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Starlikeness of a new general integral operator for meromorphic multivalent functions

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Abstract In the present paper, we introduce a new general integral operator of meromorphic multivalent functions. The starlikeness of this integral operator is determined. Several special cases are also discussed in the form of corollaries.

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1. Introduction

Let Σ_p denote the class of all meromorphic functions of the form

$$f(z) = \frac{1}{z^p} + \sum_{k=1-p}^{\infty} a_k z^k \quad (p \in \mathbb{N} := \{1, 2, \ldots\}), \tag{1}$$

which are analytic and p-valent in the punctured unit disk

$\mathbb{U}^* = \{z \in \mathbb{C} : 0 < |z| < 1\} = \mathbb{U} \setminus \{0\}$,

where $\mathbb{U}$ is the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and

$$\mathbb{H}(\mathbb{U}) = \left\{ f : \mathbb{U} \xrightarrow{f} \mathbb{C} \quad \text{holomorphic in } \mathbb{U} \right\}.$$

For $a \in \mathbb{C}$ and $n \in \mathbb{N}$, let

$\mathbb{H}[a, n] = \{f \in \mathbb{H}(\mathbb{U}) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots, \ z \in \mathbb{U}\}$.

A function $f \in \Sigma_p$ is said to be meromorphic p-valent starlike of order α $(0 \leqslant \alpha < p)$, if it satisfies the inequality

$$-\Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha. \tag{2}$$

We denote this class by $\Sigma\mathcal{S}_p^*(\alpha)$.

A function $f \in \Sigma_p$ is said to be meromorphic p-valent convex of order α $(0 \leqslant \alpha < p)$, if it satisfies the inequality

$$-\Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha. \tag{3}$$

We denote this class by $\Sigma\mathcal{K}_p(\alpha)$.

We note that $f \in \Sigma\mathcal{K}_p(\alpha)$ if and only if $-\frac{zf'}{p} \in \Sigma\mathcal{S}_p^*(\alpha)$.

Recently, many authors introduced and studied various integral operators of analytic and univalent functions in the open unit disk $\mathbb{U}$ (see, for example, [1–11]).

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In the present paper, we introduce the following new general integral operator $\mathcal{F}(z)$ of meromorphic multivalent functions.

Definition 1.1. *Let $n, p \in \mathbb{N}$, $c > 0$, $\gamma_i > 0$ $(i = 1, 2, \ldots, n)$.* We define the integral operator

$$\mathcal{J}^c_{p,\gamma_1,\ldots,\gamma_n}(f_1, \ldots, f_n) : \Sigma_p^n \longrightarrow \Sigma_p,$$

$$\mathcal{F}(z) = \mathcal{J}^c_{p,\gamma_1,\ldots,\gamma_n}(f_1, \ldots, f_n)(z) = \frac{c}{z^{p+c}} \int_0^z u^{c-1} \prod_{i=1}^n \left(-\frac{u^{p+1}}{p} f_i'(u)\right)^{\gamma_i} du \quad (z \in \mathbb{U}^*). \tag{4}$$

Remark 1.1. We note that if $c = 1$, then the integral operator $\mathcal{F}(z)$ reduces to the integral operator

$$\mathcal{J}_{p,\gamma_1,\ldots,\gamma_n}(z) = \frac{1}{z^{p+1}} \int_0^z \prod_{i=1}^n \left(-\frac{u^{p+1}}{p} f_i'(u)\right)^{\gamma_i} du \tag{5}$$

introduced by Mohammed and Darus [12]. If $n = 1, \gamma_1 = \gamma$ and $f_1 = f$, then the integral operator $\mathcal{F}(z)$ reduces to the integral operator

$$\mathcal{I}^c_{p,\gamma}(f)(z) = \frac{c}{z^{p+c}} \int_0^z u^{c-1} \left(-\frac{u^{p+1}}{p} f'(u)\right)^{\gamma} du. \tag{6}$$

For $p = 1$, the integral operator defined in (5) is introduced and studied by Mohammad and Darus [13].

For the starlikeness of the integral operator $\mathcal{F}(z)$ defined in Definition 1.1, we need to use following lemma.

Lemma 1.1 14. *Let $n \in \mathbb{N} \setminus \{0\}, \alpha, \delta \in \mathbb{R}, \gamma \in \mathbb{C}$ with $\Re\{\gamma - \alpha\delta\} \geqslant 0$. If $h \in \mathbb{H}[h(0), n]$ with $h(0) \in \mathbb{R}$ and $h(0) > \alpha$, then we have*

$$\Re\left\{h(z) + \frac{zh'(z)}{\gamma - \delta h(z)}\right\} > \alpha \Rightarrow \Re\{h(z)\} > \alpha (z \in \mathbb{U}).$$

2. Starlikeness of the operator $\mathcal{F}(z)$

In this section, we investigate sufficient conditions for the meromorphically starlikeness of the integral operator $\mathcal{F}(z)$ which is defined in Definition 1.1.

Theorem 2.1. *For $i = 1, 2, \ldots, n$, let $\gamma_i > 0$ and $f_i \in \Sigma\mathcal{K}_p(\alpha_i)$ $(0 \leqslant \alpha_i < p)$. If*

$$0 < \sum_{i=1}^n \gamma_i (p - \alpha_i) \leqslant p, \tag{7}$$

then the general integral operator $\mathcal{F}(z)$ defined in Definition 1.1 is meromorphic p-valent starlike of order

$$p - \sum_{i=1}^n \gamma_i (p - \alpha_i).$$

Proof. From (4), it is easy to see that

$$\frac{1}{c} z^{p+1} \mathcal{F}'(z) + \frac{p+c}{c} z^p \mathcal{F}(z) = \prod_{i=1}^n \left(-\frac{z^{p+1} f_i'(z)}{p}\right)^{\gamma_i}. \tag{8}$$

Differentiate the above equality with respect to z, we have

$$\frac{1}{c} z^{p+1} \mathcal{F}''(z) + \frac{2p+c+1}{c} z^p \mathcal{F}'(z) + \frac{p(p+c)}{c} z^{p-1} \mathcal{F}(z) = \sum_{i=1}^n \gamma_i \left(-\frac{z^{p+1} f_i'(z)}{p}\right)^{\gamma_i} \left(\frac{p+1}{z} + \frac{f_i''(z)}{f_i'(z)}\right) \prod_{j=1, j\neq i}^n \left(-\frac{z^{p+1} f_j'(z)}{p}\right)^{\gamma_j}. \tag{9}$$

From (8) and (9), we get

$$\frac{z^{p+1}\mathcal{F}''(z) + (2p+c+1) z^p \mathcal{F}'(z) + p(p+c) z^{p-1} \mathcal{F}(z)}{z^{p+1}\mathcal{F}'(z) + (p+c) z^p \mathcal{F}(z)} = \sum_{i=1}^n \left\{\gamma_i \left(\frac{p+1}{z} + \frac{f_i''(z)}{f_i'(z)}\right)\right\} \tag{10}$$

or equivalently

$$\frac{z^2\mathcal{F}''(z) + (2p+c+1) z \mathcal{F}'(z) + p(p+c) \mathcal{F}(z)}{z\mathcal{F}'(z) + (p+c) \mathcal{F}(z)} = \sum_{i=1}^n \left\{\gamma_i \left(p+1 + \frac{z f_i''(z)}{f_i'(z)}\right)\right\}. \tag{11}$$

After some calculations, we have

$$-\frac{z^2\mathcal{F}''(z) + (p+c+1) z \mathcal{F}'(z)}{z\mathcal{F}'(z) + (p+c) \mathcal{F}(z)} = p - \sum_{i=1}^n \left\{\gamma_i \left(p+1 + \frac{z f_i''(z)}{f_i'(z)}\right)\right\}. \tag{12}$$

We can write left-hand side of (12) as the following:

$$-\frac{\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}\left(\frac{z\mathcal{F}''(z)}{\mathcal{F}'(z)} + p + c + 1\right)}{\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)} + p + c} = p - \sum_{i=1}^n \left\{\gamma_i \left(p+1 + \frac{z f_i''(z)}{f_i'(z)}\right)\right\}. \tag{13}$$

Now we define a regular function $h(z)$ by

$$h(z) = -\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}, \tag{14}$$

and $h(0) = p$. Differentiating (14) logarithmically with respect to z, we obtain

$$-h(z) + \frac{zh'(z)}{h(z)} = 1 + \frac{z\mathcal{F}''(z)}{\mathcal{F}'(z)}. \tag{15}$$

From (13)–(15) , we have

$$\frac{h(z) + zh'(z)}{-h(z) + p + c = p - \sum_{i=1}^n \left\{\gamma_i \left(p+1 + \frac{z f_i''(z)}{f_i'(z)}\right)\right\}}. \tag{16}$$

Since $f_i \in \Sigma\mathcal{K}_p(\alpha_i) (0 \leqslant \alpha_i < p)$ for $i = 1, 2, \ldots, n$, we get

$$\Re\left\{h(z) + \frac{zh'(z)}{-h(z) + p + c}\right\} > p - \sum_{i=1}^n \gamma_i (p - \alpha_i). \tag{17}$$

It is clear that the conditions of Lemma 1.1 are satisfied. So we obtain

$$\Re\{h(z)\} > p - \sum_{i=1}^n \gamma_i (p - \alpha_i),$$

which is equivalent to

$$-\Re\left\{\frac{z\mathcal{F}'(z)}{\mathcal{F}(z)}\right\} > p - \sum_{i=1}^n \gamma_i (p - \alpha_i),$$

that is, $\mathcal{F}(z)$ is meromorphic p-valent starlike of order $p - \sum_{i=1}^{n}\gamma_i(p-\alpha_i)$.

3. Some consequences of main result

In this section, we will give some consequences of main theorem in the form of Corollaries.

Putting $c = 1$ in Theorem 2.1, we get

Corollary 3.1 12, Theorem 2.3. *For $i = 1, 2, \ldots, n$, let $\gamma_i > 0$ and $f_i \in \Sigma\mathcal{K}_p(\alpha_i)$ $(0 \leqslant \alpha_i < p)$. If*

$$0 < \sum_{i=1}^{n}\gamma_i(p-\alpha_i) \leqslant p,$$

then the general integral operator $\mathcal{J}_{p,\gamma_1,\ldots,\gamma_n}(z)$ defined in (5) is meromorphic p -valent starlike of order

$$p - \sum_{i=1}^{n}\gamma_i(p-\alpha_i).$$

If we set

$$p = 1 \quad \text{and} \quad \alpha_i = -\frac{1}{n\gamma_i} + 1 \quad (i = 1, 2, \ldots, n)$$

in Corollary 3.1, then we have [13, Theorem 2.1].

Taking $n = 1, \gamma_1 = \gamma$ and $f_1 = f$ in Theorem 2.1, we get

Corollary 3.2. *Let $\gamma > 0$ and $f \in \Sigma\mathcal{K}_p(\alpha)(0 \leqslant \alpha < p)$. If*

$$0 < \gamma(p-\alpha) \leqslant p,$$

then the general integral operator $\mathcal{I}^{c}_{p,\gamma}(f)$ defined in (6) is meromorphic p -valent starlike of order

$$p - \gamma(p-\alpha).$$

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