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Some inequalities of Qi type for double integrals

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Abstract In the paper, the authors establish some new inequalities of Qi type for double integrals on a rectangle, from which some known integral inequalities of Qi type may be derived.

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1. Introduction

In [1] and its preprint [2], the following interesting integral inequality was obtained.

Theorem 1.1 [1, Proposition1.3] and [2, Proposition2]. *Let $n \in \mathbb{N}$ and the n-th order derivative of f be continuous on $[a,b] \subseteq \mathbb{R}$. If $f^{(i)}(a) \geqslant 0$ for $0 \leqslant i \leqslant n-1$ and $f^{(n)}(x) \geqslant n!$ on $[a,b]$, then*

$$\int_a^b f^{n+2}(x)\mathrm{d}x \geqslant \left[\int_a^b f(x)\mathrm{d}x\right]^{n+1}. \tag{1.1}$$

At the end of [1,2], the following open problem was posed.

Open Problem 1.1 [1, Theorem1.5 (OpenProblem)] and [2, OpenProblem]. Under what conditions does the inequality

$$\int_a^b f^t(x)\mathrm{d}x \geqslant \left[\int_a^b f(x)\mathrm{d}x\right]^{t-1} \tag{1.2}$$

hold for some $t > 1$?

Thereafter, many mathematicians devoted to finding answers to Open Problem 1.1 and to generalizing the integral inequality (1.1). See [3–11] and plenty of references therein. For a collection of over forty articles, please refer to the list of references in the recently published paper [12].

Motivated by Open Problem 1.1, we now naturally pose the following questions.

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Open Problem 1.2. Let $f(x,y)$ be a positive and continuous function defined on a rectangle $[a,b]\times[c,d]\subseteq\mathbb{R}^2$. Under what conditions does the inequality

$$\int_a^b\int_c^d f^t(x,y)\,\mathrm{d}x\,\mathrm{d}y \geqslant \left[\int_a^b\int_c^d f(x,y)\,\mathrm{d}x\mathrm{d}y\right]^{t-1} \tag{1.3}$$

hold for some $t\in\mathbb{R}$?

The aim of this paper is to provide several affirmative answers to Open Problem 1.2. In other words, some new inequalities for double integrals on a rectangle $[a,b]\times[c,d]$, from which some integral inequalities of Qi type may be derived, will be established in this paper.

2. A definition and a lemma

For providing affirmative answers to Open Problem 1.2, we need a definition and a lemma which are not common knowledge.

Definition 2.1 (*[13,14]*). Let $I\subseteq\mathbb{R}_+=(0,\infty)$ be an interval and $r\in\mathbb{R}$. A function $f:I\to\mathbb{R}_+$ is said to be r-mean convex on I if

$$f\left([\lambda x^r+(1-\lambda)y^r]^{1/r}\right)\leqslant[\lambda f^r(x)+(1-\lambda)f^r(y)]^{1/r},\quad r\neq 0 \tag{2.1}$$

or

$$f(x^\lambda y^{1-\lambda})\leqslant f^\lambda(x)f^{1-\lambda}(y),\quad r=0 \tag{2.2}$$

for all $x,y\in I$ and $\lambda\in[0,1]$. If the above inequalities reverse, then we say that the function f is r-mean concave on I.

Remark 2.1. The 0-mean convex (0-mean concave, respectively) functions are the well known geometrically convex (geometrically concave, respectively) functions.

Lemma 2.1 (*[13,14]*). *Let $I\subseteq\mathbb{R}_+$ be an interval and $r\in\mathbb{R}$. A function $f:I\to\mathbb{R}_+$ is r-mean convex (or r-mean concave, respectively) on I if and only if*

$$f\left(\left[\sum_{k=1}^n\lambda_k x_k^r\right]^{1/r}\right)\lesseqgtr\left[\sum_{k=1}^n\lambda_k f^r(x_k)\right]^{1/r},\quad r\neq 0 \tag{2.3}$$

or

$$f\left(\prod_{k=1}^n x_k^{\lambda_k}\right)\lesseqgtr\prod_{k=1}^n f^{\lambda_k}(x_k),\quad r=0 \tag{2.4}$$

holds for all $x=(x_1,x_2,\ldots c,x_n)\in I^n$ and $\lambda_k\geqslant 0$ satisfying $\sum_{k=1}^n\lambda_k=1$.

3. New inequalities of Qi type for double integrals

Now we are in a position to establish some new inequalities of Qi type for double integrals on the rectangle $[a,b]\times[c,d]$.

Theorem 3.1. *For $I\subseteq\mathbb{R}_0=[0,\infty)$ being an interval, let $f:[a,b]\times[c,d]\to I$ be continuous and not identically zero, and let $g:I\to\mathbb{R}_0$ be convex (or concave, respectively). If*

$$g((b-a)(d-c)u)\lesseqgtr g((b-a)(d-c))g(u) \tag{3.1}$$

for $u\in I$ and

$$\int_a^b\int_c^d f(x,y)\,\mathrm{d}x\,\mathrm{d}y\gtreqless\frac{g((b-a)(d-c))}{(b-a)(d-c)}, \tag{3.2}$$

then we have

$$\int_a^b\int_c^d g(f(x,y))\,\mathrm{d}x\,\mathrm{d}y\gtreqless\frac{g\left(\int_a^b\int_c^d f(x,y)\,\mathrm{d}x\,\mathrm{d}y\right)}{\int_a^b\int_c^d f(x,y)\,\mathrm{d}x\,\mathrm{d}y}. \tag{3.3}$$

Proof. Let

$$(x_k,y_k)=\left(a+\frac{k}{n}(b-a),c+\frac{k}{n}(d-c)\right),\quad 1\leqslant k\leqslant n. \tag{3.4}$$

By the convexity of g, by inequalities (3.1) and (3.2), and by Lemma 2.1, we have

$$\begin{aligned}
g\left(\int_a^b\int_c^d f(x,y)\,\mathrm{d}x\,\mathrm{d}y\right)&=g\left((b-a)(d-c)\lim_{n\to\infty}\frac{1}{n^2}\sum_{i=1}^n\sum_{j=1}^n f(x_i,y_j)\right)\\
&\leqslant g((b-a)(d-c))\lim_{n\to\infty}g\left(\frac{1}{n^2}\sum_{i=1}^n\sum_{j=1}^n f(x_i,y_j)\right)\\
&\leqslant g((b-a)(d-c))\lim_{n\to\infty}\frac{1}{n^2}\sum_{i=1}^n\sum_{j=1}^n g(f(x_i,y_j))\\
&=\frac{g((b-a)(d-c))}{(b-a)(d-c)}\int_a^b\int_c^d g(f(x,y))\,\mathrm{d}x\,\mathrm{d}y\\
&\leqslant\left(\int_a^b\int_c^d f(x,y)\,\mathrm{d}x\,\mathrm{d}y\right)\int_a^b\int_c^d g(f(x,y))\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}$$

Thus, the inequality (3.3) in the direction $\geqslant$ is true.

If $g(u)$ is a concave function on I, the proof is similar. This completes the proof of Theorem 3.1. □

Corollary 3.1. *Let $f(x,y)$ be a positive continuous function on $[a,b]\times[c,d]\subseteq\mathbb{R}^2$.*

1. *If $t\geq 1$ or $t<0$ and*

$$\int_a^b\int_c^d f(x,y)\,\mathrm{d}x\,\mathrm{d}y\geqslant[(b-a)(d-c)]^{t-1},$$

then

$$\int_a^b\int_c^d f^t(x,y)\,\mathrm{d}x\,\mathrm{d}y\geqslant\left[\int_a^b\int_c^d f(x,y)\,\mathrm{d}x\mathrm{d}y\right]^{t-1}. \tag{3.5}$$

2. *If $0<t\leq 1$ and*

$$\int_a^b\int_c^d f(x,y)\,\mathrm{d}x\,\mathrm{d}y\leqslant[(b-a)(d-c)]^{t-1},$$

then

$$\int_a^b\int_c^d f^t(x,y)\,\mathrm{d}x\,\mathrm{d}y\leqslant\left[\int_a^b\int_c^d f(x,y)\,\mathrm{d}x\,\mathrm{d}y\right]^{t-1}. \tag{3.6}$$

3. *If $t\notin[0,1)$ and $f(x,y)\geqslant[(b-a)(d-c)]^{t-2}$ for $(x,y)\in[a,b]\times[c,d]$, then the inequality (3.5) is valid.*
4. *If $0<t\leqslant 1$ and $f(x,y)\leqslant[(b-a)(d-c)]^{t-2}$ for $(x,y)\in[a,b]\times[c,d]$, then the inequality (3.5) is reversed.*
5. *If $t\geq 2$ and $f(x,y)\geqslant(t-1)^2[(x-a)(y-c)]^{t-2}$ for $(x,y)\in[a,b]\times[c,d]$, then the inequality (3.5) is valid.*

Proof. This follows from applying $g(u)=u^t$ for $u>0$ and $t\in\mathbb{R}$ in Theorem 3.1. □

Theorem 3.2. *Let $I \subseteq \mathbb{R}_+$ be an interval and $r \neq 0$, and let $f : [a,b] \times [c,d] \subseteq \mathbb{R}^2 \to I$ be a continuous function and $g : I \to \mathbb{R}_+$. If $g(u)$ is r-mean convex (or r-mean concave, respectively) and satisfies*

$$g\left([(b-a)(d-c)]^{1/r}u\right) \lesseqgtr g\left([(b-a)(d-c)]^{1/r}\right)g(u), \quad u \in I \tag{3.7}$$

and

$$\int_a^b \int_c^d f(x,y)\mathrm{d}x\,\mathrm{d}y \gtreqless \frac{g\left([(b-a)(d-c)]^{1/r}\right)}{[(b-a)(d-c)]^{1/r}}, \tag{3.8}$$

then

$$\left[\int_a^b \int_c^d g^r(f(x,y))\mathrm{d}x\,\mathrm{d}y\right]^{1/r} \gtreqless \frac{g\left(\left(\int_a^b \int_c^d f^r(x,y)\mathrm{d}x\,\mathrm{d}y\right)^{1/r}\right)}{\int_a^b \int_c^d f(x,y)\mathrm{d}x\,\mathrm{d}y}. \tag{3.9}$$

Proof. Making use of the r-mean convexity of g, adopting notations in (3.4), and employing Lemma 2.1 lead to

$$\begin{aligned}
&g\left(\left[\int_a^b \int_c^d f^r(x,y)\mathrm{d}x\,\mathrm{d}y\right]^{1/r}\right)\\
&= g\left([(b-a)(d-c)]^{1/r}\left(\lim_{n\to\infty}\frac{1}{n^2}\sum_{i=1}^n\sum_{j=1}^n f^r(x_i,y_j)\right)^{1/r}\right)\\
&\leqslant g\left([(b-a)(d-c)]^{1/r}\right)\lim_{n\to\infty}\left(\left(\frac{1}{n^2}\sum_{i=1}^n\sum_{j=1}^n f^r(x_i,y_j)\right)^{1/r}\right)\\
&\leqslant g([(b-a)(d-c)]^{1/r})\left[\lim_{n\to\infty}\frac{1}{n^2}\sum_{i=1}^n\sum_{j=1}^n g^r(f(x_i,y_j))\right]^{1/r}\\
&= \frac{g([(b-a)(d-c)]^{1/r})}{[(b-a)(d-c)]^{1/r}}\left[\int_a^b g^r(f(x,y))\mathrm{d}x\,\mathrm{d}y\right]^{1/r}\\
&\leqslant \left(\int_a^b \int_c^d f(x,y)\mathrm{d}x\,\mathrm{d}y\right)\left[\int_a^b g^r(f(x,y))\mathrm{d}x\,\mathrm{d}y\right]^{1/r}.
\end{aligned}$$

The inequality (3.9) is thus proved.

The rest can be proved similarly. The proof of Theorem 3.2 is complete. □

Theorem 3.3. *For $I \subseteq \mathbb{R}_+$ being an interval, let $f : [a,b] \times [c,d] \subseteq \mathbb{R}^2 \to I$ be a continuous function, and let $g : I \to \mathbb{R}_+$ be a geometrically convex (or geometrically concave, respectively) function. If*

$$g(e^{(b-a)(d-c)u}) \lesseqgtr g(e^{(b-a)(d-c)})g(e^u), \quad u \in I \tag{3.10}$$

and

$$\int_a^b \int_c^d f(x,y)\mathrm{d}x\,\mathrm{d}y \gtreqless g\left(e^{(b-a)(d-c)}\right), \tag{3.11}$$

then

$$\exp\left(\frac{1}{(b-a)(d-c)}\int_a^b \int_c^d \ln g(f(x,y))\mathrm{d}x\,\mathrm{d}y\right) \gtreqless \frac{g\left(\exp\left(\int_a^b \int_c^d \ln f(x,y)\mathrm{d}x\,\mathrm{d}y\right)\right)}{\int_a^b \int_c^d f(x,y)\mathrm{d}x\,\mathrm{d}y}. \tag{3.12}$$

Proof. Utilizing the geometric convexity of g and using Lemma 2.1 result in

$$\begin{aligned}
&g\left(\exp\left(\int_a^b \int_c^d \ln f(x,y)\mathrm{d}x\,\mathrm{d}y\right)\right)\\
&= g\left(\exp\left((b-a)(d-c)\lim_{n\to\infty}\frac{1}{n^2}\sum_{i=1}^n\sum_{j=1}^n \ln f(x_i,y_j)\right)\right)\\
&\leqslant g(e^{(b-a)(d-c)})\lim_{n\to\infty} g\left(\prod_{i=1}^n\prod_{j=1}^n [f(x_i,y_j)]^{1/n^2}\right)\\
&\leqslant g(e^{(b-a)(d-c)})\lim_{n\to\infty}\left(\prod_{i=1}^n\prod_{j=1}^n g(f(x_i,y_j))\right)^{1/n^2}\\
&= g(e^{(b-a)(d-c)})\exp\left(\frac{1}{(b-a)(d-c)}\int_a^b \int_c^d \ln g(f(x,y))\mathrm{d}x\mathrm{d}y\right)\\
&\leqslant \left(\int_a^b \int_c^d f(x,y)\mathrm{d}x\,\mathrm{d}y\right)\exp\left(\frac{1}{(b-a)(d-c)}\int_a^b \int_c^d \ln g(f(x,y))\mathrm{d}x\mathrm{d}y\right).
\end{aligned}$$

Consequently, the inequality (3.12) is true.

The rest can be proved similarly. The proof of Theorem 3.3 is complete. □

Remark 3.1. We remark that, as an example, Theorems 3.1, 3.2, and 3.3 generalize Theorem 3.4 below.

Theorem 3.4 [15, Theorem1.1], [16, Proposition1], and [17, Theorem1]. *Let $t > 1$ and f be a continuous function on $[a,b] \subseteq \mathbb{R}$ such that*

$$\int_a^b f(x)\mathrm{d}x \geqslant (b-a)^{t-1}. \tag{3.13}$$

Then the inequality (1.2) is valid.

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