Journal of the Egyptian Mathematical Society (2014) 22, 330–336

Egyptian Mathematical Society

Journal of the Egyptian Mathematical Society

www.etms-eg.org
www.elsevier.com/locate/joems

On some generalizations of the Hilbert–Hardy type discrete inequalities

S.A.A. El-Marouf *

Permanent address: Department of Mathematics, Faculty of Science, Minoufiya University, Shebin El-Koom, Egypt
Current address: Department of Mathematics, Faculty of Science, Taibah University, Kingdom of Saudi Arabia

Received 8 November 2012; revised 16 December 2012; accepted 10 October 2013
Available online 8 December 2013

KEYWORDS

Hilbert's inequality;
Hölder's inequality;
Hardy's inequality;
Beta and gamma functions;
Weight functions

Abstract New Hilbert-type discrete inequalities are presented by using new techniques in proof. By specializing the weight coefficient functions in the hypothesis and the parameters, we obtain many special cases which include, in particular, the discrete inequality derived by Hilbert and Hardy. Many improvements and generalizations of known results are given in this paper.

2010 MATHEMATICAL SUBJECT CLASSIFICATION: Primary 26D15; Secondary 47A07

1. Introduction

The Hilbert's double series inequality is given as follows: (see [1,2]).

Let $p > 1, q > 1, \frac{1}{p} + \frac{1}{q} = 1$ *and* $a_m, b_n > 0$. *If* $0 < \sum_{m=1}^{\infty} a_m^p < \infty$, *and* $0 < \sum_{n=1}^{\infty} b_n^q < \infty$, *then we have*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} \leqslant \frac{\pi}{\sin\left(\frac{\pi}{p}\right)} \left(\sum_{m=1}^{\infty} a_m^p\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} b_n^q\right)^{\frac{1}{q}}, \tag{1.1}$$

where $\pi/\sin(\pi/p)$ is the best possible constant.

Many inequalities, in general and different versions of the Hilbert inequality, in particular play a major role in mathematical analysis and applications. In recent years, considerable attention has been given to various extensions and improvements of the Hilbert inequality (1.1) (see Refs. [3–10]). The main purpose of this paper is to obtain some extensions of (1.1).

2. The main results

First, we introduce some lemmas.

Lemma 2.1. *For* $p > 1, \alpha \geqslant 0, \beta > \frac{1}{\gamma p}$ *and* $a > 0$ *such that* $\frac{1}{p} + \frac{1}{q} = 1$, *define the weight coefficient function* $w_1(m)$ *as follows:*

$$w_1(m) = \sum_{n=1}^{\infty} \frac{1}{(a + m^\alpha n^\beta)^\gamma} \left(\frac{m}{n}\right)^{\frac{1}{q}}. \tag{2.1}$$

Then we get

$$w_1(m) \leqslant \frac{1}{\beta} a^{\frac{1}{\beta p} - \gamma} m^{\frac{\beta p - \beta - \alpha}{\beta p}} B\left(\frac{1}{\beta p}, \gamma - \frac{1}{\beta p}\right), \tag{2.2}$$

where $B(a,b)$ *is the Beta function for* $a > 0$ *and* $b > 0$.

* Current address: Department of Mathematics, Faculty of Science, Taibah University, Kingdom of Saudi Arabia. Tel./fax: +20 483486398.
E-mail address: sobhy_2000_99@yahoo.com

Peer review under responsibility of Egyptian Mathematical Society.

Production and hosting by Elsevier

1110-256X © 2013 Production and hosting by Elsevier B.V. on behalf of Egyptian Mathematical Society. Open access under CC BY-NC-ND license.
http://dx.doi.org/10.1016/j.joems.2013.10.007

Proof. From (2.1), we have

$$w_1(m)=\sum_{n=1}^{\infty}\frac{1}{a^{\gamma}\left(1+\frac{m^{\alpha}n^{\beta}}{a}\right)^{\gamma}}\left(\frac{m}{n}\right)^{\frac{1}{q}}\leqslant\frac{1}{a^{\gamma}}\int_0^{\infty}\frac{1}{\left(1+\frac{m^{\alpha}y^{\beta}}{a}\right)^{\gamma}}\left(\frac{m}{y}\right)^{\frac{1}{q}}dy.$$

Using the change of variable $u=\frac{m^{\alpha}y^{\beta}}{a}$, we have $dy=\frac{1}{\beta}\frac{a^{\frac{1}{\beta}}u^{\frac{1}{\beta}-1}}{m^{\alpha/\beta}}du$ and $0\leqslant u<\infty$.

Substituting u and dy in the right hand side of the above inequality, we get

$$w_1(m)\leqslant\frac{1}{a^{\gamma}}\int_0^{\infty}\frac{1}{(1+u)^{\gamma}}\left(\frac{m^{1+\frac{\alpha}{\beta}}}{a^{\frac{1}{\beta}}u^{\frac{1}{\beta}}}\right)^{\frac{1}{q}}\frac{1}{\beta}\frac{a^{\frac{1}{\beta}}u^{\frac{1}{\beta}-1}}{m^{\frac{\alpha}{\beta}}}du$$

$$=\frac{1}{\beta}a^{\frac{1}{\beta}-\frac{1}{\beta q}-\gamma}m^{\frac{1}{q}+\frac{\alpha}{\beta q}-\frac{\alpha}{\beta}}\int_0^{\infty}\frac{u^{\frac{1}{\beta}-\frac{1}{\beta q}-1}}{(1+u)^{\gamma}}du\cdot$$

It follows from [9] and $\frac{1}{p}+\frac{1}{q}=1$, that

$$w_1(m)\leqslant\frac{1}{\beta}a^{\frac{1}{\beta p}-\gamma}m^{\frac{\beta p-\beta-\alpha}{\beta p}}B\left(\frac{1}{\beta p},\gamma-\frac{1}{\beta p}\right).$$

Hence the lemma is proved. □

By a similar manner we can prove the following lemma.

Lemma 2.2. *For $p>1,\alpha>\frac{1}{\gamma q},\beta\geqslant0$ and $a>0$ such that $\frac{1}{p}+\frac{1}{q}=1$, define the weight coefficient*

$w_2(n)$ as:

$$w_2(n)=\sum_{m=1}^{\infty}\frac{1}{(a+m^{\alpha}n^{\beta})^{\gamma}}\left(\frac{n}{m}\right)^{\frac{1}{p}}.\tag{2.3}$$

Then we get

$$w_2(n)\leqslant\frac{1}{\alpha}a^{\frac{1}{\alpha q}-\gamma}n^{\frac{\alpha q-\alpha-\beta}{\alpha q}}B\left(\frac{1}{\alpha q},\gamma-\frac{1}{\alpha q}\right).\tag{2.4}$$

Theorem 2.3. *If $\alpha>\frac{1}{\gamma q},\beta>\frac{1}{\gamma p}$ and $(p>1)$, such that $\frac{1}{p}+\frac{1}{q}=1,\{a_m\}$ and $\{b_n\}\geqslant0$, satisfy that*

$$0<\sum_{m=1}^{\infty}m^{\frac{\beta p-\beta-\alpha}{\beta p}}a_m^p<\infty \text{ and } 0<\sum_{n=1}^{\infty}n^{\frac{\alpha q-\alpha-\beta}{\alpha q}}b_n^q<\infty.$$

Then for $(a,\gamma>0)$.

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_mb_n}{(a+m^{\alpha}n^{\beta})^{\gamma}}\leqslant a^{\frac{\alpha(q-\gamma\beta q-1)+\beta(p-\gamma\alpha p-1)}{\alpha\beta pq}}\left(\frac{1}{\beta}B(\frac{1}{\beta p},\gamma-\frac{1}{\beta p})\right)^{\frac{1}{p}}\left(\frac{1}{\alpha}B\left(\frac{1}{\alpha q},\gamma-\frac{1}{\alpha q}\right)\right)^{\frac{1}{q}}$$
$$\times\left(\sum_{m=1}^{\infty}m^{\frac{\beta p-\beta-\alpha}{\beta p}}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}n^{\frac{\alpha q-\alpha-\beta}{\alpha q}}b_n^q\right)^{\frac{1}{q}}.\tag{2.5}$$

Proof. Using Hölder's inequality, we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_mb_n}{(a+m^{\alpha}n^{\beta})^{\gamma}}$$
$$=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m}{(a+m^{\alpha}n^{\beta})^{\frac{\gamma}{p}}}\left(\frac{m}{n}\right)^{\frac{1}{pq}}\frac{b_n}{(a+m^{\alpha}n^{\beta})^{\frac{\gamma}{q}}}\left(\frac{n}{m}\right)^{\frac{1}{pq}}$$
$$\leqslant\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m^p}{(a+m^{\alpha}n^{\beta})^{\gamma}}\left(\frac{m}{n}\right)^{\frac{1}{q}}\right)^{\frac{1}{p}}\times\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{b_n^q}{(a+m^{\alpha}n^{\beta})^{\gamma}}\left(\frac{n}{m}\right)^{\frac{1}{p}}\right)^{\frac{1}{q}}$$
$$=\left(\sum_{m=1}^{\infty}a_m^p\left(\sum_{n=1}^{\infty}\frac{1}{(a+m^{\alpha}n^{\beta})^{\gamma}}\left(\frac{m}{n}\right)^{\frac{1}{q}}\right)\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}b_n^q\left(\sum_{m=1}^{\infty}\frac{1}{(a+m^{\alpha}n^{\beta})^{\gamma}}\left(\frac{n}{m}\right)^{\frac{1}{p}}\right)\right)^{\frac{1}{q}}.\tag{2.6}$$

By (2.1), (2.3) and (2.6), we get

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_mb_n}{(a+m^{\alpha}n^{\beta})^{\gamma}}\leqslant\left(\sum_{m=1}^{\infty}a_m^pw_1(m)\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}b_n^qw_2(n)\right)^{\frac{1}{q}}.\tag{2.7}$$

Substituting by (2.2) and (2.4) in (2.7), we obtain

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_mb_n}{(a+m^{\alpha}n^{\beta})^{\gamma}}\leqslant\left(\sum_{m=1}^{\infty}a_m^p\frac{1}{\beta}a^{\frac{1}{\beta p}-\gamma}m^{\frac{\beta p-\beta-\alpha}{\beta p}}B\left(\frac{1}{\beta p},\gamma-\frac{1}{\beta p}\right)\right)^{\frac{1}{p}}$$
$$\times\left(\sum_{n=1}^{\infty}b_n^q\frac{1}{\alpha}a^{\frac{1}{\alpha q}-\gamma}n^{\frac{\alpha q-\alpha-\beta}{\alpha q}}B\left(\frac{1}{\alpha q},\gamma-\frac{1}{\alpha q}\right)\right)^{\frac{1}{q}}.$$

Since $\frac{1}{p}+\frac{1}{q}=1$, we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_mb_n}{(a+m^{\alpha}n^{\beta})^{\gamma}}\leqslant a^{\frac{\alpha(q-\gamma\beta q-1)+\beta(p-\gamma\alpha p-1)}{\alpha\beta pq}}\left(\frac{1}{\beta}B\left(\frac{1}{\beta p},\gamma-\frac{1}{\beta p}\right)\right)^{\frac{1}{p}}$$
$$\left(\frac{1}{\alpha}B\left(\frac{1}{\alpha q},\gamma-\frac{1}{\alpha q}\right)\right)^{\frac{1}{q}}\times\left(\sum_{m=1}^{\infty}m^{\frac{\beta p-\beta-\alpha}{\beta p}}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}n^{\frac{\alpha q-\alpha-\beta}{\alpha q}}b_n^q\right)^{\frac{1}{q}}.$$

This completes the proof. □

Remark 2.1.

1. Let $p=q=2$ in (2.5), then we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_mb_n}{(a+m^{\alpha}n^{\beta})^{\gamma}}\leqslant\left(\frac{1}{\alpha\beta}\right)^{1/2}a^{\frac{\alpha-4\alpha\beta\gamma+\beta}{4\alpha\beta}}\left(B\left(\frac{1}{2\beta},\gamma-\frac{1}{2\beta}\right)B\left(\frac{1}{2\alpha},\gamma-\frac{1}{2\alpha}\right)\right)^{\frac{1}{2}}$$
$$\times\left(\sum_{m=1}^{\infty}m^{\frac{\beta-\alpha}{2\beta}}a_m^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty}n^{\frac{\alpha-\beta}{2\alpha}}b_n^2\right)^{\frac{1}{2}}.\tag{2.8}$$

2. Let $\gamma=1$ in (2.5). Then we get

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_mb_n}{a+m^{\alpha}n^{\beta}}\leqslant\frac{\pi a^{\frac{\alpha(q-\beta q-1)+\beta(p-\alpha p-1)}{\alpha\beta pq}}}{\left(\beta\sin\frac{\pi}{\beta p}\right)^{\frac{1}{p}}\left(\alpha\sin\frac{\pi}{\alpha q}\right)^{\frac{1}{q}}}\left(\sum_{m=1}^{\infty}m^{\frac{\beta p-\beta-\alpha}{\beta p}}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}n^{\frac{\alpha q-\alpha-\beta}{\alpha q}}b_n^q\right)^{\frac{1}{q}},\tag{2.9}$$

which is a new Hilbert-type inequality.

3. Let $\alpha=\beta=1$ in (2.9), then we obtain

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_mb_n}{a+mn}\leqslant\frac{\pi a^{\frac{-2}{pq}}}{\sin\frac{\pi}{p}}\left(\sum_{m=1}^{\infty}m^{1-\frac{2}{p}}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}n^{1-\frac{2}{q}}b_n^q\right)^{\frac{1}{q}}.\tag{2.10}$$

4. Let $a=1$ in (2.9), then we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_mb_n}{1+m^{\alpha}n^{\beta}}\leqslant\frac{\pi}{\left(\beta\sin\frac{\pi}{\beta p}\right)^{\frac{1}{p}}\left(\alpha\sin\frac{\pi}{\alpha q}\right)^{\frac{1}{q}}}\left(\sum_{m=1}^{\infty}m^{\frac{\beta p-\beta-\alpha}{\beta p}}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}n^{\frac{\alpha q-\alpha-\beta}{\alpha q}}b_n^q\right)^{\frac{1}{q}}.\tag{2.11}$$

5. Let $\alpha=\beta=1$ in (2.11), then we find

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_mb_n}{1+mn}\leqslant\frac{\pi}{\sin\frac{\pi}{p}}\left(\sum_{m=1}^{\infty}m^{1-\frac{2}{p}}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}n^{1-\frac{2}{q}}b_n^q\right)^{\frac{1}{q}}.$$

6. Let $a=4$ in (2.10), then we get

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_mb_n}{4+mn}\leqslant\frac{\pi4^{\frac{-2}{pq}}}{\sin\frac{\pi}{p}}\left(\sum_{m=1}^{\infty}m^{1-\frac{2}{p}}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}n^{1-\frac{2}{q}}b_n^q\right)^{\frac{1}{q}}.$$

7. Let $\alpha = \beta = 1$, $a = 1$ in (2.8), then we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(1+mn)^{\gamma}} \leqslant \frac{\sqrt{\pi}\,\Gamma(\gamma-\frac{1}{2})}{\Gamma(\gamma)}\left(\sum_{m=1}^{\infty}a_m^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty}b_n^2\right)^{\frac{1}{2}}. \tag{2.12}$$

8. Let $\gamma = 2$ in (2.12), then we find

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(1+mn)^{2}} \leqslant \frac{\pi}{2}\left(\sum_{m=1}^{\infty}a_m^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty}b_n^2\right)^{\frac{1}{2}}.$$

Lemma 2.4. *For* $\alpha \geqslant 0, \beta > \frac{1}{\gamma p}, \gamma > 0$ *and* $p > 1$ *such that* $\frac{1}{p}+\frac{1}{q} = 1$, *define the weight coefficient* $w_1(m)$ *as follows:*

$$w_1(m) = \sum_{n=1}^{\infty}\frac{1}{(m^{\alpha}+n^{\beta})^{\gamma}}\left(\frac{m}{n}\right)^{\frac{1}{q}}. \tag{2.13}$$

Then we get

$$w_1(m) \leqslant \frac{1}{\beta}m^{\frac{\beta p-\beta+\alpha-\alpha\gamma}{\beta p}}B\left(\frac{1}{\beta p},\gamma-\frac{1}{\beta p}\right). \tag{2.14}$$

Proof. From (2.13), we have

$$w_1(m) = \sum_{n=1}^{\infty}\frac{1}{m^{\alpha\gamma}(1+n^{\beta})^{\gamma}}\left(\frac{m}{n}\right)^{\frac{1}{q}} \leqslant \frac{1}{m^{\alpha\gamma}}\int_0^{\infty}\frac{1}{\left(1+\frac{y^{\beta}}{m^{\alpha}}\right)^{\gamma}}\left(\frac{m}{y}\right)^{\frac{1}{q}}dy.$$

Let $u = \frac{y^{\beta}}{m^{\alpha}}$, then we have $dy = \frac{1}{\beta}m^{\frac{\alpha}{\beta}}u^{\frac{1}{\beta}-1}du$ and $0 \leqslant u < \infty$.
Hence

$$w_1(m) \leqslant \frac{1}{m^{\alpha\gamma}}\int_0^{\infty}\frac{1}{(1+u)^{\gamma}}\left(\frac{m^{1-\frac{\alpha}{\beta}}}{u^{\frac{1}{\beta}}}\right)^{\frac{1}{q}}\frac{1}{\beta}m^{\frac{\alpha}{\beta}}u^{\frac{1}{\beta}-1}du$$
$$= \frac{1}{\beta}m^{\frac{1}{q}-\frac{\alpha}{\beta q}+\frac{\alpha}{\beta}-\alpha\gamma}\int_0^{\infty}\frac{u^{\frac{1}{\beta}-\frac{1}{\beta q}-1}}{(1+u)^{\gamma}}du\cdot$$

Using definition of Beta function and $\frac{1}{p}+\frac{1}{q} = 1$, we have

$$w_1(m) \leqslant \frac{1}{\beta}m^{\frac{\beta p-\beta+\alpha-\alpha\beta\gamma p}{\beta p}}B\left(\frac{1}{\beta p},\gamma-\frac{1}{\beta p}\right).$$

Hence the lemma is proved. □

Lemma 2.5. *For* $\beta \geqslant 0, \alpha > \frac{1}{\gamma q}, \gamma > 0$ *and* $p > 1$ *such that* $\frac{1}{p}+\frac{1}{q} = 1, p > 1, q > 1, \frac{1}{p}+\frac{1}{q} = 1$, *define the weight coefficient* $w_2(n)$ *as:*

$$w_2(n) = \sum_{m=1}^{\infty}\frac{1}{(m^{\alpha}+n^{\beta})^{\gamma}}\left(\frac{n}{m}\right)^{\frac{1}{p}}. \tag{2.15}$$

Then we get

$$w_2(n) \leqslant \frac{1}{\alpha}n^{\frac{\alpha q-\alpha+\beta-\alpha\beta\gamma q}{\alpha q}}B\left(\frac{1}{\alpha q},\gamma-\frac{1}{\alpha q}\right), \tag{2.16}$$

where $B(a,b)$ *is the Beta function,* $a > 0$ *and* $b > 0$.

Proof. The proof is similar to the proof of Lemma 2.4, so it is omitted. □

Theorem 2.6. *If* $(p > 1), \alpha \geqslant \frac{1}{\gamma q}$ *and* $\beta \geqslant \frac{1}{\gamma p}$ *such that* $\frac{1}{p}+\frac{1}{q} = 1$, *and* $f(x) \geqslant 0$, $g(y) \geqslant 0$, *satisfy that* $0 < \sum_{m=1}^{\infty} m^{\frac{\beta p-\beta+\alpha-\alpha\beta\gamma p}{\beta p}}a_m^p < \infty$ *and* $< \sum_{n=1}^{\infty}n^{\frac{\alpha q-\alpha+\beta-\alpha\beta\gamma q}{\alpha q}}b_n^q < \infty\cdot$

Then for $(\gamma > 0)$

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m^{\alpha}+n^{\beta})^{\gamma}} \leqslant \left(\frac{1}{\beta}\right)^{\frac{1}{p}}\left(\frac{1}{\alpha}\right)^{\frac{1}{q}}\left(B\left(\frac{1}{\beta p},\gamma-\frac{1}{\beta p}\right)\right)^{\frac{1}{p}}\left(B\left(\frac{1}{\alpha q},\gamma-\frac{1}{\alpha q}\right)\right)^{\frac{1}{q}}$$
$$\times\left(\sum_{m=1}^{\infty}m^{\frac{\beta(p-\alpha\gamma p-1)+\alpha}{\beta p}}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}n^{\frac{\alpha(q-\beta\gamma q-1)+\beta}{\alpha q}}b_n^q\right)^{\frac{1}{q}}. \tag{2.17}$$

Proof. Put the left hand side of the inequality (2.17) in the form:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m^{\alpha}+n^{\beta})^{\gamma}} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m}{(m^{\alpha}+n^{\beta})^{\frac{\gamma}{p}}}\left(\frac{m}{n}\right)^{\frac{1}{pq}}\frac{b_n}{(m^{\alpha}+n^{\beta})^{\frac{\gamma}{q}}}\left(\frac{n}{m}\right)^{\frac{1}{pq}}.$$

Applying Hölder's inequality to get the right hand side of the above inequality as follows:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m^{\alpha}+n^{\beta})^{\gamma}} \leqslant \left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m^p}{(m^{\alpha}+n^{\beta})^{\gamma}}\left(\frac{m}{n}\right)^{\frac{1}{q}}\right)^{\frac{1}{p}}$$
$$\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{b_n^q}{(m^{\alpha}+n^{\beta})^{\gamma}}\left(\frac{n}{m}\right)^{\frac{1}{p}}\right)^{\frac{1}{q}} = \left(\sum_{m=1}^{\infty}a_m^p\left(\sum_{n=1}^{\infty}\frac{1}{(m^{\alpha}+n^{\beta})^{\gamma}}\left(\frac{m}{n}\right)^{\frac{1}{q}}\right)\right)^{\frac{1}{p}}$$
$$\left(\sum_{n=1}^{\infty}b_n^q\left(\sum_{m=1}^{\infty}\frac{1}{(m^{\alpha}+n^{\beta})^{\gamma}}\left(\frac{n}{m}\right)^{\frac{1}{p}}\right)\right)^{\frac{1}{q}}. \tag{2.18}$$

By (2.13), (2.15) and (2.18), we get

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m^{\alpha}+n^{\beta})^{\gamma}} \leqslant \left(\sum_{m=1}^{\infty}a_m^p w_1(m)\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}b_n^q w_2(n)\right)^{\frac{1}{q}}. \tag{2.19}$$

Substituting (2.14) and (2.16) of Lemmas 2.4 and 2.5 in (2.19), we obtain

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m^{\alpha}+n^{\beta})^{\gamma}} \leqslant \left(\sum_{m=1}^{\infty}a_m^p\frac{1}{\beta}m^{\frac{\beta p-\beta+\alpha-\alpha\beta\gamma p}{\beta p}}B\left(\frac{1}{\beta p},\gamma-\frac{1}{\beta p}\right)\right)^{\frac{1}{p}}$$
$$\times\left(\sum_{n=1}^{\infty}b_n^q\frac{1}{\alpha}n^{\frac{\alpha q-\alpha+\beta-\alpha\beta\gamma q}{\alpha q}}B\left(\frac{1}{\alpha q},\gamma-\frac{1}{\alpha q}\right)\right)^{\frac{1}{q}}.$$

Then

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m^{\alpha}+n^{\beta})^{\gamma}} \leqslant \left(\frac{1}{\beta}\right)^{\frac{1}{p}}\left(\frac{1}{\alpha}\right)^{\frac{1}{q}}\left(B\left(\frac{1}{\beta p},\gamma-\frac{1}{\beta p}\right)\right)^{\frac{1}{p}}\left(B\left(\frac{1}{\alpha q},\gamma-\frac{1}{\alpha q}\right)\right)^{\frac{1}{q}}$$
$$\times\left(\sum_{m=1}^{\infty}m^{\frac{\beta(p-\alpha\gamma p-1)+\alpha}{\beta p}}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}n^{\frac{\alpha(q-\beta\gamma q-1)+\beta}{\alpha q}}b_n^q\right)^{\frac{1}{q}}.$$

This completes the proof. □

Now, we discuss some special values for the parameters inequality (2.17) in order to obtain some known inequalities as special cases from our result.

Remark 2.2.

1. Let $p = q = 2$ in (2.17), then we get

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m^{\alpha}+n^{\beta})^{\gamma}} \leqslant \left(\frac{1}{\alpha\beta}\right)^{\frac{1}{2}}\left(B\left(\frac{1}{2\beta},\gamma-\frac{1}{2\beta}\right)\right)^{\frac{1}{2}}\left(B\left(\frac{1}{2\alpha},\gamma-\frac{1}{2\alpha}\right)\right)^{\frac{1}{2}}$$
$$\times\left(\sum_{m=1}^{\infty}m^{\frac{\beta+\alpha-2\alpha\beta\gamma}{2\beta}}a_m^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty}n^{\frac{\alpha+\beta-2\alpha\beta\gamma}{2\alpha}}b_n^2\right)^{\frac{1}{2}}.$$

2. Let $\alpha = \beta = 1$ in (2.17), then we obtain

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m+n)^{\gamma}} \leqslant \left(B\left(\frac{1}{p},\gamma-\frac{1}{p}\right)\right)^{\frac{1}{p}}\left(B\left(\frac{1}{q},\gamma-\frac{1}{q}\right)\right)^{\frac{1}{q}} \left(\sum_{m=1}^{\infty} m^{1-\gamma}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} n^{1-\gamma}b_n^q\right)^{\frac{1}{q}}. \quad (2.20)$$

3. Let $p = q = 2$ in (2.20), then we get

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m+n)^{\gamma}} \leqslant \left(B\left(\frac{1}{2},\gamma-\frac{1}{2}\right)\right)\left(\sum_{m=1}^{\infty} m^{1-\gamma}a_m^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty} n^{1-\gamma}b_n^2\right)^{\frac{1}{2}} = \frac{\sqrt{\pi}\,\Gamma\left(\gamma-\frac{1}{2}\right)}{\Gamma(\gamma)}\left(\sum_{m=1}^{\infty} m^{1-\gamma}a_m^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty} n^{1-\gamma}b_n^2\right)^{\frac{1}{2}}. \quad (2.21)$$

4. Let $\gamma = 1$ in (2.21), then we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} \leqslant \pi\left(\sum_{m=1}^{\infty} a_m^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty} b_n^2\right)^{\frac{1}{2}},$$

which is Hilbert's double series inequality.

5. Let $\gamma = 2$ in (2.21), then we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m+n)^2} \leqslant \frac{\pi}{2}\left(\sum_{m=1}^{\infty} m^{-1}a_m^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty} n^{-1}b_n^2\right)^{\frac{1}{2}}.$$

6. Let $\alpha = \beta = 2$ in (2.17), then we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m^2+n^2)^{\gamma}} \leqslant \frac{1}{2}\left(B\left(\frac{1}{2p},\gamma-\frac{1}{2p}\right)\right)^{\frac{1}{p}}\left(B\left(\frac{1}{2q},\gamma-\frac{1}{2q}\right)\right)^{\frac{1}{q}} \times\left(\sum_{m=1}^{\infty} m^{1-2\gamma}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} n^{1-2\gamma}b_n^q\right)^{\frac{1}{q}}. \quad (2.22)$$

7. Let $p = q = 2$ in (2.22), then we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m^2+n^2)^{\gamma}} \leqslant \frac{1}{2}\left(B\left(\frac{1}{4},\gamma-\frac{1}{4}\right)\right)\left(\sum_{m=1}^{\infty} m^{1-2\gamma}a_m^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty} n^{1-2\gamma}b_n^2\right)^{\frac{1}{2}}. \quad (2.23)$$

8. Let $\gamma = 1$ in (2.23), then we get

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m^2+n^2)} \leqslant \frac{\pi}{\sqrt{2}}\left(\sum_{m=1}^{\infty} m^{-1}a_m^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty} n^{-1}b_n^2\right)^{\frac{1}{2}}.$$

9. Let $\alpha = \beta = \mu$ in (2.17), then we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m^{\mu}+n^{\mu})^{\gamma}} \leqslant \left(\frac{1}{\mu}\right)\left(B\left(\frac{1}{\mu p},\gamma-\frac{1}{\mu p}\right)\right)^{\frac{1}{p}} \times\left(B\left(\frac{1}{\mu q},\gamma-\frac{1}{\mu q}\right)\right)^{\frac{1}{q}}\left(\sum_{m=1}^{\infty} m^{1-\mu\gamma}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} n^{1-\mu\gamma}b_n^q\right)^{\frac{1}{q}}. \quad (2.24)$$

10. Let $p = q = 2$ in (2.24), then we find

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m^{\mu}+n^{\mu})^{\gamma}} \leqslant \left(\frac{1}{\mu}\right)\left(B\left(\frac{1}{2\mu},\gamma-\frac{1}{2\mu}\right)\right)\left(\sum_{m=1}^{\infty} m^{1-\mu\gamma}a_m^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty} n^{1-\mu\gamma}b_n^2\right)^{\frac{1}{2}}. \quad (2.25)$$

11. Let $\gamma = 1$ in (2.25), then we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m^{\mu}+n^{\mu}} \leqslant \frac{\pi}{\mu\sin\frac{\pi}{2\mu}}\left(\sum_{m=1}^{\infty} m^{1-\mu}a_m^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty} n^{1-\mu}b_n^2\right)^{\frac{1}{2}}.$$

12. Let $\gamma = 1$ in (2.17), then we get

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m^{\alpha}+n^{\beta}} \leqslant \frac{\pi}{\left(\beta\sin\frac{\pi}{\beta p}\right)^{\frac{1}{p}}\left(\alpha\sin\frac{\pi}{\alpha q}\right)^{\frac{1}{q}}}\left(\sum_{m=1}^{\infty} m^{\frac{\beta p-\beta+\alpha-\alpha\beta p}{\beta p}}a_m^p\right)^{\frac{1}{p}} \times\left(\sum_{n=1}^{\infty} n^{\frac{\alpha q-\alpha+\beta-\alpha\beta q}{\alpha q}}b_n^q\right)^{\frac{1}{q}}.$$

13. Let $\alpha = 1$ in (2.17), then we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m+n^{\beta})^{\gamma}} \leqslant \left(\frac{1}{\beta}\right)^{\frac{1}{p}}\left(B(\frac{1}{\beta p},\gamma-\frac{1}{\beta p})\right)^{\frac{1}{p}}\left(B\left(\frac{1}{q},\gamma-\frac{1}{q}\right)\right)^{\frac{1}{q}} \times\left(\sum_{m=1}^{\infty} m^{\frac{\beta p-\beta-\beta\gamma p+1}{\beta p}}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} n^{\frac{q+\beta-\beta\gamma q-1}{q}}b_n^q\right)^{\frac{1}{q}}. \quad (2.26)$$

14. Let $p = q = 2$ in (2.26), then we get

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{(m+n^{\beta})^{\gamma}} \leqslant \left(\frac{1}{\beta}\right)^{\frac{1}{2}}\left(B\left(\frac{1}{2\beta},\gamma-\frac{1}{2\beta}\right)\right)^{\frac{1}{2}}\left(B\left(\frac{1}{2},\gamma-\frac{1}{2}\right)\right)^{\frac{1}{2}} \times\left(\sum_{m=1}^{\infty} m^{\frac{\beta-2\beta\gamma+1}{2\beta}}a_m^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty} n^{\frac{\beta-2\beta\gamma+1}{2}}b_n^2\right)^{\frac{1}{2}}. \quad (2.27)$$

15. Let $\gamma = 1$ in (2.27), then we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n^{\beta}} \leqslant \frac{\pi}{\left(\beta\sin\frac{\pi}{2\beta}\right)^{\frac{1}{2}}}\left(\sum_{m=1}^{\infty} m^{\frac{1-\beta}{2\beta}}a_m^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty} n^{\frac{1-\beta}{2}}b_n^2\right)^{\frac{1}{2}}. \quad (2.28)$$

16. Let $\beta = 2$ in (2.28), then we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n^2} \leqslant \frac{\pi}{(2)^{\frac{1}{4}}}\left(\sum_{m=1}^{\infty} m^{\frac{-1}{4}}a_m^2\right)^{\frac{1}{2}}\left(\sum_{n=1}^{\infty} n^{\frac{-1}{2}}b_n^2\right)^{\frac{1}{2}}.$$

By introducing some parameters, a new form of Hardy–Hilbert's inequality is given as follows:

Theorem 2.7. *If $a, b, c > 0$, $0 < \alpha < pq$, $0 < \beta < qr$, $0 < \gamma < pr$ and $(p > 1)$, such that $\frac{1}{p}+\frac{1}{q}+\frac{1}{r} = 1$. Also, if $0 < \sum_0^{\infty} m^{\frac{\alpha}{\beta}+\frac{\alpha}{\gamma}-\alpha\lambda}a_m^p < \infty, 0 < \sum_0^{\infty} n^{\frac{\beta}{\gamma}+\frac{\beta}{\alpha}-\beta\lambda}b_n^q < \infty$ and $0 < \sum_0^{\infty} t^{\frac{\gamma}{\alpha}+\frac{\gamma}{\beta}-\gamma\lambda}c_t^r < \infty$. Then, for any $\lambda > \max\left\{\frac{1}{\beta}+\frac{1}{\gamma}-\frac{1}{qr},\frac{1}{\gamma}+\frac{1}{\alpha}-\frac{1}{pr},\frac{1}{\alpha}+\frac{1}{\beta}-\frac{1}{pq}\right\}$,*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{t=1}^{\infty}\frac{a_m b_n c_t}{(am^{\alpha}+bn^{\beta}+ct^{\gamma})^{\lambda}} \leqslant \left(\frac{a^{\frac{1}{\beta}+\frac{1}{\gamma}-\lambda}}{\beta b^{\frac{1}{\beta}}\gamma c^{\frac{1}{\gamma}}}\right)^{\frac{1}{p}}\left(\frac{b^{\frac{1}{\gamma}+\frac{1}{\alpha}-\lambda}}{\alpha a^{\frac{1}{\alpha}}\gamma c^{\frac{1}{\gamma}}}\right)^{\frac{1}{q}} \left(\frac{c^{\frac{1}{\alpha}+\frac{1}{\beta}-\lambda}}{\alpha a^{\frac{1}{\alpha}}\beta b^{\frac{1}{\beta}}}\right)^{\frac{1}{r}} \times\left(B\left(\frac{1}{\beta}-\frac{1}{qr},\lambda-\frac{1}{\beta}+\frac{1}{qr}\right)B\left(\frac{1}{qr}-\frac{1}{\beta}-\frac{1}{\gamma}+\lambda,\frac{1}{\gamma}\right)\right)^{\frac{1}{p}}$$
$$\times\left(B\left(\frac{1}{\gamma}-\frac{1}{pr},\lambda-\frac{1}{\gamma}+\frac{1}{pr}\right)B\left(\frac{1}{pr}-\frac{1}{\gamma}-\frac{1}{\alpha}+\lambda,\frac{1}{\alpha}\right)\right)^{\frac{1}{q}} \times\left(B\left(\frac{1}{\alpha}-\frac{1}{pq},\lambda-\frac{1}{\alpha}+\frac{1}{pq}\right)B\left(\frac{1}{pq}-\frac{1}{\alpha}-\frac{1}{\beta}+\lambda,\frac{1}{\beta}\right)\right)^{\frac{1}{r}}$$
$$\times\left(\sum_{m=1}^{\infty} m^{\frac{\alpha}{\beta}+\frac{\alpha}{\gamma}-\alpha\lambda}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty} n^{\frac{\beta}{\gamma}+\frac{\beta}{\alpha}-\beta\lambda}b_n^q\right)^{\frac{1}{q}}\left(\sum_{t=1}^{\infty} t^{\frac{\gamma}{\alpha}+\frac{\gamma}{\beta}-\gamma\lambda}c_t^r\right)^{\frac{1}{r}}. \quad (2.29)$$

Proof. Apply Hölder's inequality to estimate the right hand side of the above inequality as follows:

$$\begin{aligned}
&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{t=1}^{\infty}\frac{a_m b_n c_t}{(am^{\alpha}+bn^{\beta}+ct^{\gamma})^{\lambda}}\\
&=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{t=1}^{\infty}\frac{a_m}{(am^{\alpha}+bn^{\beta}+ct^{\gamma})^{\frac{\lambda}{p}}}\left(\frac{am^{\alpha}}{bn^{\beta}}\right)^{\frac{1}{pqr}}\\
&\quad\times\frac{b_n}{(am^{\alpha}+bn^{\beta}+ct^{\gamma})^{\frac{\lambda}{q}}}\left(\frac{bn^{\beta}}{ct^{\gamma}}\right)^{\frac{1}{pqr}}\frac{c_t}{(am^{\alpha}+bn^{\beta}+ct^{\gamma})^{\frac{\lambda}{r}}}\left(\frac{ct^{\gamma}}{am^{\alpha}}\right)^{\frac{1}{pqr}}.\\
&\leqslant\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{t=1}^{\infty}\frac{a_m^p}{(am^{\alpha}+bn^{\beta}+ct^{\gamma})^{\lambda}}\left(\frac{am^{\alpha}}{bn^{\beta}}\right)^{\frac{1}{qr}}\right)^{\frac{1}{p}}\\
&\quad\times\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{t=1}^{\infty}\frac{b_n^q}{(am^{\alpha}+bn^{\beta}+ct^{\gamma})^{\lambda}}\left(\frac{bn^{\beta}}{ct^{\gamma}}\right)^{\frac{1}{pr}}\right)^{\frac{1}{q}}\\
&\quad\times\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{t=1}^{\infty}\frac{c_t^r}{(am^{\alpha}+bn^{\beta}+ct^{\gamma})^{\lambda}}\left(\frac{ct^{\gamma}}{am^{\alpha}}\right)^{\frac{1}{pq}}\right)^{\frac{1}{r}}\\
&=S^{1/p}T^{1/q}R^{1/r}.
\end{aligned}\tag{2.30}$$

Such that here

$$S=\sum_{m=1}^{\infty}(am^{\alpha})^{\frac{1}{qr}}a_m^p\sum_{t=1}^{\infty}\sum_{n=1}^{\infty}\frac{(bn^{\beta})^{-\frac{1}{qr}}}{(am^{\alpha}+bn^{\beta}+ct^{\gamma})^{\lambda}}.$$

Since $\sum_{n=1}^{\infty}f(n)\leqslant\int_0^{\infty}f(y)dy$, then

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{(bn^{\beta})^{-\frac{1}{qr}}}{(am^{\alpha}+bn^{\beta}+ct^{\gamma})^{\lambda}}&\leqslant\int_0^{\infty}\frac{(by^{\beta})^{-\frac{1}{qr}}}{(am^{\alpha}+by^{\beta}+ct^{\gamma})^{\lambda}}dy\\
&=\frac{1}{(am^{\alpha}+ct^{\gamma})^{\lambda+\frac{1}{qr}}}\int_0^{\infty}\frac{\left(\frac{by^{\beta}}{am^{\alpha}+ct^{\gamma}}\right)^{-\frac{1}{qr}}}{\left(1+\frac{by^{\beta}}{am^{\alpha}+ct^{\gamma}}\right)^{\lambda}}dy.
\end{aligned}$$

By putting $u=\frac{by^{\beta}}{am^{\alpha}+ct^{\gamma}}$, then we find $dy=\frac{1}{\beta}\left(\left(\frac{am^{\alpha}+ct^{\gamma}}{b}\right)u\right)^{\frac{1}{\beta}-1}\left(\frac{am^{\alpha}+ct^{\gamma}}{b}\right)du$ and $0\leqslant u<\infty$. Then

$$\int_0^{\infty}\frac{(by^{\beta})^{-\frac{1}{qr}}}{(am^{\alpha}+by^{\beta}+ct^{\gamma})^{\lambda}}dy=\frac{1}{\beta b^{\frac{1}{\beta}}(am^{\alpha}+ct^{\gamma})^{\lambda+\frac{1}{qr}-\frac{1}{\beta}}}\int_0^{\infty}\frac{u^{\frac{1}{\beta}-\frac{1}{qr}-1}}{(1+u)^{\lambda}}du.$$

From definition of Beta function, we get

$$\int_0^{\infty}\frac{(by^{\beta})^{-\frac{1}{qr}}}{(am^{\alpha}+by^{\beta}+ct^{\gamma})^{\lambda}}dy=\frac{1}{\beta b^{\frac{1}{\beta}}(am^{\alpha}+ct^{\gamma})^{\lambda+\frac{1}{qr}-\frac{1}{\beta}}}B\left(\frac{1}{\beta}-\frac{1}{qr},\lambda-\frac{1}{\beta}+\frac{1}{qr}\right).$$

Therefore, we have

$$S=\frac{1}{\beta b^{\frac{1}{\beta}}}B\left(\frac{1}{\beta}-\frac{1}{qr},\lambda-\frac{1}{\beta}+\frac{1}{qr}\right)\sum_{m=1}^{\infty}(am^{\alpha})^{\frac{1}{qr}}a_m^p\sum_{t=1}^{\infty}\frac{1}{(am^{\alpha}+ct^{\gamma})^{\lambda+\frac{1}{qr}-\frac{1}{\beta}}}.$$

Now,

$$\begin{aligned}
\sum_{t=1}^{\infty}\frac{1}{(am^{\alpha}+ct^{\gamma})^{\lambda+\frac{1}{qr}-\frac{1}{\beta}}}&\leqslant\int_0^{\infty}\frac{1}{(am^{\alpha}+cz^{\gamma})^{\lambda+\frac{1}{qr}-\frac{1}{\beta}}}dz\\
&=\int_0^{\infty}\frac{(cz^{\gamma})^{\frac{1}{\beta}-\frac{1}{qr}-\lambda}}{(1+\frac{am^{\alpha}}{cz^{\gamma}})^{\lambda+\frac{1}{qr}-\frac{1}{\beta}}}dz.
\end{aligned}$$

Using the change in variables $u=\frac{am^{\alpha}}{cz^{\gamma}},dz=-\frac{1}{\gamma}\left(\frac{am^{\alpha}}{c}u^{-1}\right)^{\frac{1}{\gamma}-1}\frac{am^{\alpha}}{c}u^{-2}du.$

Hence

$$\int_0^{\infty}\frac{1}{(am^{\alpha}+cz^{\gamma})^{\lambda+\frac{1}{qr}-\frac{1}{\beta}}}dz=\frac{(am^{\alpha})^{\frac{1}{\beta}-\frac{1}{qr}+\frac{1}{\gamma}-\lambda}}{\gamma c^{\frac{1}{\gamma}}}\int_0^{\infty}\frac{u^{\frac{1}{qr}-\frac{1}{\beta}-\frac{1}{\gamma}+\lambda-1}}{(1+u)^{\lambda+\frac{1}{qr}-\frac{1}{\beta}}}du.$$

Using definition of Beta function, we have

$$\int_0^{\infty}\frac{1}{(ax^{\alpha}+cz^{\gamma})^{\lambda+\frac{1}{qr}-\frac{1}{\beta}}}dz=\frac{(am^{\alpha})^{\frac{1}{\beta}-\frac{1}{qr}+\frac{1}{\gamma}-\lambda}}{\gamma c^{\frac{1}{\gamma}}}B\left(\frac{1}{qr}-\frac{1}{\beta}-\frac{1}{\gamma}+\lambda,\frac{1}{\gamma}\right).$$

Then

$$S=\frac{a^{\frac{1}{\beta}+\frac{1}{\gamma}-\lambda}}{\beta b^{\frac{1}{\beta}}\gamma c^{\frac{1}{\gamma}}}B\left(\frac{1}{\beta}-\frac{1}{qr},\lambda-\frac{1}{\beta}+\frac{1}{qr}\right)B\left(\frac{1}{qr}-\frac{1}{\beta}-\frac{1}{\gamma}+\lambda,\frac{1}{\gamma}\right)\sum_{m=1}^{\infty}m^{\frac{\alpha}{\beta}+\frac{\alpha}{\gamma}-\alpha\lambda}a_m^p.$$

Similarly, we can write T and R as follows:

$$T=\frac{b^{\frac{1}{\gamma}+\frac{1}{\alpha}-\lambda}}{\alpha a^{\frac{1}{\alpha}}\gamma c^{\frac{1}{\gamma}}}B\left(\frac{1}{\gamma}-\frac{1}{pr},\lambda-\frac{1}{\gamma}+\frac{1}{pr}\right)B\left(\frac{1}{pr}-\frac{1}{\gamma}-\frac{1}{\alpha}+\lambda,\frac{1}{\alpha}\right)\sum_{n=1}^{\infty}n^{\frac{\beta}{\gamma}+\frac{\beta}{\alpha}-\beta\lambda}b_n^q.$$

$$R=\frac{c^{\frac{1}{\alpha}+\frac{1}{\beta}-\lambda}}{\alpha a^{\frac{1}{\alpha}}\beta b^{\frac{1}{\gamma}}}B\left(\frac{1}{\alpha}-\frac{1}{pq},\lambda-\frac{1}{\alpha}+\frac{1}{pq}\right)B\left(\frac{1}{pq}-\frac{1}{\alpha}-\frac{1}{\beta}+\lambda,\frac{1}{\beta}\right)\sum_{t=1}^{\infty}t^{\frac{\gamma}{\alpha}+\frac{\gamma}{\beta}-\gamma\lambda}c_t^r.$$

$$\begin{aligned}
\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{t=1}^{\infty}\frac{a_m b_n c_t}{(am^{\alpha}+bn^{\beta}+ct^{\gamma})^{\lambda}}&\leqslant\left(\frac{a^{\frac{1}{\beta}+\frac{1}{\gamma}-\lambda}}{\beta b^{\frac{1}{\beta}}\gamma c^{\frac{1}{\gamma}}}\right)^{\frac{1}{p}}\left(\frac{b^{\frac{1}{\gamma}+\frac{1}{\alpha}-\lambda}}{\alpha a^{\frac{1}{\alpha}}\gamma c^{\frac{1}{\gamma}}}\right)^{\frac{1}{q}}\left(\frac{c^{\frac{1}{\alpha}+\frac{1}{\beta}-\lambda}}{\alpha a^{\frac{1}{\alpha}}\beta b^{\frac{1}{\gamma}}}\right)^{\frac{1}{r}}\\
&\times\left(B\left(\tfrac{1}{\beta}-\tfrac{1}{qr},\lambda-\tfrac{1}{\beta}+\tfrac{1}{qr}\right)B\left(\tfrac{1}{qr}-\tfrac{1}{\beta}-\tfrac{1}{\gamma}+\lambda,\tfrac{1}{\gamma}\right)\right)^{\frac{1}{p}}\\
&\times\left(B\left(\tfrac{1}{\gamma}-\tfrac{1}{pr},\lambda-\tfrac{1}{\gamma}+\tfrac{1}{pr}\right)B\left(\tfrac{1}{pr}-\tfrac{1}{\gamma}-\tfrac{1}{\alpha}+\lambda,\tfrac{1}{\alpha}\right)\right)^{\frac{1}{q}}\\
&\times\left(B\left(\tfrac{1}{\alpha}-\tfrac{1}{pq},\lambda-\tfrac{1}{\alpha}+\tfrac{1}{pq}\right)B\left(\tfrac{1}{pq}-\tfrac{1}{\alpha}-\tfrac{1}{\beta}+\lambda,\tfrac{1}{\beta}\right)\right)^{\frac{1}{r}}\\
&\times\left(\sum_{m=1}^{\infty}m^{\frac{\alpha}{\beta}+\frac{\alpha}{\gamma}-\alpha\lambda}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}n^{\frac{\beta}{\gamma}+\frac{\beta}{\alpha}-\beta\lambda}b_n^q\right)^{\frac{1}{q}}\left(\sum_{t=1}^{\infty}t^{\frac{\gamma}{\alpha}+\frac{\gamma}{\beta}-\gamma\lambda}c_t^r\right)^{\frac{1}{r}}.
\end{aligned}$$

Or

$$\begin{aligned}
\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{t=1}^{\infty}\frac{a_m b_n c_t}{(am^{\alpha}+bn^{\beta}+ct^{\gamma})^{\lambda}}&\leqslant\frac{1}{\Gamma(\lambda)}\left(\frac{a^{\frac{1}{\beta}+\frac{1}{\gamma}-\lambda}}{\beta b^{\frac{1}{\beta}}\gamma c^{\frac{1}{\gamma}}}\right)^{\frac{1}{p}}\left(\frac{b^{\frac{1}{\gamma}+\frac{1}{\alpha}-\lambda}}{\alpha a^{\frac{1}{\alpha}}\gamma c^{\frac{1}{\gamma}}}\right)^{\frac{1}{q}}\left(\frac{c^{\frac{1}{\alpha}+\frac{1}{\beta}-\lambda}}{\alpha a^{\frac{1}{\alpha}}\beta b^{\frac{1}{\beta}}}\right)^{\frac{1}{r}}\\
&\times\left(\Gamma\left(\tfrac{1}{\gamma}\right)\Gamma\left(\tfrac{1}{\beta}-\tfrac{1}{qr}\right)\Gamma\left(\tfrac{1}{qr}-\tfrac{1}{\beta}-\tfrac{1}{\gamma}+\lambda\right)\right)^{\frac{1}{p}}\\
&\times\left(\Gamma\left(\tfrac{1}{\alpha}\right)\Gamma\left(\tfrac{1}{\gamma}-\tfrac{1}{pr}\right)\Gamma\left(\tfrac{1}{pr}-\tfrac{1}{\gamma}-\tfrac{1}{\alpha}+\lambda\right)\right)^{\frac{1}{q}}\\
&\times\left(\Gamma\left(\tfrac{1}{\beta}\right)\Gamma\left(\tfrac{1}{\alpha}-\tfrac{1}{pq}\right)\Gamma\left(\tfrac{1}{pq}-\tfrac{1}{\alpha}-\tfrac{1}{\beta}+\lambda\right)\right)^{\frac{1}{r}}\\
&\times\left(\sum_{m=1}^{\infty}m^{\frac{\alpha}{\beta}+\frac{\alpha}{\gamma}-\alpha\lambda}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}n^{\frac{\beta}{\gamma}+\frac{\beta}{\alpha}-\beta\lambda}b_n^q\right)^{\frac{1}{q}}\left(\sum_{t=1}^{\infty}t^{\frac{\gamma}{\alpha}+\frac{\gamma}{\beta}-\gamma\lambda}c_t^r\right)^{\frac{1}{r}}.
\end{aligned}\tag{2.31}$$

This completes the proof. □

Remark 2.3.

1. Setting $p=q=r=3$ in (2.29), we get

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{t=1}^{\infty}\frac{a_m b_n c_t}{(am^{\alpha}+bn^{\beta}+ct^{\gamma})^{\lambda}} \leqslant \frac{1}{\Gamma(\lambda)}\left(\frac{a^{\frac{1}{\beta}+\frac{1}{\gamma}-\lambda}}{\beta b^{\frac{1}{\beta}}\gamma c^{\frac{1}{\gamma}}}\right)^{\frac{1}{3}}\left(\frac{b^{\frac{1}{\gamma}+\frac{1}{\alpha}-\lambda}}{\alpha a^{\frac{1}{\alpha}}\gamma c^{\frac{1}{\gamma}}}\right)^{\frac{1}{3}}\left(\frac{c^{\frac{1}{\alpha}+\frac{1}{\beta}-\lambda}}{\alpha a^{\frac{1}{\alpha}}\beta b^{\frac{1}{\beta}}}\right)^{\frac{1}{3}}$$
$$\times\left(\Gamma\left(\tfrac{1}{\gamma}\right)\Gamma\left(\tfrac{1}{\beta}-\tfrac{1}{9}\right)\Gamma\left(\tfrac{1}{9}-\tfrac{1}{\beta}-\tfrac{1}{\gamma}+\lambda\right)\right)^{\frac{1}{3}}$$
$$\times\left(\Gamma\left(\tfrac{1}{\alpha}\right)\Gamma\left(\tfrac{1}{\gamma}-\tfrac{1}{9}\right)\Gamma\left(\tfrac{1}{9}-\tfrac{1}{\gamma}-\tfrac{1}{\alpha}+\lambda\right)\right)^{\frac{1}{3}}$$
$$\times\left(\Gamma\left(\tfrac{1}{\beta}\right)\Gamma\left(\tfrac{1}{\alpha}-\tfrac{1}{9}\right)\Gamma\left(\tfrac{1}{9}-\tfrac{1}{\alpha}-\tfrac{1}{\beta}+\lambda\right)\right)^{\frac{1}{3}}$$
$$\times\left(\sum_{m=1}^{\infty}m^{\frac{\alpha}{\beta}+\frac{\alpha}{\gamma}-\alpha\lambda}a_m^3\right)^{\frac{1}{3}}\left(\sum_{n=1}^{\infty}n^{\frac{\beta}{\gamma}+\frac{\beta}{\alpha}-\beta\lambda}b_n^3\right)^{\frac{1}{3}}\left(\sum_{t=1}^{\infty}t^{\frac{\gamma}{\alpha}+\frac{\gamma}{\beta}-\gamma\lambda}c_t^3\right)^{\frac{1}{3}}. \quad (2.32)$$

2. Let $\alpha = \beta = \gamma = 1$ in (2.29), then we get

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{t=1}^{\infty}\frac{a_m b_n c_t}{(am+bn+ct)^{\lambda}}$$
$$\leqslant a^{\frac{3-\lambda}{p}-1}b^{\frac{3-\lambda}{q}-1}c^{\frac{3-\lambda}{r}-1}\left(B\left(1-\frac{1}{qr},\lambda-1+\frac{1}{qr}\right)B\left(\frac{1}{qr}+\lambda-2,1\right)\right)^{\frac{1}{p}},$$
$$\times\left(B\left(1-\frac{1}{pr},\lambda-1+\frac{1}{pr}\right)B\left(\frac{1}{pr}+\lambda-2,1\right)\right)^{\frac{1}{q}}$$
$$\times\left(B\left(1-\frac{1}{pq},\lambda-1+\frac{1}{pq}\right)B\left(\frac{1}{pq}+\lambda-2,1\right)\right)^{\frac{1}{r}}$$
$$\times\left(\sum_{m=1}^{\infty}m^{2-\lambda}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}n^{2-\lambda}b_n^q\right)^{\frac{1}{q}}\left(\sum_{t=1}^{\infty}t^{2-\lambda}c_t^r\right)^{\frac{1}{r}}. \quad (2.33)$$

3. For $a = b = c = \lambda = 1$ in (2.33), one has the following inequality:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{t=1}^{\infty}\frac{a_m b_n c_t}{m+n+t}\leqslant\frac{\pi(qr)^{\frac{1}{p}}(pr)^{\frac{1}{q}}(pq)^{\frac{1}{r}}}{\left((1-qr)\sin\frac{\pi}{qr}\right)^{\frac{1}{p}}\left((1-pr)\sin\frac{\pi}{pr}\right)^{\frac{1}{q}}\left((1-pq)\sin\frac{\pi}{pq}\right)^{\frac{1}{r}}}$$
$$\times\left(\sum_{m=1}^{\infty}ma_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}nb_n^q\right)^{\frac{1}{q}}\left(\sum_{t=1}^{\infty}tc_t^r\right)^{\frac{1}{r}}.$$

4. Let $\lambda = 1$ in (2.29), then we get

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{t=1}^{\infty}\frac{a_m b_n c_t}{am^{\alpha}+bn^{\beta}+ct^{\gamma}}\leqslant\left(\frac{a^{\frac{1}{\beta}+\frac{1}{\gamma}-1}}{\beta b^{\frac{1}{\beta}}\gamma c^{\frac{1}{\gamma}}}\right)^{\frac{1}{p}}\left(\frac{b^{\frac{1}{\gamma}+\frac{1}{\alpha}-1}}{\alpha a^{\frac{1}{\alpha}}\gamma c^{\frac{1}{\gamma}}}\right)^{\frac{1}{q}}\left(\frac{c^{\frac{1}{\alpha}+\frac{1}{\beta}-1}}{\alpha a^{\frac{1}{\alpha}}\beta b^{\frac{1}{\beta}}}\right)^{\frac{1}{r}}$$
$$\times\left(B\left(\tfrac{1}{\beta}-\tfrac{1}{qr},1-\tfrac{1}{\beta}+\tfrac{1}{qr}\right)B\left(\tfrac{1}{qr}-\tfrac{1}{\beta}-\tfrac{1}{\gamma}+1,\tfrac{1}{\gamma}\right)\right)^{\frac{1}{p}}$$
$$\times\left(B\left(\tfrac{1}{\gamma}-\tfrac{1}{pr},1-\tfrac{1}{\gamma}+\tfrac{1}{pr}\right)B\left(\tfrac{1}{pr}-\tfrac{1}{\gamma}-\tfrac{1}{\alpha}+1,\tfrac{1}{\alpha}\right)\right)^{\frac{1}{q}}$$
$$\times\left(B\left(\tfrac{1}{\alpha}-\tfrac{1}{pq},1-\tfrac{1}{\alpha}+\tfrac{1}{pq}\right)B\left(\tfrac{1}{pq}-\tfrac{1}{\alpha}-\tfrac{1}{\beta}+1,\tfrac{1}{\beta}\right)\right)^{\frac{1}{r}}$$
$$\times\left(\sum_{m=1}^{\infty}m^{\frac{\alpha}{\beta}+\frac{\alpha}{\gamma}-\alpha}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}y^{\frac{\beta}{\gamma}+\frac{\beta}{\alpha}-\beta}b_n^q\right)^{\frac{1}{q}}\left(\sum_{t=1}^{\infty}t^{\frac{\gamma}{\alpha}+\frac{\gamma}{\beta}-\gamma}c_t^r\right)^{\frac{1}{r}}.$$

5. For $a = b = c = 1$ in (2.29), we obtain

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{t=1}^{\infty}\frac{a_m b_n c_t}{(m^{\alpha}+n^{\beta}+t^{\gamma})^{\lambda}}\leqslant\left(\frac{1}{\beta\gamma}\right)^{\frac{1}{p}}\left(\frac{1}{\alpha\gamma}\right)^{\frac{1}{q}}\left(\frac{1}{\alpha\beta}\right)^{\frac{1}{r}}$$
$$\times\left(B\left(\tfrac{1}{\beta}-\tfrac{1}{qr},\lambda-\tfrac{1}{\beta}+\tfrac{1}{qr}\right)B\left(\tfrac{1}{qr}-\tfrac{1}{\beta}-\tfrac{1}{\gamma}+\lambda,\tfrac{1}{\gamma}\right)\right)^{\frac{1}{p}}$$
$$\times\left(B\left(\tfrac{1}{\gamma}-\tfrac{1}{pr},\lambda-\tfrac{1}{\gamma}+\tfrac{1}{pr}\right)B\left(\tfrac{1}{pr}-\tfrac{1}{\gamma}-\tfrac{1}{\alpha}+\lambda,\tfrac{1}{\alpha}\right)\right)^{\frac{1}{q}} \quad (2.34)$$
$$\times\left(B\left(\tfrac{1}{\alpha}-\tfrac{1}{pq},\lambda-\tfrac{1}{\alpha}+\tfrac{1}{pq}\right)B\left(\tfrac{1}{pq}-\tfrac{1}{\alpha}-\tfrac{1}{\beta}+\lambda,\tfrac{1}{\beta}+\lambda,\tfrac{1}{\beta}\right)\right)^{\frac{1}{r}}$$
$$\times\left(\sum_{m=1}^{\infty}m^{\frac{\alpha}{\beta}+\frac{\alpha}{\gamma}-\alpha\lambda}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}n^{\frac{\beta}{\gamma}+\frac{\beta}{\alpha}-\beta\lambda}\right)^{\frac{1}{q}}\left(\sum_{t=1}^{\infty}t^{\frac{\gamma}{\alpha}+\frac{\gamma}{\beta}-\gamma\lambda}c_t^r\right)^{\frac{1}{r}}.$$

6. Substituting $\lambda = 1$ in (2.34), then we obtain,

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{t=1}^{\infty}\frac{a_m b_n c_t}{m^{\alpha}+n^{\beta}+t^{\gamma}}$$
$$\leqslant\left(\frac{1}{\beta\gamma}\right)^{\frac{1}{p}}\left(\frac{1}{\alpha\gamma}\right)^{\frac{1}{q}}\left(\frac{1}{\alpha\beta}\right)^{\frac{1}{r}}\left(B\left(\frac{1}{\beta}-\frac{1}{qr},1-\frac{1}{\beta}+\frac{1}{qr}\right)B\left(\frac{1}{qr}-\frac{1}{\beta}-\frac{1}{\gamma}+1,\frac{1}{\gamma}\right)\right)^{\frac{1}{p}}$$
$$\times\left(B\left(\frac{1}{\gamma}-\frac{1}{pr},1-\frac{1}{\gamma}+\frac{1}{pr}\right)B\left(\frac{1}{pr}-\frac{1}{\gamma}-\frac{1}{\alpha}+1,\frac{1}{\alpha}\right)\right)^{\frac{1}{q}}$$
$$\times\left(B\left(\frac{1}{\alpha}-\frac{1}{pq},1-\frac{1}{\alpha}+\frac{1}{pq}\right)B\left(\frac{1}{pq}-\frac{1}{\alpha}-\frac{1}{\beta}+1,\frac{1}{\beta}\right)\right)^{\frac{1}{r}}$$
$$\times\left(\sum_{m=1}^{\infty}m^{\frac{\alpha}{\beta}+\frac{\alpha}{\gamma}-\alpha}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}n^{\frac{\beta}{\gamma}+\frac{\beta}{\alpha}-\beta}\right)^{\frac{1}{q}}\left(\sum_{t=1}^{\infty}t^{\frac{\gamma}{\alpha}+\frac{\gamma}{\beta}-\gamma}c_t^r\right)^{\frac{1}{r}}.$$

7. Let $\alpha = \beta = 1$ in (2.29), then we get

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{t=1}^{\infty}\frac{a_m b_n c_t}{(am+bn+ct^{\gamma})^{\lambda}}\leqslant\left(\frac{a^{1-\lambda+\frac{1}{\gamma}}}{b\gamma c^{\frac{1}{\gamma}}}\right)^{\frac{1}{p}}\left(\frac{b^{1-\lambda+\frac{1}{\gamma}}}{a\gamma c^{\frac{1}{\gamma}}}\right)^{\frac{1}{q}}\left(\frac{c^{2-\lambda}}{ab}\right)^{\frac{1}{r}}$$
$$\times\left(B\left(1-\tfrac{1}{qr},\lambda-1+\tfrac{1}{qr}\right)B\left(\tfrac{1}{qr}-\tfrac{1}{\gamma}-1+\lambda,\tfrac{1}{\gamma}\right)\right)^{\frac{1}{p}}$$
$$\times\left(B\left(\tfrac{1}{\gamma}-\tfrac{1}{pr},\lambda-\tfrac{1}{\gamma}+\tfrac{1}{pr}\right)B\left(\tfrac{1}{pr}-\tfrac{1}{\gamma}-1+\lambda,1\right)\right)^{\frac{1}{q}}$$
$$\times\left(B\left(1-\tfrac{1}{pq},\lambda-1+\tfrac{1}{pq}\right)B\left(\tfrac{1}{pq}-2+\lambda,1\right)\right)^{\frac{1}{r}}$$
$$\times\left(\sum_{m=1}^{\infty}m^{\frac{1}{\gamma}+1-\lambda}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}n^{\frac{1}{\gamma}+1-\lambda}b_n^q\right)^{\frac{1}{q}}\left(\sum_{t=1}^{\infty}t^{(2-\lambda)\gamma}c_t^r\right)^{\frac{1}{r}}. \quad (2.35)$$

8. For $\lambda = 2, \gamma = 2$ in (2.35), we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{t=1}^{\infty}\frac{a_m b_n c_t}{(am+bn+ct^2)^2}\leqslant 2^{\frac{1-r}{r}}a^{\frac{1-2p}{2p}}b^{\frac{1-2q}{2q}}c^{\frac{1-r}{2r}}\left(B\left(1-\tfrac{1}{qr},1+\tfrac{1}{qr}\right)B\left(\tfrac{1}{qr}+\tfrac{1}{2},\tfrac{1}{2}\right)\right)^{\frac{1}{p}}$$
$$\times\left(B\left(\tfrac{1}{2}-\tfrac{1}{pr},\tfrac{3}{2}+\tfrac{1}{pr}\right)B\left(\tfrac{1}{pr}+\tfrac{1}{2},1\right)\right)^{\frac{1}{q}}\left(B\left(1-\tfrac{1}{pq},1+\tfrac{1}{pq}\right)B\left(\tfrac{1}{pq},1\right)\right)^{\frac{1}{r}}$$
$$\times\left(\sum_{m=1}^{\infty}m^{-\frac{1}{2}}a_m^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}n^{-\frac{1}{2}}b_n^q\right)^{\frac{1}{q}}\left(\sum_{t=1}^{\infty}c_t^r\right)^{\frac{1}{r}}.$$

Acknowledgment

This work is supported by Deanship of Scientific Research, Taibah University, Kingdom of Saudi Arabia No. (3090-1434).

References

[1] G.H. Hardy, J.E. Littlewood, G. Polya, Inequalities, Cambridge Univ. Press, Cambridge, 1964.

[2] M. Krnic, J. Pecaric, Extension of Hilbert's inequality, J. Math. Anal. Appl. 324 (2006) 150–160.

[3] S.A.A. El-Marouf, Generalization of Hilbert inequality with some parameters, Math. Slovaca (2014) (in press).

[4] S.A.A. El-Marouf, On some generalization of Hilbert–Hardy type integral inequalities, Indian J. Sci. Technol. 6 (2) (2014) 4098–4111.

[5] B.G. Pachpatte, Inequalities similar to certain extensions of Hilbert's inequality, J. Math. Anal. Appl. 243 (2000) 217–227.
[6] Z. Xie, Z. Zheng, A Hilbert-type integral inequality whose kernel is a homogeneous form of degree -3, J. Math. Anal. Appl. 339 (2008) 324–331.
[7] B. Yang, T.M. Rassias, On The way of weight coefficient and research for the Hilbert-type inequalities, Math. Inequal. Appl. 6 (4) (2003) 625–658.
[8] B. Yang, On a relation between Hilbert's inequality and a Hilbert-type inequality, Appl. Math. Lett. 21 (2008) 483–488.
[9] L. Zhongxue, G. Mingzhe, L. Debnath, On new generalizations of the Hilbert integral inequality, J. Math. Anal. Appl. 326 (2007) 1452–1457.
[10] L.C. Andrews, Special Functions of mathematics for engineers, second ed., McGraw-Hill International Editions, 1985.