Journal of the Egyptian Mathematical Society (2014) 22, 379–385

Egyptian Mathematical Society

Journal of the Egyptian Mathematical Society

www.etms-eg.org
www.elsevier.com/locate/joems

ORIGINAL ARTICLE

# Dynamical behavior of a delayed diffusive predator–prey model with competition and type III functional response ☆

**Changjin Xu** [a,*], **Qiming Zhang** [b]

[a] *Guizhou Key Laboratory of Economics System Simulation, Guizhou University of Finance and Economics, Guiyang 550004, PR China*
[b] *College of Science, Hunan University of Technology, Zhuzhou 412007, PR China*

Received 17 August 2013; revised 13 November 2013; accepted 16 November 2013
Available online 22 December 2013

**KEYWORDS**

Predator–prey model;
Permanence;
Global asymptotic stability;
Type III functional response

**Abstract** In this paper, a delayed diffusive predator–prey model with competition and type III functional response is investigated. By using inequality analytical technique, some sufficient conditions which ensure the permanence of the model have been derived. By Lyapunov functional method, a series of sufficient conditions which assure the global asymptotic stability of the system are established. The paper ends with some numerical simulations that illustrate our analytical predictions.

**2010 MATHEMATICS SUBJECT CLASSIFICATION:** 34K20; 34C25; 92D25



## 1. Introduction

Since Lotka [1] and Volterra [2] introduced the first predator–prey model, numerous complicated but realistic predator–prey models have been formulated by ecologists and mathematicians. In 1992, Berryman [3] argued that the dynamic relationship between predators and their preys has been one dominant theme in both ecology and mathematical ecology due to its universal existence and importance. Dynamics of predator–prey models has been discussed by a lot of papers. It is well known that in many applications, the nature of permanence and global asymptotic stability of predator–prey models is of great interest. Recently, Samanta [4] investigated the permanence and global asymptotic stability of a delay predator–prey model with disease in the prey. Fan and Li [5] gave a theoret-

* Corresponding author. Tel.: +86 08512276595.
E-mail address: xcj403@126.com (C. Xu).

☆ This work is supported by National Natural Science Foundation of China (Nos. 11261010 and 11201138), Soft Science and Technology Program of Guizhou Province (No. 2011LKC2030), Scientific Research Fund of Hunan Provincial Education Department (No. 12B034), Natural Science and Technology Foundation of Guizhou Province (J[2012]2100), Governor Foundation of Guizhou Province ([2012]53) and Doctoral Foundation of Guizhou University of Finance and Economics (2010).
Peer review under responsibility of Egyptian Mathematical Society.

ELSEVIER Production and hosting by Elsevier


http://dx.doi.org/10.1016/j.joems.2013.11.006

ical study on permanence of a delayed ratio-dependent predator–prey model with Holling type functional response. Chen [6] focused on the permanence and global attractivity of Lotka–Volterra competition system with feedback control. Wang and Zhu [7] analyzed the permanence and global asymptotic stability for a delayed predator–prey system with Hassell–Varley type functional response. Teng et al. [8] addressed the permanence criteria for a delayed discrete nonautonomous-species Kolmogorov systems. For more research on the this topic of predator–prey models, one can see [9–17].

In 2008, Liu [18] investigated the permanence and almost periodic solution to the following delayed predator–prey system with diffusion and type III functional response

$$\begin{cases} \frac{dx_1(t)}{dt} = x_1[a_{10}(t) - a_{11}(t)x_1] - \frac{\alpha_1(t)x_1^2x_3}{1+\beta_1(t)x_1^2} - \frac{\alpha_2(t)x_1^2x_4}{1+\beta_2(t)x_1^2} + D_1(t)(x_2 - x_1), \\ \frac{dx_2(t)}{dt} = x_2[a_{20}(t) - a_{21}(t)x_2] + D_2(t)(x_1 - x_2), \\ \frac{dx_3(t)}{dt} = x_3\left[-a_{30}(t) + a_{31}(t)\frac{\alpha_1(t)x_1^2(t-\tau_1)}{1+\beta_1(t)x_1^2(t-\tau_1)} - a_{32}(t)x_3 - a_{34}(t)x_4\right], \\ \frac{dx_4(t)}{dt} = x_4\left[-a_{40}(t) + a_{41}(t)\frac{\alpha_2(t)x_1^2(t-\tau_2)}{1+\beta_2(t)x_1^2(t-\tau_2)} - a_{42}(t)x_4 - a_{43}(t)x_3\right]. \end{cases} \quad (1.1)$$

with the initial condition

$$x_1(s) = \phi_1(s) \in C([-\tau, 0], R_+), \quad s \in C([-\tau, 0], \quad \phi_1(0) \geqslant 0, \quad x_1(0) = \phi_i \geqslant 0(\text{constants}), \quad i = 1,2,3,4, \quad (1.2)$$

where $x_i(t)$ $(i = 1,2)$ describe the densities of the prey population in Patch 1 and Patch 2, respectively, $x_j$ $(j = 3,4)$ describe the densities of the predator population in Patch 1 with competition, $a_{10}(t)$ and $a_{i1}(t)$ $(i = 1,2)$ represent the intrinsic growth rate and the intra-specific interference coefficient of the prey population $x_i$ $(i = 1,2)$, respectively. We then assume that the death rate of the predator population $x_i$ $(i = 3,4)$ in Patch 1 is proportional to both the existing predator population with the proportional functions $a_{30}(t)t$ and, respectively, $a_{40}(t)$ and to its square with the proportional functions $a_{32}(t)$ and, respectively, $a_{42}(t)$. The predator consumes the prey according to Holling type III functional response [19,20], that is, $\frac{\alpha_1(t)x_1^2x_3}{1+\beta_1(t)x_1^2}$ and $\frac{\alpha_2(t)x_1^2x_4}{1+\beta_2(t)x_1^2}$. $\tau_i$ $(i = 1,2)$ is the time to digest food in the predator organism. Applying inequality theory and Liapunov–Razumikhin technique, Liu [18] obtained some sufficient conditions which guarantee the uniform permanence and the existence and uniqueness of the positive almost periodic solution which is globally asymptotically stable of system (1.1).

In this paper, we will focus on the permanence and global asymptotic stability of model (1.1). It shall be pointed that although Liu [18] had investigated the permanence of model, the sufficient conditions they obtained are different from the ones in this paper. Moreover, the global asymptotic stability of model (1.1) has not still been studied in Liu [18].

Let $f(t)$ be a bounded continuous functions on interval $[0, +\infty)$, we define

$$f^l = \inf_{t\in R} f(t), \quad f^u = \sup_{t\in R} f(t).$$

In the following discussion, we always assume that system (1.1) satisfies the following assumptions:

(H1) $a_{i0}(t)$, $a_{i1}(t)$ $(i = 1,2,3,4), \alpha_j, \beta_j, D_j(j = 1,2), a_{n1}(t), a_{n2}(t)(n = 3,4), a_{34}(t), a_{43}(t)$ are all bounded continuous functions on the interval $[0, +\infty)$ and strictly for periodic functions and satisfy:

(1) $\min\left\{a_{i0}^l, a_{i1}^l, \alpha_j^l, \beta_j^l, a_{n1}^l, a_{n2}^l, a_{34}^l, a_{43}^l\right\} > 0$;

(2) $\max\left\{a_{i0}^l, a_{i1}^l, \alpha_j^l, \beta_j^l, a_{n1}^l, a_{n2}^l, a_{34}^l, a_{43}^l\right\} < +\infty$, $i = 1,2,3,4$; $j = 1,2$; $n = 3,4$.

(H2)$a_{31}^u\alpha_1^u > a_{30}^l\beta_1^l, a_{41}^u\alpha_2^u > a_{40}^l\beta_2^l$.

We denote $X = (x_1, x_2, x_3, x_4) \in R_+^4 = \{(x_1, x_2, x_3, x_4) | x_i \geqslant 0, \ i = 1,2,3,4\}$. For the point of view of biology, system (1.1) is discussed in $R_+^4$.

The organization of this paper is as follows. In the next Section 2, Basic definitions and Lemmas are given, some sufficient conditions for the permanence of the delayed diffusive predator–prey model with competition and type III functional response in consideration are established. A series of sufficient conditions which guarantee the existence and global stability of positive periodic solution of the delayed diffusive predator–prey model with competition and type III functional response are included in Section 3. In Section 4, we give an example which shows the feasibility of the main results. Conclusions are presented in Section 5.

## 2. Permanence

In order to obtain the main result of this paper, we shall first state the definition of permanence and several lemmas which will be useful in the proving the main result.

**Definition 2.1** [21]. We say that system (1.1) is permanence if there are positive constants $M$ and $m$ such that for each positive solution $(x_1(t), x_2(t), x_3(t), x_4(t))$ of system (1.1) satisfies

$$m \leqslant \lim_{t\to+\infty} \inf x_i(t) \leqslant \lim_{t\to+\infty} \sup x_i(t) \leqslant M \quad (i = 1,2,3,4).$$

**Lemma 2.1** [22]. *If $a > 0$, $b > 0$ and $\dot{x} \geqslant x(b - ax)$, when $t \geqslant 0$ and $x(0) > 0$, we have*

$$\lim_{t\to+\infty} \inf x(t) \geqslant \frac{b}{a}.$$

If $a > 0, b > 0$ and $\dot{x} \leqslant x(b - ax)$, when $t \geqslant 0$ and $x(0) > 0$, we have

$$\lim_{t\to+\infty} \sup x(t) \leqslant \frac{b}{a}.$$

**Lemma 2.2.** *Let $X(t) = (x_1(t), x_2(t), x_3(t), x_4(t))$ denote any solution of system (1.1) with initial conditions (1.2). If the condition (H1) and (H2) hold, then there exists a positive constant $T$ such that*

$$x_i(t) \leqslant M \quad (i = 1,2,3,4), \quad \text{for } t \geqslant T,$$

*where*

$$M > M^*, M^* = \max\left\{\frac{a_{10}^u}{a_{11}^l}, \frac{a_{20}^u}{a_{21}^l}, \frac{-a_{30}^l + \frac{a_{31}^u\alpha_1^u}{\beta_1^l}}{a_{32}^l}, \frac{-a_{40}^l + \frac{a_{41}^u\alpha_2^u}{\beta_2^l}}{a_{42}^l}\right\}.$$

**Proof.** It follows from system (1.1) with initial conditions (1.2) that

$$\begin{cases} \left.\frac{dx_1(t)}{dt}\right|_{x_1=0} = D_1(t)x_2 > 0, \\ \left.\frac{dx_2(t)}{dt}\right|_{x_2=0} = D_2(t)x_1 > 0, \\ x_3(t) = x_3(0)\int_0^t \left[-a_{30}(s) + a_{31}(s)\frac{\alpha_1(s)x_1^2(s-\tau_1)}{1+\beta_1(s)x_1^2(s-\tau_1)} - a_{32}(s)x_3 - a_{34}(s)x_4\right]ds > 0, \\ x_4(t) = x_4(0)\int_0^t \left[-a_{40}(s) + a_{41}(s)\frac{\alpha_2(s)x_1^2(s-\tau_2)}{1+\beta_2(s)x_1^2(s-\tau_2)} - a_{42}(s)x_4 - a_{43}(s)x_3\right]ds > 0. \end{cases} \tag{2.1}$$

Thus $X = (x_1, x_2, x_3, x_4) \in R_+^4 = \{(x_1, x_2, x_3, x_4)|x_i \geqslant 0, i = 1, 2, 3, 4\}$ is a positively invariant set of system (1.1). We define

$$V(t) = \max\{x_1(t), x_2(t), x_3(t), x_4(t)\}. \tag{2.2}$$

(1) If $V(t) = x_1(t)$, then

$$D^+V(t) = \frac{dx_1(t)}{dt} \leqslant x_1[a_{10}(t) - a_{11}(t)x_1] \leqslant V(t)\left[a_{10}^u - a_{11}^l V(t)\right]. \tag{2.3}$$

(2) If $V(t) = x_2(t)$, then

$$D^+V(t) = \frac{dx_2(t)}{dt} \leqslant x_2[a_{20}(t) - a_{21}(t)x_2] \leqslant V(t)\left[a_{20}^u - a_{21}^l V(t)\right]. \tag{2.4}$$

(3) If $V(t) = x_3(t)$, then

$$D^+V(t) = \frac{dx_3(t)}{dt} \leqslant x_3\left[-a_{30}(t) + \frac{a_{31}(t)\alpha_1(t)}{\beta_1(t)} - a_{32}(t)x_3\right] \leqslant V(t)\left[-a_{30}^l + \frac{a_{31}^u\alpha_1^u}{\beta_1^l} - a_{32}^l V(t)\right]. \tag{2.5}$$

(4) If $V(t) = x_4(t)$, then

$$D^+V(t) = \frac{dx_4(t)}{dt} \leqslant x_4\left[-a_{40}(t) + \frac{a_{41}(t)\alpha_2(t)}{\beta_2(t)} - a_{42}(t)x_4\right] \leqslant V(t)\left[-a_{40}^l + \frac{a_{41}^u\alpha_2^u}{\beta_2^l} - a_{42}^l V(t)\right]. \tag{2.6}$$

Let

$$\begin{cases} \theta_1^u = a_{10}^u, \theta_2^u = a_{20}^u, \theta_3^u = -a_{30}^l + \frac{a_{31}^u\alpha_1^u}{\beta_1^l}, \\ \theta_4^u = -a_{40}^l + \frac{a_{41}^u\alpha_2^u}{\beta_2^l}, \delta_1^l = a_{11}^l, \delta_2^l = a_{21}^l, \delta_3^l = a_{32}^l, \delta_4^l = a_{42}^l. \end{cases} \tag{2.7}$$

It follows from (2.2)–(2.7) that

$$D^+V(t) \leqslant V(t)\left[\theta_i^u - \delta_i^l V(t)\right], \quad i = 1, 2, 3, 4. \tag{2.8}$$

Applying the comparison theorem we derive from the above inequality that:

(1) If $\max\{x_1(0), x_2(0), x_3(0), x_4(0)\} \leqslant M$, then $\max\{x_1(t), x_2(t), x_3(t), x_4(t)\} \leqslant M$, $t \geqslant 0$.

(2) If $\max\{x_1(0), x_2(0), x_3(0), x_4(0)\} > M$, let $-\gamma = \max\left\{M\left(\theta_i^u - \delta_i^l M\right)\right\}$ $(i = 1, 2, 3, 4), \gamma > 0$. When $D^+V(t) \leqslant V(t)\left[\theta_i^u - \delta_i^l V(t)\right] < -\gamma < 0$, $i = 1, 2, 3, 4$, by continuous dependence of the initial value there exists a positive constant $\varepsilon$ such that $V(t) > M$ for $t \in [0, \varepsilon)$, then

$$D^+V(t) \leqslant V(t)\left[\theta_i^u - \delta_i^l V(t)\right] < -\gamma < 0, \quad i = 1, 2, 3, 4.$$

In view of Lemma 2.1, there exists a constant $T > 0$ such that $\max\{x_1(t), x_2(t), x_3(t), x_4(t)\} \leqslant M$ for $t \geqslant T$. The proof of Lemma 2.2 is complete. □

In order to facilitate the calculation, we define

$$m^* = \min\{\rho_1, \rho_2, \rho_3, \rho_4\}, \tag{2.9}$$

where

$$\rho_1 = \frac{\left(a_{10}^l - D_1^u - \left(\alpha_1^u + \alpha_2^u\right)M^2\right.}{a_{11}^u}, \quad \rho_2 = \frac{a_{20}^l}{a_{21}^u + D_2^u},$$
$$\rho_3 = \frac{\frac{a_{31}^l\alpha_1^l m_1^2}{1+\beta_1(t)M^2} - \left(a_{30}^u + a_{34}^u M\right)}{a_{32}^u}, \quad \rho_4 = \frac{\frac{a_{41}^l\alpha_2^l m_1^2}{1+\beta_2(t)M^2} - \left(a_{40}^u + a_{43}^u M\right)}{a_{42}^u}.$$

We assume that

(H3)$m^* > 0$.

**Theorem 2.1.** *Suppose that the conditions (H2)–(H3) hold true, then system (1.1) is permanent.*

**Proof.** It is easy to see that system (1.1) with the initial value condition $(x_1(0), x_2(0), x_3(0), X_4(0))$ has positive solution $(x_1(t), x_2(t), x_3(t), x_4(t))$ passing through $(x_1(0), x_2(0), x_3(0), x_4(0))$. Let $(x_1(t), x_2(t), x_3(t), x_4(t))$ be any positive solution of system (1.1) with the initial condition $(x_1(0), x_2(0), x_3(0), x_4(0))$. It follows from the first equation of system (1.1) that

$$\begin{aligned} \frac{dx_1(t)}{dt} &\geqslant x_1\left[a_{10}(t) - D_1(t) - a_{11}(t)x_1 - \frac{\alpha_1(t)x_1x_3}{1+\beta_1(t)x_1^2} - \frac{\alpha_2(t)x_1x_4}{1+\beta_2(t)x_1^2}\right] \\ &\geqslant x_1[a_{10}(t) - D_1(t) - a_{11}(t)x_1 - \alpha_1(t)x_1x_3 - \alpha_2(t)x_1x_4] \\ &\geqslant x_1\{[a_{10}(t) - D_1(t) - (\alpha_1(t)+\alpha_2(t))M^2] - a_{11}(t)x_1\} \\ &\geqslant x_1\left\{\left[a_{10}^l - D_1^u - \left(\alpha_1^u + \alpha_2^u\right)M^2\right] - a_{11}^u x_1\right\}. \end{aligned} \tag{2.10}$$

It follows from Lemma 2.1 that

$$\liminf_{t\to+\infty} x_1(t) \geqslant \frac{a_{10}^l - D_1^u - \left(\alpha_1^u + \alpha_2^u\right)M^2}{a_{11}^u} := m_1. \tag{2.11}$$

From the second equation of system (1.1) that

$$\frac{dx_2(t)}{dt} \geqslant x_2[a_{20}(t) - (a_{21}(t) + D_2(t))x_2] \geqslant x_2[a_{20}^l - (a_{21}^u + D_2^u)x_2] \tag{2.12}$$

It follows from Lemma 2.1 that

$$\lim_{t\to+\infty}\inf x_2(t) \geqslant \frac{a_{20}^l}{a_{21}^u + D_2^u} := m_2. \tag{2.13}$$

From the third equation of system (1.1) that

$$\begin{aligned}\frac{dx_3(t)}{dt} &\geqslant x_3\left[-a_{30}(t) + \frac{a_{31}(t)\alpha_1(t)m_1^2}{1+\beta_1(t)M^2} - a_{32}(t)x_3 - a_{34}(t)M\right]\\ &\geqslant x_3\left[\frac{a_{31}^l\alpha_1^l m_1^2}{1+\beta_1(t)M^2} - (a_{30}^u + a_{34}^u M) - a_{32}^u x_3\right]\end{aligned} \tag{2.14}$$

From the third equation of system (1.1) that

$$\begin{aligned}\frac{dx_4(t)}{dt} &\geqslant x_4\left[-a_{40}(t) + \frac{a_{41}(t)\alpha_2(t)m_1^2}{1+\beta_2(t)M^2} - a_{42}(t)x_4 - a_{43}(t)M\right]\\ &\geqslant x_4\left[\frac{a_{41}^l\alpha_2^l m_1^2}{1+\beta_2(t)M^2} - (a_{40}^u + a_{43}^u M) - a_{42}^u x_4\right].\end{aligned} \tag{2.15}$$

Under the condition (H3), we get

$$\left(a_{10}^l - D_1^u - (\alpha_1^u + \alpha_2^u)M^2 > 0,\right.$$

$$\frac{a_{31}^l\alpha_1^l m_1^2}{1+\beta_1(t)M^2} - \left(a_{30}^u + a_{34}^u M > 0,\right.$$

$$\frac{a_{41}^l\alpha_2^l m_1^2}{1+\beta_2(t)M^2} - \left(a_{40}^u + a_{43}^u M\right) > 0.$$

Choose $m$ satisfying $0 < m < m^*$ and close enough to $m^*$. Define

$$\widetilde{V}(t) = \min\{x_1(t), x_2(t), x_3(t), x_4(t)\}. \tag{2.16}$$

Calculating the right derivative $D_+\widetilde{V}(t)$ of $\widetilde{V}(t)$ along the solution of system (1.1), we obtain

$$D_+\widetilde{V}(t) \geqslant \widetilde{V}(t)\left\{\left[a_{10}^l - D_1^u - (\alpha_1^u + \alpha_2^u)M^2\right] - a_{11}^u\widetilde{V}(t)\right\}, \tag{2.17}$$

$$D_+\widetilde{V}(t) \geqslant \widetilde{V}(t)\left[a_{20}^l - (a_{21}^u + D_2^u)\widetilde{V}(t)\right], \tag{2.18}$$

$$D_+\widetilde{V}(t) \geqslant \widetilde{V}(t)\left[\frac{a_{31}^l\alpha_1^l m_1^2}{1+\beta_1(t)M^2} - (a_{30}^u + a_{34}^u M) - a_{32}^u\widetilde{V}(t)\right], \tag{2.19}$$

$$D_+\widetilde{V}(t) \geqslant \widetilde{V}(t)\left[\frac{a_{41}^l\alpha_2^l m_1^2}{1+\beta_2(t)M^2} - (a_{40}^u + a_{43}^u M) - a_{42}^u\widetilde{V}(t)\right]. \tag{2.20}$$

(1) If $\widetilde{V}(0) = \min\{x_1(0), x_2(0), x_3(0), x_4(0)\} \geqslant m$, then $\widetilde{V}(t) = \min\{x_1(t), x_2(t), x_3(t), x_4(t)\} \geqslant m$.
(2) If $\widetilde{V}(0) = \min\{x_1(0), x_2(0), x_3(0), x_4(0)\} < m$, then let

$$\mu = \min\left\{\rho_1^{(0)}, \rho_2^{(0)}, \rho_3^{(0)}, \rho_4^{(0)}\right\},$$

where

$$\rho_1^{(0)} = x_1(0)\left\{\left[a_{10}^l - D_1^u - (\alpha_1^u + \alpha_2^u)M^2\right] - a_{11}^u m\right\},$$
$$\rho_2^{(0)} = x_2(0)\left[a_{20}^l - (a_{21}^u + D_2^u)m\right],$$
$$\rho_3^{(0)} = x_3(0)\left[\frac{a_{31}^l\alpha_1^l m_1^2}{1+\beta_1(t)M^2} - (a_{30}^u + a_{34}^u M) - a_{32}^u x_3(0)\right],$$
$$\rho_4^{(0)} = x_4(0)\left[\frac{a_{41}^l\alpha_2^l m_1^2}{1+\beta_2(t)M^2} - (a_{40}^u + a_{43}^u M) - a_{42}^u m\right].$$

If $\widetilde{V}(0) < m$ holds, by dependence of initial value then there exists $\varepsilon > 0$ such that if $t \in [0, \varepsilon)$, then $\widetilde{V}(t) < m$ and we get $D_+\widetilde{V}(t) > \mu > 0$. Thus there exists $\overline{T} > T > 0$ such that $\min\{x_1(t), x_2(t), x_3(t), x_4(t)\} \geqslant m$ for $t > \overline{T}$. Let

$$\Lambda = \{(x_1(t), x_2(t), x_3(t), x_4(t)) | m \leqslant x_i(t) \leqslant M(i = 1, 2, 3, 4)\},$$

then $\Lambda$ is a bounded compact region in $R_+^4$ which has a positive distance from coordinate planes. According to the analysis above, we obtain that there exists a constant $\overline{T} > 0$, if $t > \overline{T}$, then every positive solution of system (1.1) eventually enters and remains in the region $\Lambda$. The proof of Theorem 2.1 is complete. □

## 3. The existence and global asymptotic stability of positive periodic solution

In this section, we will derive sufficient conditions for the existence of periodic solution of system (1.1). Firstly, we use the fixed point theorem of Brouwer.

**Lemma 3.1** [*Brouwer*]. *Suppose that the continuous operator $P$ maps closed bounded convex set $Q \in R^n$ onto itself, then the operator $P$ has at least one fixed point in set $Q$.*

**Theorem 3.1.** *Suppose that the conditions (H1)–(H3) hold, then there is at least one positive periodic solution of system (1.1).*

**Proof.** Let the unique of periodic system (1.1) for initial value $X^0 = (x_1^0, x_2^0, x_3^0, x_4^0)$ be denoted as

$$X(t, X^0) = (x_1(t, X^0), x_2(t, X^0), x_3(t, X^0), x_4(t, X^0)).$$

In the sequel, we define the Poincare map $P: R_+^4 \to R_+^4$ is $P(x^0) = P(\omega, X^0)$, where $\omega$ is the period of periodic system (1.1). If (H1)-(H2) are fulfilled, then from Theorem 2.1 we know that there exists $m > 0$ such that

$$x_i(t) \geqslant m \quad (i = 1, 2, 3, 4).$$

Then the compact region $\Lambda \in R_+^4$ is a positive invariant set of system (1.1), and $\Lambda$ is also a closed bounded convex set. Therefore we have $X(t, X^0) \in \Lambda$ when $X^0 \in \Lambda$, also $X(\omega, X^0) \subset \Lambda$. Thus $P\Lambda \subset \Lambda$. The operator $P$ is continuous because the solution is continuous with respect to the initial value. Applying the fixed point theorem of Brouwer, we obtain that $P$ has at least one positive $\omega$-periodic solution of system (1.1). This completes the proof of Theorem 3.1. □

**Definition 3.1.** A bounded positive solution $(u_1(t), u_2(t), u_3(t), u_4(t))^T$ of system (1.1) is said to be globally asymptotically stable, if for any other positive bounded solution $(x_1(t), x_2(t), x_3(t), x_4(t))^T$ of system (1.1), the following equality holds,

$$\lim_{t\to+\infty}\left[\sum_{i=1}^{4}|x_i(t) - u_i(t)|\right] = 0.$$

**Definition 3.2** [23]. Let $h$ be a real number and $f$ be a non-negative function defined on $[h, +\infty)$ such that $f$ is integrable on $[h, +\infty)$ and is uniformly continuous on $[h, +\infty)$, then $\lim_{t\to+\infty} f(t) = 0$.

**Theorem 3.2.** *In addition to* $(H1)$-$(H3)$ , *assume further that*

$(H4) K_i > 0$,

*where* $K_i$ $(i = 1,2,3,4)$ *are defined by* (3.20)–(3.22) and (3.23) *respectively*. *Then system* (1.1) *has a unique positive solution* $(x_1(t), x_2(t), x_3(t), x_4(t))^T$ *which is global attractivity*.

**Proof.** According to Theorem 3.1 we have obtained that if (H1)–(H3) hold true, then system (1.1) has at least one strictly positive $\omega$-periodic solution $X(t) = (x_1(t), x_2(t), x_3(t), x_4(t))$. Let $U(t) = (u_1(t), u_2(t), u_3(t), u_4(t))$ be any positive solution of system (1.1). From Theorem 3.1, there exist positive constants $m, M$ such that

$$m \leqslant x_i \leqslant M, \quad m \leqslant u_i \leqslant M, \quad i = 1,2,3,4, \quad \text{for } t > \overline{T}.$$

Let

$$\bar{x}_i(t) = \ln x_i(t), \quad \bar{u}_i(t) = \ln u_i(t), \quad i = 1,2,3,4.$$

Define

$$V_i(t) = |\ln \bar{x}_i(t) - \ln \bar{u}_i(t)|, \quad i = 1,2,3,4. \tag{3.1}$$

Then the upper-right derivative of $V_i(t)$ along the solution of (1.1) are given below:

$$\begin{aligned} D^+V_1(t) = {} & \left(\frac{\bar{x}_1{}'(t)}{\bar{x}_1(t)} - \frac{\bar{u}_1{}'(t)}{\bar{u}_1(t)}\right)\operatorname{sgn}(\bar{x}_1(t) - \bar{u}_1(t)) = \operatorname{sgn}(\bar{x}_1(t) \\ & - \bar{u}_1(t))\Bigg[-a_{11}(t)(x_1(t) - u_1(t)) \\ & -\alpha_1(t)\left(\frac{x_1^2(t)x_3(t)}{1+\beta_1(t)x_1^2(t)} - \frac{u_1^2(t)u_3(t)}{1+\beta_1(t)u_1^2(t)}\right) \\ & -\alpha_2(t)\left(\frac{x_1^2(t)x_4(t)}{1+\beta_2(t)x_1^2(t)} - \frac{u_1^2(t)u_4(t)}{1+\beta_2(t)u_1^2(t)}\right)\Bigg] \\ & + D_1(t)\left(\frac{x_2(t)}{x_1(t)} - \frac{u_2(t)}{u_1(t)}\right)\Bigg], \end{aligned} \tag{3.2}$$

$$\begin{aligned} D^+V_2(t) = {} & \left(\frac{\bar{x}_2{}'(t)}{\bar{x}_2(t)} - \frac{\bar{u}_2{}'(t)}{\bar{u}_2(t)}\right)\operatorname{sgn}(\bar{x}_1(t) - \bar{u}_1(t)) \\ = {} & \operatorname{sgn}(\bar{x}_2(t) - \bar{u}_2 1(t))[-a_{21}(t)(x_2(t) - u_2(t)) \\ & + D_2(t)\left(\frac{x_1(t)}{x_2(t)} - \frac{u_1(t)}{u_2(t)}\right)\Bigg], \end{aligned} \tag{3.3}$$

$$\begin{aligned} D^+V_3(t) = {} & \left(\frac{\bar{x}_3{}'(t)}{\bar{x}_3(t)} - \frac{\bar{u}_3{}'(t)}{\bar{u}_3(t)}\right)\operatorname{sgn}(\bar{x}_3(t) - \bar{u}_3(t)) \\ = {} & \operatorname{sgn}(\bar{x}_3(t) - \bar{u}_3(t))\Bigg[a_{31}(t)\alpha_1(t)\Bigg(\frac{x_1^2(t-\tau_1)}{1+\beta_1(t)x_1^2(t-\tau_1)} \\ & - \frac{x_1^2(t-\tau_1)}{1+\beta_1(t)x_1^2(t-\tau_1)}\Bigg) - a_{32}(t)(x_3(t) - u_3(t)) \\ & - a_{34}(t)(x_4(t) - u_4(t))\Bigg] \end{aligned} \tag{3.4}$$

$$\begin{aligned} D^+V_4(t) = {} & \left(\frac{\bar{x}_4{}'(t)}{\bar{x}_4(t)} - \frac{\bar{u}_4{}'(t)}{\bar{u}_4(t)}\right)\operatorname{sgn}(\bar{x}_4(t) - \bar{u}_4(t)) \\ = {} & \operatorname{sgn}(\bar{x}_4(t) - \bar{u}_4(t))\Bigg[a_{41}(t)\alpha_2(t)\Bigg(\frac{x_1^2(t-\tau_2)}{1+\beta_2(t)x_1^2(t-\tau_2)} \\ & - \frac{x_1^2(t-\tau_1)}{1+\beta_2(t)x_1^2(t-\tau_1)}\Bigg) - a_{42}(t)(x_4(t) - u_4(t)) \\ & - a_{43}(t)(x_3(t) - u_3(t))\Bigg]. \end{aligned} \tag{3.5}$$

Let

$$\overline{D}_1(t) = D_1(t)\operatorname{sgn}(\bar{x}_1(t) - \bar{u}_1(t))\left(\frac{x_2(t)}{x_1(t)} - \frac{u_2(t)}{u_1(t)}\right),$$

$$\overline{D}_2(t) = D_2(t)\operatorname{sgn}(\bar{x}_2(t) - \bar{u}_2(t))\left(\frac{x_1(t)}{x_2(t)} - \frac{u_1(t)}{u_2(t)}\right).$$

If $x_1(t) > u_1(t)$, then

$$\begin{aligned} \overline{D}_1(t) = D_1(t)\left(\frac{x_2(t)}{x_1(t)} - \frac{u_2(t)}{u_1(t)}\right) & \leqslant \frac{D_1(t)}{u_1(t)}(x_2(t) - u_2(t)) \\ & \leqslant \frac{D_1^u}{m}|x_2(t) - u_2(t)|. \end{aligned} \tag{3.6}$$

If $x_1(t) < u_1(t)$, then

$$\begin{aligned} \overline{D}_1(t) = D_1(t)\left(\frac{u_2(t)}{u_1(t)} - \frac{x_2(t)}{x_1(t)}\right) & \leqslant \frac{D_1(t)}{x_1(t)}(u_2(t) - x_2(t)) \\ & \leqslant \frac{D_1^u}{m}|x_2(t) - u_2(t)|. \end{aligned} \tag{3.7}$$

It follows from (3.6) and (3.7) that

$$\overline{D}_1(t) \leqslant \frac{D_1^u}{m}|x_2(t) - u_2(t)|. \tag{3.8}$$

Similarly we have

$$\overline{D}_2(t) \leqslant \frac{D_2^u}{m}|x_1(t) - u_1(t)|. \tag{3.9}$$

Then we have

$$\begin{aligned} D^+V_1(t) \leqslant {} & -a_{11}^l|x_1(t) - u_1(t)| + \frac{\alpha_1^u(1+\beta_1^u M^3}{1+\beta_1^l m^2}|x_3(t) - u_3(t)| \\ & + \frac{\alpha_2^u(1+\beta_1^u M^3}{1+\beta_1^l m^2}|x_1(t) - u_1(t)| + \frac{D_1^u}{m}|x_2(t) - u_2(t)|, \\ = {} & \left(-a_{11}^l + \frac{\alpha_2^u(1+\beta_1^u M^3}{1+\beta_1^l m^2}\right)|x_1(t) - u_1(t)| \\ & + \frac{D_1^u}{m}|x_2(t) - u_2(t)| + \frac{\alpha_1^u(1+\beta_1^u M^3}{1+\beta_1^l m^2}|x_3(t) - u_3(t)|, \end{aligned} \tag{3.10}$$

$$D^+V_2(t) \leqslant -a_{21}^l|(x_2(t) - u_2(t)| + \frac{D_2^u}{m}|x_1(t) - u_1(t)|, \tag{3.11}$$

$$\begin{aligned} D^+V_3(t) \leqslant {} & \frac{2a_{31}^u\alpha_1^u M}{(1+\beta_1^l m)^2}|x_1(t-\tau_1) - u_1(t-\tau_1)| \\ & - a_{32}^l|x_3 - u_3(t)| + a_{34}^u|x_4 - u_4(t)|, \end{aligned} \tag{3.12}$$

$$\begin{aligned} D^+V_4(t) \leqslant {} & \frac{2a_{41}^u\alpha_2^u M}{(1+\beta_2^l m)^2}|x_1(t-\tau_2) - u_1(t-\tau_2)| \\ & - a_{42}^l|x_4 - u_4(t)| + a_{43}^u|x_3 - u_3(t)|. \end{aligned} \tag{3.13}$$

Define

$$V_5(t) = \frac{2a_{31}^u\alpha_1^u M}{(1+\beta_1^l m)^2}\int_{t-\tau_1}^t |x_1(s) - u_1(s)|ds \tag{3.14}$$

and

$$V_6(t) = \frac{2a_{41}^u\alpha_2^u M}{(1+\beta_2^l m)^2}\int_{t-\tau_2}^t |x_1(s) - u_1(s)|ds \tag{3.15}$$

Calculating the right-upper derivative of $V_5(t)$ and $V_6(t)$ along the solution of system (1.1), we derive

$$D^+V_5(t) = \frac{2a_{31}^u\alpha_1^u M}{(1+\beta_1^l m)^2}|x_1(s)-u_1(s)| - \frac{2a_{31}^u\alpha_1^u M}{(1+\beta_1^l m)^2}|x_1(t-\tau_1)-u_1(t-\tau_1)|, \quad (3.16)$$

$$D^+V_6(t) = \frac{2a_{41}^u\alpha_2^u M}{(1+\beta_2^l m)^2}|x_1(s)-u_1(s)| - \frac{2a_{41}^u\alpha_2^u M}{(1+\beta_2^l m)^2}|x_1(t-\tau_2)-u_1(t-\tau_2)|. \quad (3.17)$$

Let

$$V(t) = \sigma_1 V_1(t) + \sigma_2 V_2(t) + \sigma_3(V_3(t)+V_5(t)) + \sigma_4(V_4(t)+V_6(t)). \quad (3.18)$$

It follows (3.10)–(3.17) that

$$D^+V(t) \leqslant -\sum_{i=1}^{4} K_i|x_i(t)-u_i(t)|, \quad t \geqslant \overline{T}, \quad (3.19)$$

where $\overline{T}$ is defined in Theorem 2.1 and $K_i$ $(i=1,2,3,4)$ are defined in (3.20)–(3.22) and (3.23), respectively.

$$K_1 = \sigma_1\left[\frac{\alpha_2^u(1+\beta_1^u M^3)}{1+\beta_1^l m^2} - a_{11}^l\right] - \frac{\sigma_2 D_2^u}{m} - \frac{2\sigma_3 a_{31}^u\alpha_1^u M}{1+\beta_1^l m^2} - \frac{2\sigma_4 a_{41}^u\alpha_2^u M}{1+\beta_2^l m^2}, \quad (3.20)$$

$$K_2 = \sigma_2 a_{21}^l - \frac{\sigma_1 D_1^u}{m}, \quad (3.21)$$

$$K_3 = \sigma_3 a_{32}^l - \sigma_4 a_{43}^u - \frac{\sigma_1\alpha_1^u(1+\beta_1^u M^3)}{1+\beta_1^l m^2}, \quad (3.22)$$

$$K_4 = \sigma_4 a_{42}^l - \sigma_3 a_{34}^u. \quad (3.23)$$

Integrating both sides of (3.19) on interval $[\overline{T}, t]$ yields

$$V(t) + \sum_{i=1}^{4}\int_{\overline{T}}^{t} K_i(t)|x_i(t)-u_i(t)|ds \leqslant V(\overline{T}). \quad (3.24)$$

It follows from (3.24) that

$$\sum_{i=1}^{4}\int_{\overline{T}}^{t} K_i|x_i(t)-u_i(t)|ds \leqslant V(\overline{T}) < \infty, \quad \text{for } t \geqslant \overline{T}. \quad (3.25)$$

Since $x_i(t)$ $(i=1,2,3,4)$ are bounded for $t \geqslant \overline{T}$, so $|x_i(t)-u_i(t)|$ $(i=1,2,3,4)$ are uniformly continuous on $[\overline{T},\infty)$. By Barbalat's Lemma [23], we have

$$\lim_{t\to\infty}|x_i(t)-u_i(t)| = 0, \ (i=1,2,3,4). \quad (3.26)$$

By Theorems 7.4 and 8.2 in [24], we know that the positive solution $(x_1(t),x_2(t),x_3(t),x_4(t))^T$ of Eq. (1.1) is uniformly asymptotically stable. The proof of Theorem 3.2 is complete. □

## 4. Numerical example

To illustrate the theoretical results, we present some numerical simulations. Let us consider the following discrete system:

$$\begin{cases} \frac{dx_1(t)}{dt} = x_1[0.5+0.2\sin t-(2+\sin t)(t)x_1] - \frac{(2+\cos t)x_1^2x_3}{1+(2+\sin t)x_1^2} \\ \quad -\frac{(7-\sin t)x_1^2x_4}{1+(2-\sin t)x_1^2} + (0.9+0.4\cos t)(x_2-x_1), \\ \frac{dx_2(t)}{dt} = x_2[0.4+0.2\cos t-(7-\sin t)x_2] + (0.8+0.2\sin t)(x_1-x_2), \\ \frac{dx_3(t)}{dt} = x_3\left[-(0.8+0.4\sin t)+(1+\cos t)\frac{(2+\cos t)x_1^2(t-\tau_1)}{1+(2+\sin t)x_1^2(t-\tau_1)}\right. \\ \quad \left.-(8+0.4\cos t)x_3-(0.7-0.5\sin t)x_4\right], \\ \frac{dx_4(t)}{dt} = x_4\left[-(0.6+0.3\cos t)+(2+\sin t)\frac{(7-\sin t)x_1^2(t-\tau_2)}{1+(2-\sin t)x_1^2(t-\tau_2)}\right. \\ \quad \left.-(8+0.5\sin t)x_4-(0.9+0.2\sin t)x_3\right]. \end{cases} \quad (4.1)$$

Corresponding to system (4.1), we have

$a_{10}(t) = 0.5+0.2\sin t$, $a_{20}(t) = 0.4+0.2\cos t$, $a_{30}(t) = 0.8+0.4\sin t$,
$a_{40}(t) = 0.6+0.3\cos t$, $a_{11}(t) = 2+\sin t$, $a_{21}(t) = 7-\sin t$, $a_{31}(t) = 1+\cos t$,
$a_{41}(t) = 2+\sin t$, $\alpha_1(t) = 2+\cos t$, $\alpha_2(t) = 7-\sin t$, $\beta_1(t) = 2+\sin t$,
$\beta_2(t) = 2-\sin t$, $a_{32}(t) = 8+0.4\cos t$, $a_{34}(t) = 0.7-0.5\sin t$, $a_{42}(t) = 8+0.5\sin t$,
$a_{43}(t) = 0.9+0.2\sin t$, $D_1(t) = 0.9+0.4\cos t$, $D_2(t) = 0.8+0.2\sin t$.

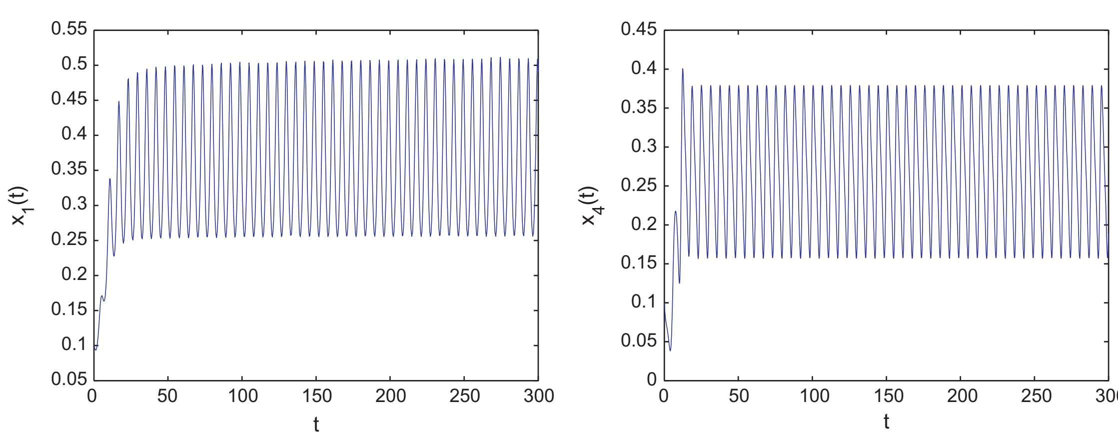


**Fig. 1** The dynamical behavior of the solution $(x_1(t),x_2(t),x_3(t),x_4(t))$ of system (4.1).

It is easy to see that

$$\begin{cases} a_{10}^u = 0.7, \quad a_{10}^l = 0.3, \quad a_{20}^u = 0.6, \quad a_{20}^l = 0.2, \quad a_{30}^u = 1.2, \quad a_{30}^l = 0.4, \quad a_{40}^u = 0.9, \\ a_{40}^l = 0.3, \quad a_{11}^u = 3, \quad a_{11}^l = 1, \quad a_{21}^u = 8, \quad a_{21}^l = 6, \quad a_{31}^u = 2, \quad a_{31}^l = 0, \quad a_{41}^u = 3, \\ a_{41}^l = 1, \quad \alpha_1^u = 3, \quad \alpha_1^l = 1, \quad \alpha_2^u = 8, \quad \alpha_2^l = 6, \quad \beta_1^u = 3, \quad \beta_1^l = 1, \quad \beta_2^u = 3, \\ \beta_2^l = 1, \quad a_{32}^u = 8.4, \quad a_{32}^l = 7.6, \quad a_{34}^u = 1.2, \quad a_{34}^l = 0.2, \quad a_{42}^u = 8.5, \quad a_{42}^l = 7.5, \\ a_{43}^u = 1.1, \quad a_{43}^l = 0.7, \quad D_1^u = 1.3, \quad D_1^l = 0.5, \quad D_2^u = 1, \quad D_2^l = 0.6. \end{cases}$$

Let $\sigma_1 = 0.2$, $\sigma_2 = 0.3$, $\sigma_3 = 0.8$ and $\sigma_4 = 0.75$. Then the coefficients of system (4.1) satisfy the conditions in Theorem 3.2. The phase diagram of system (4.1) is illustrated in Fig. 1. Numerical simulations show that system (4.1) has a unique positive periodic solution which is globally asymptotically stable.

## 5. Conclusions

In this paper, we have investigated the dynamical behavior of a delayed diffusive predator–prey model with competition and type III functional response. By using inequality analytical technique, sufficient conditions which ensure the permanence of the system are obtained. Moreover, we also analyze the positive periodic solution by mean of fixed point theorem of Brouwer. By Lyapunov functional method, we has also obtained some sufficient conditions for the global stability of positive periodic solution of the system. From the conditions (H1)–(H3) in Theorems 2.1 and 3.2., we can conclude that delay has no influence on the permanence and the global stability of the system. Numerical simulations show the feasibility of our main results.

## Acknowledgments

The authors would like to express sincere appreciation to the editor and anonymous referee for their valuable comments which have led to an improvement the presentation of this paper.

## References

[1] A. Lotka, The Elements of Physical Biology, Williams and Wilkins, Baltimore, 1925.
[2] V. Voltera, Variazioni e fluttuazioni del numero di individui in specie animali conviventi, Mem. Accd. Lincei. 2 (1926) 1–113.
[3] A.A. Berryman, The origns and evolution of predator–prey theory, Ecology 73 (5) (1992) 1530–1535.
[4] G.P. Samanta, Analysis of a delay nonautonous predator–prey system with disease in the prey, Nonlinear Anal.: Modell. Contr. 15 (1) (2010) 97–108.
[5] Y.H. Fan, W.T. Li, Permanence for a delayed discrete ratio-dependent predator–prey model with Holling type functional response, J. Math. Anal. Appl. 299 (2) (2004) 357–374.
[6] F.D. Chen, The permanence and global attractivity of Lotka–Volterra competition system with feedback control, Nonlinear Anal.: Real World Appl. 7 (1) (2006) 133–143.
[7] K. Wang, Y.L. Zhu, Permanence and global asymptotic stability of a delayed predator–prey model with Hassell–Varley type functional response, Bull. Iran. Math. Soc. 37 (3) (2011) 197–215.
[8] Z.D. Teng, Y. Zhang, S.J. Gao, Permanence criteria for general delayed discrete nonautonomous $n$-species Kolmogorov systems and its applications, Comput. Math. Appl. 59 (2) (2010) 812–828.
[9] J. Dhar, K.S. Jatav, Mathematical analysis of a delayed stage-structured predator–prey model with impulsive diffusion between two predators territories, Ecological Complexity 16 (2013) 59–67.
[10] S.Q. Liu, L.S. Chen, Necessary-sufficient conditions for permanence and extinction in Lotka–Volterra system with distributed delay, Appl. Math. Lett. 16 (6) (2003) 911–917.
[11] X.Y. Liao, S.F. Zhou, Y.M. Chen, Permanence and global stability in a discrete $n$-species competition system with feedback controls, Nonlinear Anal.: Real World Appl. 9 (4) (2008) 1661–1671.
[12] H.X. Hu, Z.D. Teng, H.J. Jiang, On the permanence in non-autonomous Lotka–Volterra competitive system with pure-delays and feedback controls, Nonlinear Anal.: Real World Appl. 10 (3) (2009) 1803–1815.
[13] Y. Muroya, Permanence and global stability in a Lotka–Volterra predator–prey system with delays, Appl. Math. Lett. 16 (8) (2003) 1245–1250.
[14] T. Kuniya, Y. Nakata, Permanence and extinction for a nonautonomous SEIRS epidemic model, Appl. Math. Comput. 218 (18) (2012) 9321–9331.
[15] Z.Y. Hou, On permanence of Lotka–Volterra systems with delays and variable intrinsic growth rates, Nonlinear Anal.: Real World Appl. 14 (2) (2013) 960–975.
[16] C.H. Li, C.C. Tsai, S.Y. Yang, Analysis of the permanence of an SIR epidemic model with logistic process and distributed time delay, Commun. Nonlinear Sci. Numer. Simul. 17 (9) (2012) 3696–3707.
[17] F.D. Chen, M.S. You, Permanence for an integrodifferential model of mutualism, Appl. Math. Comput. 186 (1) (2007) 30–34.
[18] Q. Liu, Almost periodic solution of a diffusive mixed system with time delay and type III functional response, Discr. Dyn. Nat. Soc. (2008) 13 (Article ID 706154).
[19] S. Tang, L.S. Chen, Chaos in functional response host-parasitoid ecosystem models, Chaos, Solitons Fract. 13 (4) (2002) 875–884.
[20] F. Wei, K. Wang, Uniform persistence of asymptotically periodic multispecies competition predator–prey systems with Holling III type functional response, Appl. Math. Comput. 170 (2) (2005) 994–998.
[21] X. Lv, S.P. Lu, P. Lu, Existence and global attractivity of positive periodic solutions of Lotka–Volterra predator–prey systems with deviating arguments, Nonlinear Anal.: Real World Appl. 11 (5-6) (2010) 574–583.
[22] F. Montes de Oca, M. Vivas, Extinction in a two dimensional Lotka–Volterra system with infinite delay, Nonlinear Anal.: Real World Appl. 7 (5) (2006) 1042–1047.
[23] K. Gopalsamy, Stability and Oscillations in Delay Differential Equations of Population Dynamics, Kluwer Academic, Dordrecht, The Netherlands, 1992.
[24] T. Yoshizawa, Stability Theory by Lyapunov' Second Method, Math. Soc. Japan, Tokyo, 1966.