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ORIGINAL ARTICLE

# Continuous and integrable solutions of a nonlinear Cauchy problem of fractional order with nonlocal conditions

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**Abstract** In this article, we discuss the existence of at least one solution as well as uniqueness for a nonlinear fractional differential equation with weighted initial data and nonlocal conditions. The existence of at least one $L_1$ and continuous solution will be proved under the Caratheòdory conditions via a classical fixed point theorem of Schauder. An example is also given to illustrate the efficiency of the main theorems.

**MATHEMATICS SUBJECT CLASSIFICATION:** 26A33; 34K37; 34A08; 45E10; 47H10



## 1. Introduction

Fractional differential equations have gained considerable importance due to their application in various sciences, such as physics, mechanics, chemistry, engineering, etc. In the recent years, there has been a significant development in ordinary and partial differential equations involving fractional derivatives, see the monographs of Kilbas et al. [1], Miller and Ross [2], Podlubny [3], and the papers [4–16] and the references therein.

Let $I = (0, T], L_1 = L_1(0, T]$ be the space of Lebesgue integrable functions on $I$. and $C(0, T]$ be the space of continuous functions defined on $I$.

Consider the weighted nonlocal Cauchy type fractional problem

$$D^\alpha(p(t)u(t)) = f(t, u(t)) \quad a.e.\ t \in (0, T], T < \infty \tag{1}$$

$$\lim_{t\to 0^+} t^{1-\alpha} p(t)u(t) = \sum_{j=1}^{m} a_j u(\tau_j), \quad \tau_j \in (0, T). \tag{2}$$

where $D^\alpha$ denoted the Riemann–Liouville derivative of order $\alpha \in (0, 1]$.

Problems with non-local conditions have been extensively studied by several authors in the last two decades. The reader is referred to [7–9,17–19] and references therein.

Nonlinear fractional differential equation with weighted initial data has been carried out by various researchers. In present, there are some papers which deal with the existence and multiplicity of solutions for weighted nonlinear fractional differential equations.

In [12] Khaled et al. studied the weighted Cauchy-type problem

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$$(I)\begin{cases} D^{\alpha}u(t) = f(t,u), & t \in (0,T] \\ t^{1-\alpha}u(t)|_{t=0} = b, \end{cases}$$

where $D^{\alpha}$ is the fractional derivative (in the sense of Riemann–Liouville) of order $0 < \alpha < 1$, $f$ is a continuous nonlinear function.

In [10] Furati et al. studied the weighted Cauchy-type problem $(I)$ where $f(t,u)$ is assumed to be continuous on $R^+ \times R$ and $|f(t,u)| \leqslant t^{\mu}e^{-\sigma t}\psi(t)|u|^m$.

Also in [5] El-Sayed et al. studied the problem $(I)$ where the function $f$ satisfies Carathèodory conditions with growth condition. In [19] the existence and uniqueness of the solution of the problem $(I)$ was discussed by using the method of upper and lower solutions and its associated monotone iterative.

In [16] Weia et al. studied the existence and uniqueness of the solution of the periodic boundary value problem for a fractional differential equation involving a Riemann–Liouville fractional derivative

$$\begin{cases} D^{\alpha}u(t) = f(t,u), & t \in (0,T] \\ t^{1-\alpha}u(t)|_{t=0} = t^{1-\alpha}u(t)|_{t=T}. \end{cases}$$

by using the monotone iterative method. In [11] Jankowski discussed the existence of solutions of fractional equations of Volterra type with the Riemann–Liouville derivative,

$$\begin{cases} D^{\alpha}x(t) = f(t,x(t),\int_0^t k(t,s)x(s)ds), & t \in (0,T] \\ t^{1-\alpha}u(t)|_{t=0} = r, \end{cases}$$

existence results are obtained by using a Banach fixed point theorem with weighted norms and by a monotone iterative method.

In [4] Belmekki et al. studied the existence and uniqueness of the solution for a class of fractional differential equations

$$\begin{cases} D^{\alpha}u(t) - \lambda u(t) = f(t,u(t)), & t \in (0,1] \\ \lim_{t\to 0^+} t^{1-\alpha}u(t) = u(1) \end{cases}$$

by using the fixed point theorem of Schaeffer and the Banach contraction principle.

In this paper we will study the existence of solutions for problem (1) and (2) with certain nonlinearities, using the equivalence of the fractional differ-integral problem with the corresponding Volterra integral equation. We prove the existence of at least $L_1$ and continuous solutions of the problem (1) and (2) such that the function $f$ satisfies Carathèodory conditions and

$$|f(t,u)| \leqslant h(t), \quad \text{a.e } t \in (0,T]$$

where $h(t)$ is a Lebesgue function on $(0,T]$. Also the uniqueness of the solution will be studied.

Our problem (1) and (2) includes as a special case when $p(t) = 1$, the nonlocal fractional differential equation

$$D^{\alpha}u(t) = f(t,u(t)) \quad a.e.\ t \in (0,T], T < \infty$$

$$\lim_{t\to 0^+} t^{1-\alpha}\ u(t) = \sum_{j=1}^{m} a_j u(\tau_j), \quad \tau_j \in (0,T).$$

## 2. Preliminaries

In this section, we present some definitions, lemmas and notation which will be used in our theorems.

**Definition 2.1** (*see [2,3,13,14]*). The Riemann–Liouville fractional integral of order $\alpha > 0$ of a Lebesgue-measurable function $f : R^+ \to R$ is defined by

$$I_a^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1}f(s)ds,$$

when $a = 0$ we write $I_a^{\alpha}f(t) = I^{\alpha}f(t)$.

And we have, for $\alpha, \beta \in R^+$,

$(r_1)$ $I_a^{\alpha} : L_1 \longrightarrow L_1$,
$(r_2)$ $f(t) \in L_1, \quad I_a^{\alpha}\, I_a^{\beta}f(t) = I_a^{\alpha+\beta}f(t)$.

**Definition 2.2** (*see [2,3,13,14]*). The Riemann–Liouville fractional derivative of order $\alpha \in (0,1]$ of a Lebesgue-measurable function $f : R^+ \to R$ is defined by

$$D^{\alpha}f(t) = \frac{d}{dt}I^{1-\alpha}f(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t (t-s)^{-\alpha}f(s)ds.$$

**Theorem 2.1** (*Schauder fixed point Theorem*). *Let $S$ be a nonempty, closed , convex and bounded subset of the Banach space $X$ and let $Q : S \to S$ be a continuous and compact operator. Then the operator equation $Qu = u$ has at least one fixed-point in $S$.*

**Theorem 2.2** (*Kolmogorov compactness criterion [20]*). *Let $\Omega \subseteq L^p(0,T), 1 \leqslant p < \infty$. If*

(i) *$\Omega$ is bounded in $L^p(0,T)$ and*
(ii) *$u_h \to u$ as $h \to 0$ uniformly with respect to $u \in \Omega$, then $\Omega$ is relatively compact in $L^p(0,T)$ where*

$$u_h(t) = \frac{1}{h}\int_t^{t+h} u(s)ds.$$

**Definition 2.3.** A function $f : I \times R \to R$ is called Carathéodory function if:

(i) $t \to f(t,u)$ is measurable for all $u \in R$, and,
(ii) $u \to f(t,u)$ is continuous for all $t \in I$.
(iii) There exists a Lebesgue function $h(t)$ on $I$, and

## 3. Integral equation representation

We investigate in our paper the Cauchy problem for the nonlinear fractional differential equation with the nonlocal condition with the following assumptions.

$(h_1)$ The function $f : (0,T] \times R \to R$ is Carathéodory function.
$(h_2)$ $p(t) > 0$ for all $t \in I$ and is continuous with $\inf_{(0,T]}|p(t)| = p$.
$(h_3)$ $\sum_{j=1}^{m} \frac{a_j}{p(\tau_j)\tau_j^{1-\alpha}} \neq 1$.

**Lemma 3.1.** *The solution of the nonlocal problem (1) and (2) can be expressed by the fractional-order integral equation*

$$u(t) = \frac{At^{\alpha-1}}{p(t)} \sum_{j=1}^{m} \frac{a_j}{p(\tau_j)} \int_0^{\tau_j} \frac{(\tau_j - s)^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s))ds + \frac{1}{p(t)} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s))ds \quad (3)$$

*where* $A = \left(1 - \sum_{j=1}^{m} \frac{a_j}{p(\tau_j)\tau_j^{1-\alpha}}\right)^{-1}$

**Proof.** From the properties of Riemann–Liouville fractional derivative, Eq. (1) can be written as,

$$\frac{d}{dt} I^{1-\alpha} p(t)u(t) = f(t, u(t)),$$

integrating from 0 to $t$ both sides, we get

$$I^{1-\alpha}p(t)u(t) - I^{1-\alpha}p(t)u(t)|_{t=0} = \int_0^t f(s, u(s))ds,$$

$$I^{1-\alpha}p(t)u(t) - C = \int_0^t f(s, u(s))ds,$$

operating by $I^\alpha$ on both sides, we have

$$I^\alpha I^{1-\alpha} p(t)u(t) - I^\alpha C = I^{\alpha+1} f(t, u(t))$$

$$Ip(t)u(t) - \frac{Ct^\alpha}{\Gamma(\alpha+1)} = I^{\alpha+1} f(t, u(t)),$$

differentiate both sides, then

$$p(t)u(t) - \frac{Ct^{\alpha-1}}{\Gamma(\alpha)} = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha-1)} f(s, u(s))ds,$$
$$t^{1-\alpha}p(t)u(t) = \frac{C}{\Gamma(\alpha)} + t^{1-\alpha} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha-1)} f(s, u(s))ds, \quad (4)$$

and from (2) we have,

$$\lim_{t\to 0^+} t^{1-\alpha}p(t)u(t) = \frac{C}{\Gamma(\alpha)} = \sum_{j=1}^{m} a_j u(\tau_j). \quad (5)$$

Now from (4), putting $t = \tau_j$, we obtain

$$u(\tau_j) = \frac{C\tau_j^{\alpha-1}}{p(\tau_j)\Gamma(\alpha)} + \frac{1}{p(\tau_j)} \int_0^{\tau_j} \frac{(\tau_j - s)^{\alpha-1}}{\Gamma(\alpha-1)} f(s, u(s))ds,$$

$$\sum_{j=1}^{m} a_j u(\tau_j) = \sum_{j=1}^{m} \frac{a_j C}{p(\tau_j)\tau_j^{1-\alpha}\Gamma(\alpha)} + \sum_{j=1}^{m} \frac{a_j}{p(\tau_j)} \int_0^{\tau_j} \frac{(\tau_j - s)^{\alpha-1}}{\Gamma(\alpha-1)} f(s, u(s))ds. \quad (6)$$

From (5),we have

$$\frac{C}{\Gamma(\alpha)} = \sum_{j=1}^{m} \frac{a_j C}{p(\tau_j)\tau_j^{1-\alpha}\Gamma(\alpha)} + \sum_{j=1}^{m} \frac{a_j}{p(\tau_j)} \int_0^{\tau_j} \frac{(\tau_j - s)^{\alpha-1}}{\Gamma(\alpha-1)} f(s, u(s))ds,$$

$$\left[1 - \sum_{j=1}^{m} \frac{a_j}{p(\tau_j)\tau_j^{1-\alpha}}\right] \frac{C}{\Gamma(\alpha)} = \sum_{j=1}^{m} \frac{a_j}{p(\tau_j)} \int_0^{\tau_j} \frac{(\tau_j - s)^{\alpha-1}}{\Gamma(\alpha-1)} f(s, u(s))ds.$$

$$\frac{C}{\Gamma(\alpha)} = A \sum_{j=1}^{m} \frac{a_j}{p(\tau_j)} \int_0^{\tau_j} \frac{(\tau_j - s)^{\alpha-1}}{\Gamma(\alpha-1)} f(s, u(s))ds. \quad \square$$

Substituting in (4) we get (3).

Now we want to prove that if $u(t)$ satisfied (3), then $\lim_{t\to 0^+} t^{1-\alpha}p(t)u(t) = \sum_{j=1}^{m} a_j u(\tau_j)$, from (3) we have,

$$\lim_{t\to 0^+} t^{1-\alpha}p(t)u(t) = A \sum_{j=1}^{m} \frac{a_j}{p(\tau_j)} \int_0^{\tau_j} \frac{(\tau_j - s)^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s))ds + \lim_{t\to 0^+} t^{1-\alpha} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s))ds$$
$$= A \sum_{j=1}^{m} \frac{a_j}{p(\tau_j)} \int_0^{\tau_j} \frac{(\tau_j - s)^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s))ds$$

Also from (3),

$$\sum_{j=1}^{m} a_j u(\tau_j) = \left(A \sum_{j=1}^{m} \frac{a_j}{\tau_j^{1-\alpha} p(\tau_j)} + 1\right) \times \left(\sum_{j=1}^{m} \frac{a_j}{p(\tau_j)} \int_0^{\tau_j} \frac{(\tau_j - s)^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s))ds\right)$$
$$= \left(\frac{1}{1 - \sum_{j=1}^{m} \frac{a_j}{\tau_j^{1-\alpha} p(\tau_j)}} \sum_{j=1}^{m} \frac{a_j}{\tau_j^{1-\alpha} p(\tau_j)} + 1\right) \times \left(\sum_{j=1}^{m} \frac{a_j}{p(\tau_j)} \int_0^{\tau_j} \frac{(\tau_j - s)^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s))ds\right)$$
$$= A \sum_{j=1}^{m} \frac{a_j}{p(\tau_j)} \int_0^{\tau_j} \frac{(\tau_j - s)^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s))ds.$$

Then the integral Eq. (3) is equivalent to the nonlocal problem (1) and (2).

## 4. Existence of $L_1$ solutions

Here we study the existence of at least one $L_1$ solution of the nonlocal Cauchy problem (1) and (2).

**Definition 4.1.** By a solutions of the nonlocal Cauchy problem (1) and (2) we mean a functions $u \in L_1(0, T]$ on the interval $(0, T]$ and this functions satisfies (1) and (2).

**Theorem 4.1.** *Assume that the hypothesis* $(h_1) - (h_3)$ *holds, then the nonlocal problem* (1) and (2) *has at least one* $L_1$ *solution.*

**Proof.** Let $T$ be an operator defined by:

$$(Tu)(t) = \frac{At^{\alpha-1}}{p(t)} \sum_{j=1}^{m} \frac{a_j}{p(\tau_j)} \int_0^{\tau_j} \frac{(\tau_j - s)^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s))ds + \frac{1}{p(t)} \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s))ds. \quad (7)$$

Then from Definition 2.1 we can write it as

$$(Tu)(t) = \frac{At^{\alpha-1}}{p(t)} \sum_{j=1}^{m} \frac{a_j}{p(\tau_j)} I^\alpha f(\tau_j, u(\tau_j)) + \frac{1}{p(t)} I^\alpha f(t, u(t)).$$

Let $\beta < \alpha$, then we can write

$$(Tu)(t) = \frac{At^{\alpha-1}}{p(t)} \sum_{j=1}^{m} \frac{a_j}{p(\tau_j)} I^{\alpha-\beta} I^\beta f(\tau_j, u(\tau_j)) + \frac{1}{p(t)} I^{\alpha-\beta} I^\beta f(t, u(t)) \quad (8)$$

and

$$|(Tu)(t)| \leqslant \frac{|A|t^{\alpha-1}}{|p(t)|}\sum_{j=1}^{m}\frac{|a_j|}{|p(\tau_j)|}I^{\alpha-\beta}I^{\beta}|f(\tau_j,u(\tau_j))| + \frac{1}{|p(t)|}I^{\alpha-\beta}I^{\beta}|f(t,u(t))|$$

and from assumption $(h_1)$, we have

$$|(Tu)(t)| \leqslant \frac{|A|t^{\alpha-1}}{|p(t)|}\sum_{j=1}^{m}\frac{|a_j|}{|p(\tau_j)|}I^{\alpha-\beta}I^{\beta}h(\tau_j) + \frac{1}{|p(t)|}I^{\alpha-\beta}I^{\beta}h(t)$$

Let $M = \max_I\{I^{\beta}h(t)\}$.

Then from assumption $(h_2)$ we get

$$\begin{aligned}|(Tu)(t)| &\leqslant \frac{|A|t^{\alpha-1}}{\inf|p(t)|}\sum_{j=1}^{m}\frac{|a_j|M}{\inf|p(t)|}\int_0^{\tau_j}\frac{(\tau_j-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}ds + \frac{M}{\inf|p(t)|}\int_0^t\frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}ds\\ &\leqslant \frac{|A|t^{\alpha-1}}{p}\sum_{j=1}^{m}\frac{|a_j|M}{p}\frac{\tau_j^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{M}{p}\frac{t^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\\ &\leqslant \frac{|A|t^{\alpha-1}}{p^2}\sum_{j=1}^{m}\frac{|a_j|MT^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{M}{p}\frac{T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\end{aligned}$$

and 
$$\begin{aligned}\|Tu\|_{L_1} &= \int_0^T|Tu(t)|dt\\ &\leqslant \frac{|A|}{p^2}\sum_{j=1}^{m}\frac{|a_j|MT^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\int_0^T t^{\alpha-1}dt + \frac{M}{p}\frac{T^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)}\int_0^T dt\\ &\leqslant \frac{|A|}{p^2}\sum_{j=1}^{m}\frac{|a_j|MT^{2\alpha-\beta}}{\alpha\Gamma(\alpha-\beta+1)} + \frac{M}{p}\frac{T^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+1)},\end{aligned}$$

The last estimate shows that the operator $T$ maps $L_1$ into itself.

Let

$$r = \frac{|A|}{p^2}\sum_{j=1}^{m}\frac{|a_j|MT^{2\alpha-\beta}}{\alpha\Gamma(\alpha-\beta+1)} + \frac{M}{p}\frac{T^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+1)}, \tag{9}$$

define the subset $B_r \subset L_1(I)$ by $B_r = \{u(t), t \in I : \|u\|_{L_1} \leqslant r\}$, the set $B_r$ is nonempty, closed and convex.

Now let $u \in \partial B_r$, that is $\|u\| = r$, then $T(\partial B_r) \subset \overline{B_r}$(closureof$B_r$) if

$$\|Tu\|_{L_1} \leqslant \frac{|A|}{p^2}\sum_{j=1}^{m}\frac{|a_j|MT^{2\alpha-\beta}}{\alpha\Gamma(\alpha-\beta+1)} + \frac{M}{p}\frac{T^{\alpha-\beta+1}}{\Gamma(\alpha-\beta+1)} = r$$

and $\|Tu\|_{L_1} \leqslant r$, where $r$ is given by (9). Moreover,

$$\|f\| = \int_0^T|f(s,u(s))|\ ds \leqslant \int_0^T h(s)\ ds = \|h\|_{L_1},$$

thus $f$ is in $L_1(0,T]$. Further, $f$ is continuous in $u$ (assumption $(ii)$) and $I^{\alpha}$ maps $L_1(0,T]$ continuously into itself, then $I^{\alpha}f(t,u(t))$ is continuous in $u \in B_r$, and we have $T$ maps $B_r$ continuously into $L_1(0,T]$.

To prove that $T$ is compact, we apply Theorem 2.2. So let $\Omega$ be a bounded set of $B_r$. Then $T(\Omega)$ is bounded in $L_1(0,T]$, i.e. condition $(i)$ of Theorem 2.2 is satisfied. To prove that $(Tu)_h \to Tu$ in $L_1(0,T]$ as $h \to 0$ uniformly with respect to $u \in \Omega$, we have from (7),

$$\begin{aligned}\|(Tu)_h - Tu\|_{L_1} &= \int_0^T|(Tu)_h(t) - (Tu)(t)|dt\\ &= \int_0^T\left|\frac{1}{h}\int_t^{t+h}(Tu)(s)ds - (Tu)(t)\right|dt\\ &\leqslant \int_0^T\frac{1}{h}\int_t^{t+h}|(Tu)(s) - (Tu)(t)|dsdt\\ &\leqslant \sum_{j=1}^{m}\frac{|A||a_j|}{|p(\tau_j)|}\left(\int_0^{\tau_j}\frac{(\tau_j-y)^{\alpha-1}}{\Gamma(\alpha)}|f(y,u(y))|dy\right)\\ &\quad\times\int_0^T\frac{1}{h}\int_t^{t+h}\left|\frac{s^{\alpha-1}}{p(s)} - \frac{t^{\alpha-1}}{p(t)}\right|ds\,dt + \int_0^T\frac{1}{h}\\ &\quad\times\int_t^{t+h}\left|\frac{1}{p(s)}I^{\alpha}f(s,u(s)) - \frac{1}{p(t)}I^{\alpha}f(t,u(t))\right|ds\,dt.\end{aligned}$$

Since $f \in L_1(0,T]$, then $I^{\alpha}f(.) \in L_1(0,T]$ and $\frac{1}{p(t)}I^{\alpha}f(t,u(t)) \in L_1(0,T]$. Moreover $\frac{t^{\alpha-1}}{p(t)} \in L_1(0,T]$, so we have, (see [21])

$$\frac{1}{h}\int_t^{t+h}\left|\frac{s^{\alpha-1}}{p(s)} - \frac{t^{\alpha-1}}{p(t)}\right|ds \to 0$$

$$\frac{1}{h}\int_t^{t+h}\left|\frac{1}{p(s)}I^{\alpha}f(s,u(s)) - \frac{1}{p(t)}I^{\alpha}f(t,u(t))\right|ds \to 0$$

for a.e. $t \in (0,T]$. Therefore by Theorem 2.2, we have $T(\Omega)$ is relatively compact, that is $T$ is compact operator. Now applying Theorem 2.1, then $T$ has a fixed point.

## 5. Existence of continuous solutions

Here we study the existence of unique and at least one continuous solution of the nonlocal Cauchy problem (1) and (2).

**Definition 5.1.** By a solutions of the nonlocal Cauchy problem (1) and (2) we mean a functions $\{u : t^{1-\alpha}u(t)$ is continuous on the interval $(0,T]\}$ and this functions satisfies (1) and (2).

Let

$$C(0,T] = \{u : u(t) \text{ is continuous on } (0,T] : \|u\|_C = \max_{t\in(0,T]}|u(t)|\}$$

$$C_{1-\alpha}(0,T] = \left\{u : t^{1-\alpha}u(t) \text{ is continuous on } (0,T] \text{ with the weighted norm } \|u\|_{C_{1-\alpha}} = \|t^{1-\alpha}u(t)\|_C\right\}$$

**Theorem 5.1.** *Assume that the hypothesis $(h_1)-(h_3)$ holds. Then the nonlocal problem (1) and (2) has at least one solution $u \in C_{1-\alpha}(0,T]$.*

**Proof.** Define the subset $Q_r \subset C_{1-\alpha}(0,T]$ by

$$Q_r = \{u(t) \in C_{1-\alpha}(0,T] : \|u(t)\|_{C_{1-\alpha}(0,T]} \leq r\}$$

where

$$r = \frac{|A|}{p^2}\sum_{j=1}^{m}\frac{|a_j|MT^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{M}{p}\frac{T^{1-\beta}}{\Gamma(\alpha-\beta+1)}$$

The set $Q_r$ is nonempty, closed and convex. □

Let $T : Q_r \to Q_r$ be an operator defined by (7).

For $u \in Q_r$, then $T$ is a continuous operator, i.e, if $\{u_n(t)\}$ is a sequence in $Q_r$ converges to $u(t), \forall t \in (0, T]$, for

$$\lim_{n\to\infty} Tu_n(t) = \frac{At^{\alpha-1}}{p(t)} \sum_{j=1}^{m} \frac{a_j}{p(\tau_j)} \lim_{n\to\infty} \int_0^{\tau_j} \frac{(\tau_j - s)^{\alpha-1}}{\Gamma(\alpha)} f(s, u_n(s))\ ds + \frac{1}{p(t)} \lim_{n\to\infty} \int_0^{t} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} f(s, u_n(s))\ ds,$$

by assumption $(h_1)$ and the Lebesgue dominated convergence Theorem we deduce that

$$\lim_{n\to\infty} Tu_n(t) = Tu(t).$$

Then $T$ is continuous. Now from Eq. (8), let $u \in Q_r$, then

$$\begin{aligned} |t^{1-\alpha}(Tu)(t)| &\leqslant \frac{|A|}{|p(t)|} \sum_{j=1}^{m} \frac{|a_j|}{|p(\tau_j)|} I^{\alpha-\beta} I^{\beta} h(\tau_j) + \frac{t^{1-\alpha}}{|p(t)|} I^{\alpha-\beta} I^{\beta} h(t) \\ &\leqslant \frac{|A|}{\inf|p(t)|} \sum_{j=1}^{m} \frac{|a_j| M}{\inf|p(t)|} \int_0^{\tau_j} \frac{(\tau_j - s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\ ds + \frac{Mt^{1-\alpha}}{\inf|p(t)|} \int_0^{t} \frac{(t-s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)} ds \\ &\leqslant \frac{|A|}{p} \sum_{j=1}^{m} \frac{|a_j| M}{p} \frac{\tau_j^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} + \frac{M}{p} \frac{t^{1-\beta}}{\Gamma(\alpha-\beta+1)} \\ &\leqslant \frac{|A| M T^{\alpha-\beta}}{p^2 \Gamma(\alpha-\beta+1)} \sum_{j=1}^{m} |a_j| + \frac{M}{p} \frac{T^{1-\beta}}{\Gamma(\alpha-\beta+1)} = r. \end{aligned}$$

Then $\{Tu(t)\}$ is uniformly bounded in $Q_r$.

In what follows we show that $T$ is a completely continuous operator.

For $t_1, t_2 \in (0, T], t_1 < t_2$ such that $|t_2 - t_1| < \delta$, from (7) we have

$$\begin{aligned} &\left|t_2^{1-\alpha}(Tu)(t_2) - t_1^{1-\alpha}(Tu)(t_1)\right| \\ &\leqslant \left| \left[\frac{1}{p(t_2)} - \frac{1}{p(t_1)}\right] \sum_{j=1}^{m} \frac{Aa_j}{p(\tau_j)} \int_0^{\tau_j} \frac{(\tau_j - s)^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s))\ ds + \frac{t_2^{1-\alpha}}{p(t_2)} \int_0^{t_2} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s))\ ds - \frac{t_1^{1-\alpha}}{p(t_1)} \int_0^{t_1} \frac{(t_1 - s)^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s))\ ds \right|, \\ &\leqslant \left| \left[\frac{1}{p(t_2)} - \frac{1}{p(t_1)}\right] \sum_{j=1}^{m} \frac{Aa_j}{p(\tau_j)} \int_0^{\tau_j} \frac{(\tau_j - s)^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s)) ds + \frac{t_2^{1-\alpha}}{p(t_2)} \int_0^{t_1} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s))\ ds + \frac{t_2^{1-\alpha}}{p(t_2)} \int_{t_1}^{t_2} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s))\ ds - \frac{t_1^{1-\alpha}}{p(t_1)} \int_0^{t_1} \frac{(t_1 - s)^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s))\ ds \right|, \\ &\leqslant \left| \left[\frac{1}{p(t_2)} - \frac{1}{p(t_1)}\right] \sum_{j=1}^{m} \frac{Aa_j}{p(\tau_j)} \int_0^{\tau_j} \frac{(\tau_j - s)^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s)) ds + \frac{t_2^{1-\alpha}}{p(t_2)} \int_0^{t_1} \frac{(t_1 - s)^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s))\ ds + \frac{t_2^{1-\alpha}}{p(t_2)} \int_{t_1}^{t_2} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s))\ ds - \frac{t_1^{1-\alpha}}{p(t_1)} \int_0^{t_1} \frac{(t_1 - s)^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s))\ ds \right|, \\ &\leqslant \left| \left[\frac{1}{p(t_2)} - \frac{1}{p(t_1)}\right] \sum_{j=1}^{m} \frac{Aa_j}{p(\tau_j)} \int_0^{\tau_j} \frac{(\tau_j - s)^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s))\ ds + \left[\frac{t_2^{1-\alpha}}{p(t_2)} - \frac{t_1^{1-\alpha}}{p(t_2)}\right] \int_0^{t_1} \frac{(t_1 - s)^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s))\ ds + t_1^{1-\alpha}\left[\frac{1}{p(t_2)} - \frac{1}{p(t_1)}\right] \int_0^{t_1} \frac{(t_1 - s)^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s)) ds + \frac{t_2^{1-\alpha}}{p(t_2)} \int_{t_1}^{t_2} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} f(s, u(s))\ ds \right|, \\ &\leqslant \left|\frac{1}{p(t_2)} - \frac{1}{p(t_1)}\right| \sum_{j=1}^{m} \frac{|A||a_j|}{p} \int_0^{\tau_j} \frac{(\tau_j - s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\ ds + \frac{|t_2^{1-\alpha} - t_1^{1-\alpha}|}{|p(t_2)|} \int_0^{t_1} \frac{(t_1 - s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\ ds + t_1^{1-\alpha}\left|\frac{1}{p(t_2)} - \frac{1}{p(t_1)}\right| \int_0^{t_1} \frac{(t_1 - s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\ ds + \frac{t_2^{1-\alpha}}{|p(t_2)|} \int_{t_1}^{t_2} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\ ds, \\ &\leqslant \left|\frac{1}{p(t_2)} - \frac{1}{p(t_1)}\right| \sum_{j=1}^{m} \frac{|A||a_j|}{p} \int_0^{\tau_j} \frac{(\tau_j - s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\ ds + \frac{|t_2^{1-\alpha} - t_1^{1-\alpha}|}{p} \int_0^{t_1} \frac{(t_1 - s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\ ds + \frac{t_2^{1-\alpha}}{p} \int_{t_1}^{t_2} \frac{(t_2 - s)^{\alpha-1}}{\Gamma(\alpha)} h(s) ds + t_1^{1-\alpha}\left|\frac{1}{p(t_2)} - \frac{1}{p(t_1)}\right| \int_0^{t_1} \frac{(t_1 - s)^{\alpha-1}}{\Gamma(\alpha)} h(s)\ ds, \end{aligned}$$

which can be written as

$$\begin{aligned} \left|t_2^{1-\alpha}(Tu)(t_2) - t_1^{1-\alpha}(Tu)(t_1)\right| &\leqslant \left|\frac{1}{p(t_2)} - \frac{1}{p(t_1)}\right| \sum_{j=1}^{m} \frac{|A||a_j|}{p} I^{\alpha-\beta} I^{\beta} h(\tau_j) + \frac{|t_2^{1-\alpha} - t_1^{1-\alpha}|}{p} I^{\alpha-\beta} I^{\beta} h(t_1) + \frac{t_2^{1-\alpha}}{p} I_{t_1}^{\alpha-\beta} I_{t_1}^{\beta} h(t_2) + t_1^{1-\alpha}\left|\frac{1}{p(t_2)} - \frac{1}{p(t_1)}\right| I^{\alpha-\beta} I^{\beta} h(t_1), \\ &\leqslant \left|\frac{1}{p(t_2)} - \frac{1}{p(t_1)}\right| \sum_{j=1}^{m} \frac{|A||a_j| M}{p} \int_0^{\tau_j} \frac{(\tau_j - s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\ ds + M\frac{|t_2^{1-\alpha} - t_1^{1-\alpha}|}{p} \int_0^{t_1} \frac{(t_1 - s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\ ds + \frac{t_2^{1-\alpha} M}{p} \int_{t_1}^{t_2} \frac{(t_2 - s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\ ds + Mt_1^{1-\alpha}\left|\frac{1}{p(t_2)} - \frac{1}{p(t_1)}\right| \int_0^{t_1} \frac{(t_1 - s)^{\alpha-\beta-1}}{\Gamma(\alpha-\beta)}\ ds, \\ &\leqslant \left|\frac{1}{p(t_2)} - \frac{1}{p(t_1)}\right| \sum_{j=1}^{m} \frac{|A||a_j| M T^{\alpha-\beta}}{p\Gamma(\alpha-\beta+1)} + \frac{MT^{\alpha-\beta}|t_2 - t_1|^{1-\alpha}}{p\Gamma(\alpha-\beta+1)} + \frac{T^{1-\alpha} M (t_2 - t_1)^{\alpha-\beta}}{p\Gamma(\alpha-\beta+1)} + \frac{MT^{1-\beta}}{\Gamma(\alpha-\beta+1)} \left|\frac{1}{p(t_2)} - \frac{1}{p(t_1)}\right|. \end{aligned}$$

Hence the class $\{Tu(t)\}$ is equi-continuous, by Arzela-Ascolis Theorem then $\{Tu(t)\}$ is relatively compact. Since all conditions of Schauder fixed point Theorem are hold, then $T$ has a fixed point in $Q_r$. Therefor the nonlocal problem (1) and (2) has at least one solution $u \in C_{1-\alpha}(0, T]$.

**Theorem 5.2.** *Let* $f:(0,T]\times R\to R$ *is continuous and satisfies the Lipschitz condition*

$$|f(t,u_1)-f(t,u_2)|\leqslant L|u_1-u_2|, L>0, \text{ for all } u_1,u_2\in R.$$

If the conditions $(h_2),(h_3)$ are satisfied and

$$\frac{2^{1-2\alpha}\sqrt{\pi}L}{\Gamma(\alpha+\frac{1}{2})p}\left(\frac{|A|\sum_{j=1}^{m}|a_j|T^{2\alpha-1}}{p}+T^{\alpha}\right)<1,$$

then the nonlocal problem (1) and (2) has a unique solution $u\in C_{1-\alpha}(0,T]$.

**Proof.** Let $T$ be an operator defined by (7), then $T:C_{1-\alpha}(0,T]\to C_{1-\alpha}(0,T]$

$$\begin{aligned}
&|t^{1-\alpha}(Tu)(t)-t^{1-\alpha}(Tv)(t)|\\
&\leqslant \frac{|A|}{|p(t)|}\sum_{j=1}^{m}\frac{|a_j|}{|p(\tau_j)|}\int_0^{\tau_j}\frac{(\tau_j-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,u(s))-f(s,v(s))|ds\\
&\quad+\frac{t^{1-\alpha}}{|p(t)|}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|f(s,u(s))-f(s,v(s))|ds\\
&\leqslant \frac{L|A|}{p}\sum_{j=1}^{m}\frac{|a_j|}{p}\int_0^{\tau_j}\frac{(\tau_j-s)^{\alpha-1}}{\Gamma(\alpha)}|u(s)-v(s)|ds\\
&\quad+\frac{Lt^{1-\alpha}}{p}\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}|u(s)-v(s)|ds\\
&\leqslant \frac{L|A|}{p^2}\sum_{j=1}^{m}|a_j|\int_0^{\tau_j}\frac{s^{\alpha-1}(\tau_j-s)^{\alpha-1}}{\Gamma(\alpha)}s^{1-\alpha}|u(s)-v(s)|ds\\
&\quad+\frac{Lt^{1-\alpha}}{p}\int_0^t\frac{s^{\alpha-1}(t-s)^{\alpha-1}}{\Gamma(\alpha)}s^{1-\alpha}|u(s)-v(s)|ds\\
&\leqslant \frac{L|A|}{p^2}\|u-v\|_{C_{1-\alpha}}\sum_{j=1}^{m}|a_j|\int_0^{\tau_j}\frac{s^{\alpha-1}(\tau_j-s)^{\alpha-1}}{\Gamma(\alpha)}ds\\
&\quad+\frac{Lt^{1-\alpha}}{p}\|u-v\|_{C_{1-\alpha}}\int_0^t\frac{s^{\alpha-1}(t-s)^{\alpha-1}}{\Gamma(\alpha)}ds\\
&\leqslant \frac{L|A|}{p^2}\|u-v\|_{C_{1-\alpha}}\sum_{j=1}^{m}\frac{|a_j|\tau_j^{2\alpha-1}\Gamma(\alpha)}{\Gamma(2\alpha)}\\
&\quad+\frac{Lt^{1-\alpha}}{p}\|u-v\|_{C_{1-\alpha}}\frac{t^{2\alpha-1}\Gamma(\alpha)}{\Gamma(2\alpha)}\\
&\leqslant \frac{2^{1-2\alpha}\sqrt{\pi}L}{\Gamma(\alpha+\frac{1}{2})p}\left(\frac{|A|\sum_{j=1}^{m}|a_j|T^{2\alpha-1}}{p}+T^{\alpha}\right)\|u-v\|_{C_{1-\alpha}}.
\end{aligned}$$

This means that

$$\|t^{1-\alpha}(Tu)(t)-t^{1-\alpha}(Tv)(t)\|_C \leqslant \frac{2^{1-2\alpha}\sqrt{\pi}L}{\Gamma(\alpha+\frac{1}{2})p}\left(\frac{|A|\sum_{j=1}^{m}|a_j|T^{2\alpha-1}}{p}+T^{\alpha}\right)\|u-v\|_{C_{1-\alpha}}.$$

Then by using Banach fixed point Theorem, the operator $T$ has a unique fixed point $u(t)\in C_{1-\alpha}$.

## 6. Example

In this section we provide an example illustrating our result obtained in Theorem 4.1.

**Example 6.1.** Consider the nonlinear fractional differential problem

$$\begin{cases} D^{\frac{1}{2}}\left(\frac{1+t}{1+t^2}u(t)\right)=\sin u(t)(1+\cos^2 u(t))t^2+e^t \quad a.e.\ t\in(0,1],\\ \lim_{t\to 0^+} t^{\frac{1}{2}}\frac{1+t}{1+t^2}u(t)=3u(\frac{1}{3})-2u(\frac{1}{2}).\end{cases}$$

Observe, the above problem is a special case of (1) and (2). indeed if we put $f(t,u(t))=\sin u(t)(1+\cos^2 u(t))t^2+e^t$, $\alpha=\frac{1}{2}, p(t)=\frac{1+t}{1+t^2}$. Then we can easy check that the assumptions of Theorem 4.1 are satisfied.

Then the problem has at least one positive solution $u\in L_1(0,1]$.

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