking of the present manuscript and their helpful suggestions.

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