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Subclasses of bi-univalent functions defined by convolution

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Abstract In this paper, we introduced two new subclasses of the function class Σ of bi-univalent functions analytic in the open unit disc defined by convolution. Furthermore, we find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses.

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1. Introduction and definitions

Let $\mathcal{A}$ denote the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disc $\mathcal{U} = \{z : |z| < 1\}$. Further, by $\mathcal{S}$ we shall denote the class of all functions in $\mathcal{A}$ which are univalent in $\mathcal{U}$.

For $f(z)$ defined by (1.1) and $\Phi(z)$ defined by

$$\Phi(z) = z + \sum_{n=2}^{\infty} \phi_n z^n \quad (\phi_n \geqslant 0), \tag{1.2}$$

the Hadamard product $(f * \Phi)(z)$ of the functions $f(z)$ and $\Phi(z)$ defined by

$$(f * \Phi)(z) = z + \sum_{n=2}^{\infty} a_n \phi_n z^n = (\Phi * f)(z). \tag{1.3}$$

For $0 \leqslant \alpha < 1$ and $\lambda \geqslant 0$, we let $Q_\lambda(h, \alpha)$ be the subclass of $\mathcal{A}$ consisting of functions $f(z)$ of the form (1.1) and functions $h(z)$ given by

$$h(z) = z + \sum_{n=2}^{\infty} h_n z^n \quad (h_n > 0) \tag{1.4}$$

and satisfying the analytic criterion:

$$Q_\lambda(h,\alpha) = \left\{ f \in \mathcal{A} : \mathrm{Re}\left((1-\lambda)\frac{(f*h)(z)}{z} + \lambda (f*h)'(z) \right) > \alpha, \ \ 0 \leqslant \alpha < 1, \ \lambda \geqslant 0 \right\}. \tag{1.5}$$

It is easy to see that $Q_{\lambda_1}(h,\alpha) \subset Q_{\lambda_2}(h,\alpha)$ for $\lambda_1 > \lambda_2 \geqslant 0$. Thus, for $\lambda \geqslant 1$, $0 \leqslant \alpha < 1$, $Q_\lambda(h,\alpha) \subset Q_1(h,\alpha) = \{f, h \in \mathcal{A} : \mathrm{Re}(f*h)'(z) > \alpha, \ 0 \leqslant \alpha < 1\}$ and hence $Q_\lambda(h, \alpha)$ is univalent class (see [1–3]).

We note that $Q_\lambda\left(\frac{z}{1-z}, \alpha\right) = Q_\lambda(\alpha)$ (see Ding et al. [4]).

It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, defined by

$$f^{-1}(f(z)) = z \quad (z \in \mathcal{U})$$

and

$$f(f^{-1}(w)) = w \quad \left(|w| < r_0(f); \ r_0(f) \geqslant \frac{1}{4} \right),$$

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where

$$f^{-1}(w) = w - a_2w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots.$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathcal{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathcal{U}$.

Let Σ denote the class of bi-univalent functions in $\mathcal{U}$ given by (1.1). For a brief history and interesting examples in the class Σ, (see Srivastava et al. [5]).

Brannan and Taha [6] (see also [7]) introduced certain subclasses of the bi-univalent function class Σ similar to the familiar subclasses $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ of starlike and convex functions of order $\alpha(0 \leqslant \alpha < 1)$, respectively (see [8]). Thus, following Brannan and Taha [6] (see also [7]), a function $f \in \mathcal{A}$ is in the class $\mathcal{S}^*_\Sigma(\alpha)$ of strongly bi-starlike functions of order $\alpha(0 < \alpha \leqslant 1)$ if each of the following conditions is satisfied:

$$f \in \Sigma \quad \text{and} \quad \left|\arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leqslant 1;\ z \in \mathcal{U})$$

and

$$\left|\arg\left(\frac{zg'(w)}{g(w)}\right)\right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leqslant 1;\ w \in \mathcal{U}),$$

where g is the extension of f^{-1} to $\mathcal{U}$. The classes $\mathcal{S}^*_\Sigma(\alpha)$ and $\mathcal{K}_\Sigma(\alpha)$ of bi-starlike functions of order α and bi-convex functions of order α, corresponding (respectively) to the function classes $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$, were also introduced analogously. For each of the function classes $\mathcal{S}^*_\Sigma(\alpha)$ and $\mathcal{K}_\Sigma(\alpha)$, they found non-sharp estimates on the first two Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ (for details, see [6,7]).

The object of the present paper is to introduce two new subclasses of the function class Σ and find estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses of the function class Σ employing the techniques used earlier by Srivastava et al. [5].

In order to derive our main results, we have to recall here the following lemma.

Lemma 1 [9]. *Let $p \in \mathcal{P}$ the family of all functions p analytic in $\mathcal{U}$ for which $Re p(z) > 0$ and have the form $p(z) = 1 + p_1z + p_2z^2 + p_3z^3 + \cdots$ for $z \in \mathcal{U}$. Then $|p_n| \leqslant 2$, for each n.*

2. Coefficient bounds for the function class $\mathcal{B}(h, \alpha, \lambda)$

Definition 1. A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{B}_\Sigma(h, \alpha, \lambda)$ if the following conditions are satisfied:

$$f \in \Sigma \quad \text{and} \quad \left|\arg\left((1-\lambda)\frac{(f*h)(z)}{z} + \lambda(f*h)'(z)\right)\right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leqslant 1;\ \lambda \geqslant 1;\ z \in \mathcal{U}) \tag{2.1}$$

and

$$\left|\arg\left((1-\lambda)\frac{(f*h)^{-1}(w)}{w} + \lambda((f*h)^{-1})'(w)\right)\right| < \frac{\alpha\pi}{2} \quad (0 < \alpha \leqslant 1;\ \lambda \geqslant 1;\ w \in \mathcal{U}), \tag{2.2}$$

where the function $h(z)$ is given by (1.4) and $(f*h)^{-1}(w)$ is defined by:

$$(f*h)^{-1}(w) = w - a_2h_2w^2 + (2a_2^2h_2^2 - a_3h_3)w^3 - (5a_2^3h_2^3 - 5a_2h_2a_3h_3 + a_4h_4)w^4 + \cdots. \tag{2.3}$$

We note that for $\lambda = 1$ and $h(z) = \frac{z}{1-z}$, the class $\mathcal{B}_\Sigma(h, \alpha, \lambda)$ reduces to the class $\mathcal{H}^\alpha_\Sigma$ introduced and studied by Srivastava et al. [5]. Also for $h(z) = \frac{z}{1-z}$ the class $\mathcal{B}_\Sigma(h, \alpha, \lambda)$ reduces to the class $\mathcal{B}_\Sigma(\alpha, \lambda)$ introduced and studied by Frasin and Aouf [10].

We begin by finding the estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in the class $\mathcal{B}_\Sigma(h, \alpha, \lambda)$.

Theorem 1. *Let $f(z)$ given by (1.1) be in the class $\mathcal{B}_\Sigma(h, \alpha, \lambda), 0 < \alpha \leqslant 1$ and $\lambda \geqslant 1$. Then*

$$|a_2| \leqslant \frac{2\alpha}{h_2\sqrt{(\lambda+1)^2 + \alpha(1 + 2\lambda - \lambda^2)}} \tag{2.4}$$

and

$$|a_3| \leqslant \frac{1}{h_3}\left(\frac{4\alpha^2}{(\lambda+1)^2} + \frac{2\alpha}{(2\lambda+1)}\right). \tag{2.5}$$

Proof. It follows from (2.1) and (2.2) that

$$(1-\lambda)\frac{(f*h)(z)}{z} + \lambda(f*h)'(z) = [p(z)]^\alpha \tag{2.6}$$

and

$$(1-\lambda)\frac{(f*h)^{-1}(w)}{w} + \lambda((f*h)^{-1})'(w) = [q(w)]^\alpha, \tag{2.7}$$

where $p(z)$ and $q(w) \in \mathcal{P}$ and have the forms

$$p(z) = 1 + p_1z + p_2z^2 + p_3z^3 + \cdots \tag{2.8}$$

and

$$q(w) = 1 + q_1w + q_2w^2 + q_3w^3 + \cdots. \tag{2.9}$$

Now, equating the coefficients in (2.6) and (2.7), we get

$$(\lambda+1)a_2h_2 = \alpha p_1, \tag{2.10}$$

$$(2\lambda+1)a_3h_3 = \alpha p_2 + \frac{\alpha(\alpha-1)}{2}p_1^2, \tag{2.11}$$

$$-(\lambda+1)a_2h_2 = \alpha q_1 \tag{2.12}$$

and

$$(2\lambda+1)(2a_2^2h_2^2 - a_3h_3) = \alpha q_2 + \frac{\alpha(\alpha-1)}{2}q_1^2. \tag{2.13}$$

From (2.10) and (2.12), we get

$$p_1 = -q_1 \tag{2.14}$$

and

$$2(\lambda+1)^2a_2^2h_2^2 = \alpha^2(p_1^2 + q_1^2). \tag{2.15}$$

Now from (2.11), (2.13) and (2.15), we obtain

$$2(2\lambda+1)a_2^2h_2^2 = \alpha(p_2+q_2) + \frac{\alpha(\alpha-1)}{2}(p_1^2+q_1^2)$$
$$= \alpha(p_2+q_2) + \frac{\alpha(\alpha-1)}{2}\,\frac{2(\lambda+1)^2a_2^2h_2^2}{\alpha^2}.$$

Therefore, we have

$$a_2^2 = \frac{\alpha^2(p_2+q_2)}{2\alpha(2\lambda+1)h_2^2-(\alpha-1)(\lambda+1)^2h_2^2}.$$

Applying Lemma 1 for the coefficients p_2 and q_2, we immediately have

$$|a_2| \leqslant \frac{2\alpha}{h_2\sqrt{(\lambda+1)^2+\alpha(1+2\lambda-\lambda^2)}}.$$

This gives the bound on $|a_2|$ as asserted in (2.4).

Next, in order to find the bound on $|a_3|$, by subtracting (2.13) from (2.4), we get

$$2(2\lambda+1)a_3h_3 - 2(2\lambda+1)a_2^2h_2^2$$
$$= \alpha p_2 + \frac{\alpha(\alpha-1)}{2}p_1^2 - \left(\alpha q_2 + \frac{\alpha(\alpha-1)}{2}q_1^2\right). \tag{2.16}$$

It follows from (2.14), (2.15) and (2.16) that

$$2(2\lambda+1)a_3h_3 = \frac{\alpha^2(2\lambda+1)(p_1^2+q_1^2)}{(\lambda+1)^2} + \alpha(p_2-q_2)$$

or, equivalently,

$$a_3 = \frac{\alpha^2(p_1^2+q_1^2)}{2(\lambda+1)^2h_3} + \frac{\alpha(p_2-q_2)}{2(2\lambda+1)h_3}$$

applying Lemma 1 once again for the coefficients p_1, p_2, q_1 and q_2, we readily get

$$|a_3| \leqslant \frac{1}{h_3}\left(\frac{4\alpha^2}{(\lambda+1)^2} + \frac{2\alpha}{(2\lambda+1)}\right).$$

This completes the proof of Theorem 1. □

Remark 1

(i) Putting $\lambda = 1$ and $h(z) = \frac{z}{1-z}$ in Theorem 1, we obtain the results obtained by Srivastava et al. [5, Theorem 1].
(ii) Putting $h(z) = \frac{z}{1-z}$ in Theorem 1, we obtain the results obtained by Frasin and Aouf [10, Theorem 2.2].

Example 1

(i) For

$$h(z) = z + \sum_{n=2}^{\infty}\left[\frac{1+\ell+\gamma(n-1)}{1+\ell}\right]^m z^n \quad (\ell,\gamma \geqslant 0;\ m \in \mathbb{N}_0 = \mathbb{N}\cup\{0\}), \tag{2.17}$$

this operator contains in turn many interesting operators (see Catas et al. [11]), Theorem 1, becomes

$$|a_2| \leqslant \frac{2\alpha}{\left[\frac{1+\ell+\gamma}{1+\ell}\right]^m\sqrt{(\lambda+1)^2+\alpha(1+2\lambda-\lambda^2)}}$$

and

$$|a_3| \leqslant \frac{1}{\left[\frac{1+\ell+2\gamma}{1+\ell}\right]^m}\left(\frac{4\alpha^2}{(\lambda+1)^2} + \frac{2\alpha}{(2\lambda+1)}\right).$$

(ii) For

$$h(z) = z + \sum_{n=2}^{\infty}\Gamma_{n-1}(\alpha_1)z^n, \tag{2.18}$$

where

$$\Gamma_{n-1}(\alpha_1) = \frac{(\alpha_1)_{n-1}\cdots(\alpha_q)_{n-1}}{(\beta_1)_{n-1}\cdots(\beta_s)_{n-1}(1)_{n-1}} \quad (n \geqslant 2), \tag{2.19}$$

$q \leqslant s+1$, $\alpha_i \in \mathbb{C}(i = 1,2,\ldots,q)$ and $\beta_1 \in \mathbb{C}\setminus\mathbb{Z}_0^-(j = 1,2,\ldots,s)$, this operator contains in turn many interesting operators (see Dziok and Srivastav [12]), Theorem 1 becomes

$$|a_2| \leqslant \frac{2\alpha}{|\Gamma_1(\alpha_1)|\sqrt{(\lambda+1)^2+\alpha(1+2\lambda-\lambda^2)}}$$

and

$$|a_3| \leqslant \frac{1}{|\Gamma_2(\alpha_1)|}\left(\frac{4\alpha^2}{(\lambda+1)^2} + \frac{2\alpha}{(2\lambda+1)}\right).$$

3. Coefficient bounds for the function class $\mathcal{B}(\boldsymbol{h}, \boldsymbol{\beta}, \boldsymbol{\lambda})$

Definition 2. A function $f(z)$ given by (1.1) is said to be in the class $\mathcal{B}_\Sigma(h,\beta,\lambda)$ if the following conditions are satisfied:

$$f \in \Sigma \quad \text{and} \quad \mathrm{Re}\left((1-\lambda)\frac{(f*h)(z)}{z} + \lambda(f*h)'(z)\right) > \beta$$
$$(0 \leqslant \beta < 1;\ \lambda \geqslant 1;\ z \in \mathcal{U}) \tag{3.1}$$

and

$$\mathrm{Re}\left((1-\lambda)\frac{(f*h)^{-1}(w)}{w} + \lambda((f*h)^{-1})'(w)\right) > \beta \quad (0$$
$$\leqslant \beta < 1;\ \lambda \geqslant 1;\ w \in \mathcal{U}), \tag{3.2}$$

where the functions $h(z)$ and $(f*h)^{-1}(w)$ are defined by (1.4) and (2.3) respectively.

We note that for $\lambda = 1$ and $h(z) = \frac{z}{1-z}$, the class $\mathcal{B}_\Sigma(h,\beta,\lambda)$ reduces to the class $\mathcal{H}_\Sigma(\beta)$ introduced and studied by Srivastava et al. [5]. Also for $h(z) = \frac{z}{1-z}$, the class $\mathcal{B}_\Sigma(h,\beta,\lambda)$ reduces to the class $\mathcal{B}_\Sigma(\beta,\lambda)$ introduced and studied by Frasin and Aouf [10].

Theorem 2. *Let $f(z)$ given by (1.1) be in the class $\mathcal{B}_\Sigma(h,\beta,\lambda), 0 \leqslant \beta < 1$ and $\lambda \geqslant 1$. Then*

$$|a_2| \leqslant \frac{1}{h_2}\sqrt{\frac{2(1-\beta)}{(2\lambda+1)}} \tag{3.3}$$

and

$$|a_3| \leqslant \frac{1}{h_3}\left(\frac{4(1-\beta)^2}{(\lambda+1)^2} + \frac{2(1-\beta)}{(2\lambda+1)}\right). \tag{3.4}$$

Proof. It follows from (3.1) and (3.2) that there exist $p(z)$ and $q(z) \in \mathcal{P}$ such that

$$(1-\lambda)\frac{(f*h)(z)}{z} + \lambda(f*h)'(z) = \beta + (1-\beta)p(z) \tag{3.5}$$

and

$$(1-\lambda)\frac{(f*h)^{-1}(w)}{w} + \lambda((f*h)^{-1})'(w) = \beta + (1-\beta)q(w), \tag{3.6}$$

where $p(z)$ and $q(w)$ have the forms (2.8) and (2.9), respectively. Equating coefficients in (3.5) and (3.6) yields

$$(\lambda+1)a_2h_2 = (1-\beta)p_1, \tag{3.7}$$
$$(2\lambda+1)a_3h_3 = (1-\beta)p_2, \tag{3.8}$$
$$-(\lambda+1)a_2h_2 = (1-\beta)q_1 \tag{3.9}$$

and

$$(2\lambda+1)(2a_2^2h_2^2 - a_3h_3) = (1-\beta)q_2. \tag{3.10}$$

From (3.7) and (3.9), we get

$$p_1 = -q_1 \tag{3.11}$$

and

$$2(\lambda+1)^2a_2^2h_2^2 = (1-\beta)^2(p_1^2+q_1^2). \tag{3.12}$$

Also, from (3.8) and (3.10), we find that

$$2(2\lambda+1)a_2^2h_2^2 = (1-\beta)(p_2+q_2).$$

Thus, we have

$$|a_2^2| \leqslant \frac{(1-\beta)}{2(2\lambda+1)h_2^2}(|p_2|+|q_2|) = \frac{2(1-\beta)}{(2\lambda+1)h_2^2},$$

which is the bound on $|a_2|$ as given in (3.3).

Next, in order to find the bound on $|a_3|$, by subtracting (3.8) form (3.10), we get

$$2(2\lambda+1)a_3h_3 - 2(2\lambda+1)a_2^2h_2^2 = (1-\beta)(p_2-q_2)$$

or, equivalently,

$$a_3 = \frac{1}{h_3}\left(a_2^2h_2^2 + \frac{(1-\beta)(p_2-q_2)}{2(2\lambda+1)}\right).$$

Upon substituting the value of a_2^2 from (3.12), we obtain

$$a_3 = \frac{1}{h_3}\left(\frac{(1-\beta)^2(p_1^2+q_1^2)}{2(\lambda+1)^2} + \frac{(1-\beta)(p_2-q_2)}{2(2\lambda+1)}\right).$$

Applying Lemma 1 for the coefficients p_1, p_2, q_1 and q_2, we readily get

$$|a_3| \leqslant \frac{1}{h_3}\left(\frac{4(1-\beta)^2}{(\lambda+1)^2} + \frac{2(1-\beta)}{(2\lambda+1)}\right),$$

which is the bound on $|a_3|$ as asserted in (3.4). □

Remark 2

(i) Putting $\lambda = 1$ and $h(z) = \frac{z}{1-z}$ in Theorem 1, we obtain the results obtained by Srivastava et al. [5, Theorem 2].

(ii) Putting $h(z) = \frac{z}{1-z}$ in Theorem 1, we obtain the results obtained by Frasin and Aouf [10, Theorem 3.2].

Example 2

(i) For $h(z)$ given by (2.17), Theorem 2, becomes

$$|a_2| \leqslant \frac{1}{\left[\frac{1+\ell+\gamma}{1+\ell}\right]^m}\sqrt{\frac{2(1-\beta)}{(2\lambda+1)}}$$

and

$$|a_3| \leqslant \frac{1}{\left[\frac{1+\ell+2\gamma}{1+\ell}\right]^m}\left(\frac{4(1-\beta)^2}{(\lambda+1)^2} + \frac{2(1-\beta)}{(2\lambda+1)}\right).$$

(ii) For $h(z)$ given by (2.18), Theorem 2, becomes

$$|a_2| \leqslant \frac{1}{|\Gamma_1(\alpha_1)|}\sqrt{\frac{2(1-\beta)}{(2\lambda+1)}}$$

and

$$|a_3| \leqslant \frac{1}{|\Gamma_2(\alpha_1)|}\left(\frac{4(1-\beta)^2}{(\lambda+1)^2} + \frac{2(1-\beta)}{(2\lambda+1)}\right).$$

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