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REVIEW PAPER

On generalizing covering approximation space

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Abstract In this paper, we present the covering rough sets based on neighborhoods by approximation operations as a new type of extended covering rough set models. In fact, we have introduced generalizations to W. Zhu approaches (Zhu, 2007). Based on the notion of neighborhood induced from any binary relation, four different pairs of dual approximation operators are defined with their properties being discussed. The relationships among these operators are investigated. Finally, an interesting theorem to generate different topologies is provided. Comparisons between these topologies are discussed. In addition, several examples and counter examples to indicate counter connections are investigated.

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Contents

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1. Introduction

In order to extract useful information hidden in voluminous data, many methods in addition to classical logic have been proposed. These include fuzzy set theory [2], rough set theory [3], computing with words [4–7] and computational theory for linguistic dynamic systems [8]. Rough set theory, proposed by Pawlak in the early 1980s [3,9], is a mathematical tool to deal with uncertainty and incomplete information. Since then we have witnessed a systematic, world-wide growth of interest in rough set theory [10–26] and its applications [27–34].

Nowadays, it turns out that this approach is of fundamental importance to artificial intelligence and cognitive sciences, especially in the areas of data mining, machine learning, decision analysis, knowledge management, expert systems, and pattern recognition. Rough set theory bears on the assumption that some elements of a universe may be indiscernible in view of the available information about the elements. Thus, the indiscernibility relation is the starting point of rough set theory. Such a relation was first described by equivalence relation in the way that two elements are related by the relation if and only if they are indiscernible from each other. In this framework, a rough set is a formal approximation of a subset of the universe in terms of a pair of unions of equivalence classes which give the lower and upper approximations of the subset. However, the requirement of equivalence relation as the indiscernibility relation is too restrictive for many applications. In other words, many practical data sets cannot be handled well by classical rough sets. In light of this, equivalence relation has been generalized to characteristic relation [35–37] similarity relation [38], tolerance relation [39–42], and even arbitrary binary relation [43–49] in some extensions of the classical rough sets. Another approach is the relaxation of the partition arising from equivalence relation to a covering. The covering of a universe is used to construct the lower and upper approximations of any subset of the universe [11,15,19,25,50]. In the literature, several different types of covering-based rough sets have been proposed and investigated; see, for example, [1,23,26,51–54] and the bibliographies therein. It is well-known that coverings are a fundamental concept in topological spaces and play an important role in the study of topological properties. This motivates the research of covering rough sets from the topology point of view. Some initial attempts have already been made along the way. For example, Zhu and Wang examined the topological properties of the lower and upper approximation operations for covering generalized rough sets in [34,55]. Wu et al. combined the notion of topological spaces into rough sets and then discussed the properties of topological rough spaces [56]. In [1], neighborhoods, another elementary concept in topology, have been used to define an upper approximation; some properties of approximation operations for this type of covering rough sets have been explored as well [1,24,52,57].

So, we can say that there are two directions (see Fig. 1.1) for generalizing rough set theory one of them is replacing the equivalence relation by an arbitrary binary relation such as Yao [58]; the other direction is replacing the partition arising from the equivalence relation to cover the universe such as Zakowski [45], Pomykala [28] and Willim Zhu [1]. But most of them had failed to achieve all the properties of original rough set theory and thus they put some conditions and restrictions.

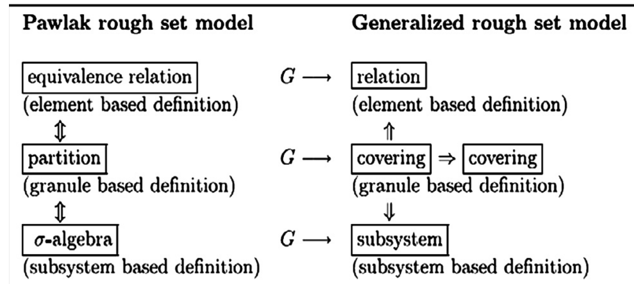


Figure 1.1 [59]: Schematic diagram of different formulations of approximation operators.

In the present paper, we introduce a framework for generalizing the two directions. In fact, we introduce the generalized covering approximation space "$\mathcal{G}_n - \boldsymbol{CAS}$" as a generalization to rough set theory and covering approximation space. Moreover, in our approaches $\mathcal{G}_n - \boldsymbol{CAS}$, four different approximations that satisfy all properties of original rough set theory without any conditions or restrictions are constructed.

Most real life situations need some sort of approximation to fit mathematical models. The beauty of using topology in approximation is achieved via obtaining approximation for qualitative concepts (i.e. subsets) without coding or using assumption. General topology is the appropriated mathematical model for every collection connected by relations. Relations were used in the construction of topological structures in several fields such as, structural analysis [60], general view of space time [61], biochemistry [62], biology [63], and rough set theory [3,9]. Recently, some topological concepts such as subbase, neighborhood and separation axioms have been applied to study covering-based rough sets. However, the topological space on covering-based rough sets and the corresponding topological properties on the topological covering-based rough space are not studied. This paper studies some of these problems. We introduce new method to generate different general topologies from any neighborhood space. The provided method can be considered an easy method to generate different topologies directly from the binary relation without using subbase or base. The used technique is useful in rough context or in covering-based rough sets since the concepts and the properties of generated topologies can be applied in rough set theory and covering-based rough set theory. We believe that the using of this method is easier in application field and it is useful for applying many topological concepts in future studies. This research not only can form the theoretical basis for further applications of topology on covering-based rough sets but also lead to the development of the rough set theory and artificial intelligence.

2. Basic concepts

In this section, we introduce the fundamental concepts which used through this paper.

Definition 2.1. "*Binary Relation*" [64]

Let A and B be sets, then a "*binary relation*" R from A to B (or between A and B) is a subset of a Cartesian product $A \times B$, namely the set of ordered pairs $(a,b) \in R$ such that $a \in A$ and $b \in B$.

The binary relation R can be from the set A to itself, and then we say that R is a binary relation on A. Moreover, if R is a binary relation from A to B (or from the set to itself) we say that $a \in A$ is **related to** $b \in B$ if $(a,b) \in R$, sometimes written aRb.

Definition 2.2. "*Inverse Relation*" [64]

Let R be a relation from A to B. Then $R^{-1} = \{(b,a)|(a,b) \in R\}$ is a relation from B to$A$ and it is called **the inverse** of the relation R.

Definition 2.3. [64] A binary relation R on a set A is:

(i) Serial if for every $a \in A, \exists b \in A, aRb$.
(ii) Reflexive if for every $a \in A, aRa$.
(iii) Symmetric if for every $a, b \in A, if\ aRb \Rightarrow bRa$.
(iv) Transitive if for every $a, b, c \in A, if\ aRb\ and\ bRc \Rightarrow aRc$.
(v) Equivalence if it is reflexive, symmetric and transitive relation.

Definition 2.4 [65]. Let U be any set, and R be any binary relation on U. Then the "*after set*" (resp.*"fore set"*) of the element $x \in U$ is the class $xR = \{y \in U : xRy\}$ (resp. $Rx = \{y \in U : yRx\}$).

Definition 2.5 [66]. A *topological space* is the pair (U, τ) consisting of a set U and family τ of subsets of U satisfying the following conditions:

(T1) $\emptyset \in \tau$ and $U \in \tau$.
(T2) τ is closed under finite intersection.
(T3) τ is closed under arbitrary union.

The pair (U, τ) is called "*space*", the elements of U are called *"points"* of the space, the subsets of U that belonging to τ are called "*open"* sets in the space and the complement of the subsets of U belonging to τ are called *"closed"* sets in the space; the family τ of open subsets of U is also called a *"topology"* for U.

Definition 2.6. "*Pawlak Approximation Space*" [3,9]

Let U be a finite set, the universe of discourse, and R be an equivalence relation on U, called an indiscernibility relation. The pair $\mathcal{A} = (U, R)$ is called Pawlak approximation space. The relation R will generate a partition $U/R = \{[x]R : x \in U\}$ on U, where $[x]R$ is the equivalence class with respect to R containing x.

For any $X \subseteq U$ the upper approximation $\overline{Apr}(X)$ and the lower approximation $\underline{Apr}(X)$ of a subset X are defined respectively as follows [3,9]:

$\overline{Apr}(X) = \cap\{Y \subseteq U/R : Y \cap X \neq \emptyset\}$ and $\underline{Apr}(X) = \cup\{Y \subseteq U/R : Y \subseteq X\}$.

Let $\emptyset$ be the empty set, X^c is the complement of X in U, we have the following properties of the Pawlak's rough sets [3,9]:

(L1) $\underline{Apr}(X) = [\overline{Apr}(X^c)]^c$.	(U1) $\overline{Apr}(X) = [\underline{Apr}(X^c)]^c$.
(L2) $\underline{Apr}(U) = U$.	(U2) $\overline{Apr}(U) = U$.
(L3) $\underline{Apr}(X \cap Y) = \underline{Apr}(X) \cap \underline{Apr}(Y)$.	(U3) $\overline{Apr}(X \cup Y) = \overline{Apr}(X) \cup \overline{Apr}(Y)$.
(L4) $\underline{Apr}(X \cup Y) \supseteq \underline{Apr}(X) \cup \underline{Apr}(Y)$.	(U4) $\overline{Apr}(X \cap Y) \subseteq \overline{Apr}(X) \cap \overline{Apr}(Y)$.
(L5) If $X \subseteq Y$, then $\underline{Apr}(X) \subseteq \underline{Apr}(Y)$.	(U5) If $X \subseteq Y$, then $\overline{Apr}(X) \subseteq \overline{Apr}(Y)$.
(L6) $\underline{Apr}(\emptyset) = \emptyset$.	(U6) $\overline{Apr}(\emptyset) = \emptyset$.
(L7) $\underline{Apr}(X) \subseteq X$.	(U7) $X \subseteq \overline{Apr}(X)$.
(L8) $\underline{Apr}(\underline{Apr}(X)) = \underline{Apr}(X)$.	(U8) $\overline{Apr}(\overline{Apr}(X)) = \overline{Apr}(X)$.
(L9) $\underline{Apr}(\overline{Apr}(X)) = \underline{Apr}(X)$.	(U9) $\overline{Apr}(\underline{Apr}(X)) = \overline{Apr}(X)$.

Definition 2.7. "*Covering*" [1]

Let U be a domain of discourse, $\mathcal{C} = \{C_k | k \in K\}$ a family of subsets of U. If none subsets in $\mathcal{C}$ is empty, and $\cup_{k \in K} C_k = U$, then $\mathcal{C}$ is called a covering of U. The pair $\langle U, \mathcal{C} \rangle$ is called a "**covering approximation space"** if $\mathcal{C}$ is a covering of U.

It follows from the above definition that any partition of U is certainly a covering of U. For convenience, the members of a general covering (not necessarily a partition) are also called elementary sets, and any union of elementary sets is called a definable set. In the literature, there are several kinds of rough sets induced by a covering [1,11,15,19,23,25,26,50,53].

For our purpose, we only recall the covering rough sets based on the following concept of neighborhoods [1].

Definition 2.8 [1]. Let $\langle U, \mathcal{C} \rangle$ be a "**covering approximation space"**. For any $x \in U$, we define the neighborhood of $x$as follows: $N(x) = \cap\{K \in \mathcal{C} | x \in K\}$.

Definition 2.9 [1]. Let $\langle U, \mathcal{C} \rangle$ be a covering approximation space. For any $X \in U$, the lower approximation of X is defined as: $X_* = \cup\{K | K \in \mathcal{C}$ and $K \subseteq X\}$

And the upper approximation of X is defined as: $X^* = X_* \cup \{N(x) | x \in X - X_*\}$.

Remark 2.1. It is clear that, we can give another representation of the upper approximation as follows:

$X^* = \cup\{N(x) | x \in X\}$.

The above representation was proved in [1].

The following proposition introduces the fundamental properties of the above approximations that were proved in [1].

Proposition 2.1 [1]. *Let* $\langle U, \mathcal{C} \rangle$*a be a covering approximation space. Then for any* $X, Y \subseteq U$*, the lower (resp. the upper) approximation of* X *have the following properties:*

(i) $U^* = U$.
(ii) $(X_*)_* = X_*$.
(iii) $\emptyset_* = \emptyset^* = \emptyset$.
(iv) $(X^*)^* = X^*$.
(v) $X_* \subseteq X \subseteq X^*$.
(vi) $X \subseteq Y \Rightarrow X_* \subseteq Y_*$.
(vii) $(X \cup Y)^* = X^* \cup Y^*$.
(viii) $X \subseteq Y \Rightarrow X^* \subseteq Y^*$.

Remark 2.2. The following properties do not hold generally:

(i) $(X \cap Y)_* = X_* \cap Y_*$.
(ii) $(-(X_*))_* = -(X_*)$.
(iii) $(-X)_* = -(X)^*$.
(iv) $(-X)^* = -(X_*)$.
(v) $(-(X^*)) = -(X)^*$.

The following example illustrates this remark.

Example 2.1. Let $\langle U, \mathcal{C} \rangle$ be a covering approximation space where $U = \{a, b, c, d\}$ and

$K_1 = \{a, b\}, K_2 = \{a, b, c\}, K_3 = \{c, d\}$ such that $\mathcal{C} = \{K_1, K_2, K_3\}$.

Now consider $X = \{a, b, c\}$ and $Y = \{c, d\}$. Thus we get

$X \cap Y = \{c\}$, so $(X \cap Y)_* = \emptyset$. But $X_* = X$ and $Y_* = Y$, so $X_* \cap Y_* \neq \emptyset$.

Also $(-X)^* = (\{d\})^* = \{c, d\} \neq X_*$.

By similar way we have: $-(X)_* = \{d\}$, so $(-X_*)_* = \emptyset$ this implies $(-X_*)_* \neq -(X)_*$.

3. Generalized covering approximation space

In this section, we introduce the new generalized covering approximation space "$\mathcal{G}_n-$ covering approximation space" $\mathcal{G}_n - \boldsymbol{CAS}$ as a generalization for covering approximation space by using binary relation. Moreover, we give some new notions of neighborhoods. In addition, four different pairs of dual approximation operators are investigated and their properties being discussed. Comparisons between our approaches and some of others approaches are discussed. Many examples and counter examples are provided.

Definition 3.1. Let $U \neq \emptyset$ be a finite set and $\mathcal{R}$ be a binary relation on U. Then, we can define two different coverings for U induced from the binary relation $\mathcal{R}$ as follows:

(*i*) Right Covering (briefly, *r*-cover): $\mathcal{C}_r = \{x\mathcal{R} : \forall x \in U$ and $U = \cup_{x \in U} x\mathcal{R}\}$.

(*ii*) Left Covering (briefly, *l*-cover): $\mathcal{C}_l = \{\mathcal{R}x : \forall x \in U$ and $U = \cup_{x \in U} \mathcal{R}x\}$.

Definition 3.2. Let $U \neq \emptyset$ be a finite set, $\mathcal{R}$ be a binary relation on U and $\mathcal{C}_n$ be *n*-cover of U associated to $\mathcal{R}$, where $n \in \{r, l\}$. Then the triple $\langle U, \mathcal{R}, \mathcal{C}_n \rangle$ is called "$\mathcal{G}_n-$ Covering approximation space" (briefly, $\mathcal{G}_n - \boldsymbol{CAS}$).

Lemma 3.1. *Let U be any set and* $\mathcal{R}$ *(resp.*$\mathcal{R}^{-1}$*) is a serial relation on U. Then we get:*$U = \cup_{x \in U} \mathcal{R}x$ *(resp.* $U = \cup_{x \in U} x\mathcal{R}$*).*

Proof. Let R be a serial relation on U, then for every $x \in U, \exists y \in U, xRy$. Thus for each $x \in U, \exists y \in U$, such that $y \in xR$ and this implies $U = \cup_{x \in U} Rx$.

By similar way, we can show that if $\mathcal{R}^{-1}$ is a serial relation on U, then $U = \cup_{x \in U} x\mathcal{R}$. □

From the above lemma, we can notice that: If $\mathcal{R}$ (resp. $\mathcal{R}^{-1}$) is a serial relation on U, then $\mathcal{R}x$ (resp. $x\mathcal{R}$) represents a left covering (resp. right covering) of U.

Remark 3.1. If $\mathcal{R}$ is a serial relation on U, then $x\mathcal{R}$ need not be right covering of U as the following example illustrates.

Example 3.1. Let $U = \{a, b, c, d\}$ and $\mathcal{R}$ be serial relation on U where,

$\mathcal{R} = \{(a, a), (b, a), (b, c), (c, c), (d, a)\}$. Thus we get

$a\mathcal{R} = \{a\}, b\mathcal{R} = \{a, c\}, c\mathcal{R} = \{c\}$ and $d\mathcal{R} = \{a\}$. Also, $\mathcal{R}a = \{a, b, d\}, \mathcal{R}b = \emptyset, \mathcal{R}c = \{b, c\}$ and $\mathcal{R}d = \emptyset$. It is clear that: $U \neq \cup_{x \in U} x\mathcal{R}$, but $U = \cup_{x \in U} Rx$.

The following definition is very interesting since it introduces different types of neighborhoods (generated from any binary relation) which represent the basic notions in our approaches.

Definition 3.3. Suppose that the triple $\langle U, \mathcal{R}, \mathcal{C}_n \rangle$ is $\mathcal{G}_n - \boldsymbol{CAS}$. For every element $x \in U$, we can define four different neighborhoods $N_j(x)$, as follows: For each $j \in \{r, l, i, u\}$

(*i*) *r*-neighborhood: $N_r(x) = \cap\{K \in \mathcal{C}_r | x \in K\}$.

(*ii*) *l*-neighborhood: $N_l(x) = \cap\{K \in \mathcal{C}_l | x \in K\}$.

(*iii*) *i*-neighborhood: $N_i(x) = N_r(x) \cap N_l(x)$.

(*iv*) *u*-neighborhood: $N_u(x) = N_r(x) \cup N_l(x)$.

Remark 3.2. In the $\mathcal{G}_n - \boldsymbol{CAS}$, $\langle U, \mathcal{R}, \mathcal{C}_n \rangle$, if $\mathcal{R}$ is an equivalence relation on U, thus both of the right and left cover of U have become a partition on U and thus they are equivalent to equivalence classes of the relation $\mathcal{R}$. In addition, all j-neighborhoods of x, $N_j(x)$ for every $j \in \{r, l, i, u\}$, are identical to equivalence classes of x, that is: $N_j(x) = [x]_{\mathcal{R}}, \forall j \in \{r, l, i, u\}$. Accordingly, in this case, $\mathcal{G}_n - \boldsymbol{CAS}, \langle U, \mathcal{R}, \mathcal{C}_n \rangle$ has become Pawlak approximation space. Thus we can say that Pawlak approach represents a special case of our approaches. Hence $\mathcal{G}_n - \boldsymbol{CAS}$ represents a generalization to Pawlak approximation space.

Lemma 3.2. *Let the triple* $\langle U, \mathcal{R}, \mathcal{C}_n \rangle$ *be* $\mathcal{G}_n - \boldsymbol{CAS}$. *Thus, for each* $j \in \{r, l, i, u\}$*:*$(i) N_j(x) \neq \emptyset, \forall x \in U. (ii) x \in N_j(x), \forall x \in U$.

Proof. From Definitions 3.1 and 3.3, $\forall x \in U$, there exists at least $y \in U$ such that $x \in y\mathcal{R}$ and $x \in \mathcal{R}y$. Thus $N_j(x) \neq \emptyset, \forall x \in U$. Similarly, $x \in N_j(x)$, $\forall x \in U$. □

Lemma 3.3. *Let the triple* $\langle U, \mathcal{R}, \mathcal{C}_n \rangle$ *be* $\mathcal{G}_n - \boldsymbol{CAS}$. *Thus, for each* $j \in \{r, l, i, u\}$ *if* $x \in N_j(y)$*, then* $N_j(x) \subseteq N_j(y)$.

Proof. (*i*) **If** $x \in N_r(y)$, **then** $N_r(x) \subseteq N_r(y)$:

Firstly, from Definitions 3.1 and 3.3, if $x \in N_r(y)$ then x belongs to every after set that contains y.

Now, let $z \in N_r(x)$, then z belongs to every after set that contains x which means that z belongs to every after set that contains y. Thus $z \in N_r(y)$ and then $N_r(x) \subseteq N_r(y)$.

(ii) **If** $x \in N_l(y)$, **then** $N_l(x) \subseteq N_l(y)$: By similar way as in(i).

(iii) **If** $x \in N_i(y)$, **then** $N_i(x) \subseteq N_i(y)$:

Firstly, if $x \in N_i(y)$, then $x \in N_r(y)$ and $x \in N_l(y)$. Thus by using (i) and (ii), we get

$N_r(x) \subseteq N_r(y)$ and $N_l(x) \subseteq N_l(y)$ which implies $N_i(x) \subseteq N_i(y)$.

(iv) **If** $x \in N_u(y)$, **then** $N_u(x) \subseteq N_u(y)$: By similar way as in (iii). □

Lemma 3.4. *Let the triple* $\langle U, \mathcal{R}, \mathcal{C}_n \rangle$ *be* $\mathcal{G}_n - CAS$. *Thus, for each* $j \in \{r, l, u, i\} : N_j(x)$ *represent different coverings of* $U, \forall x \in U$.

Proof. From Lemma 3.3,$x \in N_j(x), \forall x \in U$. Then $U = \cup_{x \in U} N_j(x)$, $\forall j \in \{r, l, u, i\}$ and hence $N_j(x)$ represent coverings of $U, \forall x \in U$. □

The following proposition introduces the relationships between different types of the j-neighborhoods.

Proposition 3.1. *Let the triple* $\langle U, \mathcal{R}, \mathcal{C}_n \rangle$ *be* $\mathcal{G}_n - \boldsymbol{CAS}$. *Then, for each* $x \in U$:

(i) $N_i(x) \subseteq N_r(x) \subseteq N_u(x)$.
(ii) $N_i(x) \subseteq N_l(x) \subseteq N_u(x)$.

Proof. From Definition 3.3, the proof is obvious. □

The following definition is very interesting since it introduces new approximation operators as a generalization to Pawlak approximations.

Definition 3.4. Suppose the triple $\langle U, \mathcal{R}, \mathcal{C}_n \rangle$ be $\mathcal{G}_n - \boldsymbol{CAS}$. For each $j \in \{r, l, i, u\}$ and $A \subseteq U$, the j-lower and the j-upper approximations of A are defined respectively as follows:

$\underline{\mathcal{R}}_j(A) = \{x \in A | N_j(x) \subseteq A\}$ and

$\overline{\mathcal{R}}_j(A) = \{x \in U | N_j(x) \cap A \neq \emptyset\}$.

Definition 3.5. Suppose the triple $\langle U, \mathcal{R}, \mathcal{C}_n \rangle$ be $\mathcal{G}_n - CAS$ and $A \subseteq U$. Thus, for each $j \in \{r, l, i, u\}$, the subset A is called "j-exact" set if $\underline{\mathcal{R}}_j(A) = \overline{\mathcal{R}}_j(A) = A$. Otherwise, A is called "j-rough set".

Definition 3.6. Suppose the triple $\langle U, \mathcal{R}, \mathcal{C}_n \rangle$ be $\mathcal{G}_n - \boldsymbol{CAS}$. For each $j \in \{r, l, i, u\}$ and $A \subseteq U$, the j-boundary, j-positive and j-negative regions of A are defined respectively as follows:

$\mathcal{B}_j(A) = \overline{\mathcal{R}}_j(A) - \underline{\mathcal{R}}_j(A)$

$POS_j(A) = \underline{\mathcal{R}}_j(A)$ and

$NEG_j(A) = U - \overline{\mathcal{R}}_j(A)$

Definition 3.7. Suppose the triple $\langle U, \mathcal{R}, \mathcal{C}_n \rangle$ be $\mathcal{G}_n - \boldsymbol{CAS}$. For each $j \in \{r, l, i, u\}$ and $A \subseteq U$, the j-accuracy of the approximations of A is defined as follows:

$\delta_j(A) = \frac{|\underline{\mathcal{R}}_j(A)|}{|\overline{\mathcal{R}}_j(A)|}$ where $|\overline{\mathcal{R}}_j(A)| \neq 0$ and $|A|$ denotes the cardinality of A.

Remarks 3.3 From the above definitions, we notice that:

(i) Obviously, $0 \leqslant \delta_j(A) \leqslant 1$, for every $A \subseteq U$.
(ii) A is j-exact set if $\delta_j(A) = 1$ and $\mathcal{B}_j(A) = \emptyset$. Otherwise, it is j-rough set.

The following proposition introduces the fundamental properties of j-approximations.

Proposition 3.2. *Let the triple* $\langle U, \mathcal{R}, \mathcal{C}_n \rangle$ *be* $\mathcal{G}_n - \boldsymbol{CAS}$ *and* $A, B \subseteq U$. *Then*

(1) $\underline{\mathcal{R}}_j(A) \subseteq A \subseteq \overline{\mathcal{R}}_j(A)$.
(2) $\underline{\mathcal{R}}_j(U) = \overline{\mathcal{R}}_j(U) = U$.
(3) $\underline{\mathcal{R}}_j(\emptyset) = \overline{\mathcal{R}}_j(\emptyset) = \emptyset$.
(4) If $A \subseteq B$ then $\underline{\mathcal{R}}_j(A) \subseteq \underline{\mathcal{R}}_j(B)$, $\overline{\mathcal{R}}_j(A) \subseteq \overline{\mathcal{R}}_j(B)$.
(5) $\underline{\mathcal{R}}_j(A) = [\overline{\mathcal{R}}_j(A^c)]^c$, where A^c is the complement of A.
(6) $\overline{\mathcal{R}}_j(A) = [\underline{\mathcal{R}}_j(A^c)]^c$, where A^c is the complement of A.
(7) $\underline{\mathcal{R}}_j(\underline{\mathcal{R}}_j(A)) = \underline{\mathcal{R}}_j(A)$ *and* $\overline{\mathcal{R}}_j(\overline{\mathcal{R}}_j(A)) = \overline{\mathcal{R}}_j(A)$.
(8) $\underline{\mathcal{R}}_j(A \cap B) = \underline{\mathcal{R}}_j(A) \cap \underline{\mathcal{R}}_j(B)$ *and* $\overline{\mathcal{R}}_j(A \cup B) = \overline{\mathcal{R}}_j(A) \cup \overline{\mathcal{R}}_j(B)$.
(9) $\underline{\mathcal{R}}_j(A) \cup \underline{\mathcal{R}}_j(B) \subseteq \underline{\mathcal{R}}_j(A \cup B)$ *and* $\overline{\mathcal{R}}_j(A \cap B) \subseteq \overline{\mathcal{R}}_j(A) \cap \overline{\mathcal{R}}_j(B)$.

Proof. First, from Definition 3.4, the proof of (1), (2) and (3) is obvious.

(4) Let $A \subseteq B, x \in \underline{\mathcal{R}}_j(A)$. Then $x \in A$ and $N_j(x) \subseteq A$, which means that $x \in B$ and $N_j(x) \subseteq B$. Thus, $x \in \underline{\mathcal{R}}_j(B)$ and this implies $\underline{\mathcal{R}}_j(A) \subseteq \underline{\mathcal{R}}_j(B)$. By the same way, $\overline{\mathcal{R}}_j(A) \subseteq \overline{\mathcal{R}}_j(B)$.
(5) $[\overline{\mathcal{R}}_j(A^c)]^c = [\{x \in U | N_j(x) \cap A^c = \emptyset\}]^c = \{x \in U | N_j(x) \cap A^c = \emptyset\} = \{x \in A | N_j(x) \subseteq A\} = \underline{\mathcal{R}}_j(A)$.
(6) By similar way , as in (5).
(7) First, it is clear that $\underline{\mathcal{R}}_j(\underline{\mathcal{R}}_j(A)) \subseteq \underline{\mathcal{R}}_j(A)$. Now, let $x \in \underline{\mathcal{R}}_j(A)$. Then $x \in A$ and $N_j(x) \subseteq A$. We must prove that $x \in \underline{\mathcal{R}}_j(A)$ and $N_j(x) \subseteq \underline{\mathcal{R}}_j(A)$ as follows: Let $z \in N_j(x)$, then $N_j(z) \subseteq N_j(x)$, (By Lemma 3.3), which implies $N_j(z) \subseteq A$. Thus $z \in \underline{\mathcal{R}}_j(A)$ and this means that $N_j(x) \subseteq \underline{\mathcal{R}}_j(A)$. Hence, $\underline{\mathcal{R}}_j(A) \subseteq \underline{\mathcal{R}}_j(\underline{\mathcal{R}}_j(A))$ and then $\underline{\mathcal{R}}_j(\underline{\mathcal{R}}_j(A)) = \underline{\mathcal{R}}_j(A)$.
(8) By similar way, as in (7).
(9) Let $x \in (\underline{\mathcal{R}}_j(A) \cap \underline{\mathcal{R}}_j(B))$, then $x \in \underline{\mathcal{R}}_j(A)$ and $x \in \underline{\mathcal{R}}_j(B)$. Thus $x \in A, N_j(x) \subseteq A$ and $x \in B$, $N_j(x) \subseteq B$ which means that $x \in A \cap B$, $N_j(x) \subseteq (A \cap B)$. Then $x \in \underline{\mathcal{R}}_j(A \cap B)$ and this implies $\underline{\mathcal{R}}_j(A) \cap \underline{\mathcal{R}}_j(B) \subseteq \underline{\mathcal{R}}_j(A \cap B)$. Now, let $x \in \underline{\mathcal{R}}_j(A \cap B)$, then $x \in (A \cap B)$ and $N_j(x) \subseteq (A \cap B)$. Thus $x \in A, N_j(x) \subseteq A$ and $x \in B, N_j(x) \subseteq B$ which implies $x \in \underline{\mathcal{R}}_j(A)$ and $x \in \underline{\mathcal{R}}_j(B)$. Then, $x \in \underline{\mathcal{R}}_j(A) \cap \underline{\mathcal{R}}_j(B)$ and thus, $\underline{\mathcal{R}}_j(A \cap B) \subseteq \underline{\mathcal{R}}_j(A) \cap \underline{\mathcal{R}}_j(B)$. Hence, $\underline{\mathcal{R}}_j(A \cap B) = \underline{\mathcal{R}}_j(A) \cap \underline{\mathcal{R}}_j(B)$. Similarly, $\overline{\mathcal{R}}_j(A \cup B) = \overline{\mathcal{R}}_j(A) \cup \underline{\mathcal{R}}_j(B)$.
(10) Since $A \subseteq A \cup B$ and $B \subseteq A \cup B$. Then $\underline{\mathcal{R}}_j(A) \subseteq \underline{\mathcal{R}}_j(A \cup B)$ and $\underline{\mathcal{R}}_j(B) \subseteq \underline{\mathcal{R}}_j(A \cup B)$ and thus $\underline{\mathcal{R}}_j(A) \cup \underline{\mathcal{R}}_j(B) \subseteq \underline{\mathcal{R}}_j(A \cup B)$. Similarly, $\underline{\mathcal{R}}_j(A \cap B) \subseteq \underline{\mathcal{R}}_j(A) \cap \underline{\mathcal{R}}_j(B)$. □

In the above proposition, the converse of the property (10) is not true in general as the following example illustrates.

Table 3.1 Comparison between our approaches and some of the others approaches. A sign $(*)$ indicates that property is satisfied.

Property of Pawlak [3,9]	Yao [58] and others [67–71,75–77,79]	Covering [1,57,59,72–74]	$\mathcal{G}_n - CAS$
(L1)	*		*
(L2)	*	*	*
(L3)	*		*
(L4)	*	*	*
(L5)	*	*	*
(L6)		*	*
(L7)		*	*
(L8)			*
(L9)			*
(U1)	*	*	*
(U2)		*	*
(U3)	*		*
(U4)	*	*	*
(U5)	*	*	*
(U6)		*	*
(U7)		*	*
(U8)			*
(U9)			*

Example 3.3. Let the triple $\langle U, \mathcal{R}, \mathcal{C}_n \rangle$ be $\mathcal{G}_n - \boldsymbol{CAS}$, where $U = \{a,b,c,d,e\}$ and $\mathcal{R} = \{(a,a),(a,d),(b,a),(b,c),(c,c),(d,e),(e,b),(e,d)\}$. Then we get $N_r(a) = \{a\}, N_r(b) = \{b,d\}, N_r(c) = \{c\}, N_r(d) = \{d\}$ *and* $N_r(e) = \{e\}$. We will give the j-approximations in the case of $j = r$ and the other cases similarly:

Now, let $X = \{a,b,c\}$ and $Y = \{c,d\}$. Then $X \cup Y = \{a,b,c,d\}, X \cap Y = \{c\}$ and thus $\underline{\mathcal{R}}_r(X) = \{a,c\}$, $\underline{\mathcal{R}}_r(Y) = \{c,d\}$, $\underline{\mathcal{R}}_r(X \cup Y) = \{a,b,c,d\}$, $\overline{\mathcal{R}}_r(X) = \{a,b,c\}$, $\overline{\mathcal{R}}_r(Y) = \{b,c,d\}$ and $\overline{\mathcal{R}}_r(X \cap Y) = \{c\}$. Clearly, $\underline{\mathcal{R}}_r(X \cup Y) \neq \underline{\mathcal{R}}_r(X) \cup \underline{\mathcal{R}}_r(Y)$ and $\overline{\mathcal{R}}_r(X \cap Y) \neq \overline{\mathcal{R}}_r(X) \cap \overline{\mathcal{R}}_r(X)$.

Remark 3.3. Proposition 3.2 is very interesting because it illustrates that our approaches $\mathcal{G}_n - \boldsymbol{CAS}$ represent the actual generalizations for Pawlak approximation space, specifically for covering-based models. Moreover, it can be considered as a one of differences between our approaches and the others generalizations such as (see: [1,57,59,67–79]). Although many authors have introduced many sorts to generalize Pawlak approximation space, but most of them had failed to achieve all properties of the original rough set theory. In our approaches most of these properties which has never been realized, is achieved. So, we can say that our approaches represent the actual generalization of Pawlak approximation space [3,9] and the other generalizations in [1,57,59,67–79].

Table 3.1 shows a comparison between our approaches and some of others generalizations.

The following example illustrates the comparison between our approaches and Yao's method [58].

Example 3.3. Let the triple $\langle U, \mathcal{R}, \mathcal{C}_n \rangle$ be $\mathcal{G}_n - \boldsymbol{CAS}$, where $U = \{a,b,c,d\}$ and

$\mathcal{R} = \{(a,a),(a,b),(b,c),(b,d),(c,a),(d,a)\}$. Then we can get:

$a\mathcal{R} = \{a,b\}, b\mathcal{R} = \{c,d\}, c\mathcal{R} = \{a\}$ and $d\mathcal{R} = \{a\}$. Also, $\mathcal{R}a = \{a,c,d\}, \mathcal{R}b = \{a\}$, $\mathcal{R}c = \{b\}$ and $\mathcal{R}d = \{b\}$ and this implies $N_r(a) = \{a\}$, $N_r(b) = \{a,b\}, N_r(c) = \{c,d\}, N_r(d) = \{c,d\}, N_l(a) = \{a\}$, $N_l(b) = \{b\}, N_l(c) = \{a,c,d\}, N_l(d) = \{a,c,d\}$,

$N_u(a) = \{a\}, N_u(b) = \{a,b\}, N_u(c) = \{a,c,d\}, N_u(d) = \{a,c,d\}$ and

$N_i(a) = \{a\}, N_i(b) = \{b\}, N_i(c) = \{c,d\}, N_i(d) = \{c,d\}$.

Yao [58] defines the approximations of any subset $X \subseteq U$ as follows:

$$\underline{apr}(X) = \{x \in U : x\mathcal{R} \subseteq X\} \text{ and } \overline{apr}(X) = \{x \in U : x\mathcal{R} \cap X \neq \emptyset\}.$$

The following table shows the differences between Yao approach and our approaches "$\mathcal{G}_n - \boldsymbol{CAS}$":

From Table 3.2, we can notice that:

(i) $\underline{apr}(X) \not\subseteq X \not\subseteq \overline{apr}(X)$, for example the subsets $\{c,d\}$ and $\{b,c,d\}$ but in our approaches $\underline{\mathcal{R}}_j(X) \subseteq X \subseteq \overline{\mathcal{R}}_j(X)$ for any $X \subseteq U$.

(ii) In Yao's approach, all subsets in U are rough (except U), but in our approaches $\mathcal{G}_n - \boldsymbol{CAS}$, there are many subsets are j-exacts such as the shaded sets in above table. Moreover, the boundary region was reduced and became smaller than Yao approach. Hence, we can say that our approaches are more accurate than Yao approach.

4. The relationships between different types of the $G_n - CAS$

The present section is devoted to introduce comparisons between different types of the $\mathcal{G}_n - \boldsymbol{CAS}$. In addition, the best approach is provided with best accuracy.

The following results, introduce the relationships between the j-approximations, j-accuracy and j-boundary respectively.

Proposition 4.1.
Let the triple $\langle U, \mathcal{R}, \mathcal{C}_n \rangle$ *be* $\mathcal{G}_n - \boldsymbol{CAS}$ *and* $A \subseteq U$. *Then*

Table 3.2 Comparisons between of Yao approach and our approaches.

$\wp(U)$	Yao approach		$\mathcal{G}_n - \boldsymbol{CAS}$							
	$\underline{apr}(A)$	$\overline{apr}(A)$	$\underline{\mathcal{R}}_r(A)$	$\overline{\mathcal{R}}_r(A)$	$\underline{\mathcal{R}}_l(A)$	$\overline{\mathcal{R}}_l(A)$	$\underline{\mathcal{R}}_u(A)$	$\overline{\mathcal{R}}_u(A)$	$\underline{\mathcal{R}}_i(A)$	$\overline{\mathcal{R}}_i(A)$
$\{a\}$	$\{c,d\}$	$\{a,c,d\}$	$\{a\}$	$\{a,b\}$	$\{a\}$	$\{a,c,d\}$	$\{a\}$	U	$\{a\}$	$\{a\}$
$\{b\}$	$\emptyset$	$\{a\}$	$\emptyset$	$\{b\}$	$\{b\}$	$\{b\}$	$\emptyset$	$\{b\}$	$\{b\}$	$\{b\}$
$\{c\}$	$\emptyset$	$\{b\}$	$\emptyset$	$\{c,d\}$	$\emptyset$	$\{c,d\}$	$\emptyset$	$\{c,d\}$	$\emptyset$	$\{c,d\}$
$\{d\}$	$\emptyset$	$\{b\}$	$\emptyset$	$\{c,d\}$	$\emptyset$	$\{c,d\}$	$\emptyset$	$\{c,d\}$	$\emptyset$	$\{c,d\}$
$\{a,b\}$	$\{a,c,d\}$	$\{a,c,d\}$	$\{a,b\}$	$\{a,b\}$	$\{a,b\}$	U	$\{a,b\}$	U	$\{a,b\}$	$\{a,b\}$
$\{a,c\}$	$\{c,d\}$	U	$\{a\}$	U	$\{a\}$	$\{a,c,d\}$	$\{a\}$	U	$\{a\}$	$\{a,c,d\}$
$\{a,d\}$	$\{c,d\}$	U	$\{a\}$	U	$\{a\}$	$\{a,c,d\}$	$\{a\}$	U	$\{a\}$	$\{a,c,d\}$
$\{b,c\}$	$\emptyset$	$\{a,b\}$	$\emptyset$	$\{b,c,d\}$	$\{b\}$	$\{b,c,d\}$	$\emptyset$	$\{b,c,d\}$	$\{b\}$	$\{b,c,d\}$
$\{b,d\}$	$\emptyset$	$\{a,b\}$	$\emptyset$	$\{b,c,d\}$	$\{b\}$	$\{b,c,d\}$	$\emptyset$	$\{b,c,d\}$	$\{b\}$	$\{b,c,d\}$
$\{c,d\}$	$\{b\}$	$\{b\}$	$\{c,d\}$	$\{c,d\}$	$\emptyset$	$\{c,d\}$	$\emptyset$	$\{c,d\}$	$\{c,d\}$	$\{c,d\}$
$\{a,b,c\}$	$\{a,c,d\}$	U	$\{a,b\}$	U	$\{a,b\}$	U	$\{a,b\}$	U	$\{a,b\}$	U
$\{a,b,d\}$	$\{a,c,d\}$	U	$\{a,b\}$	U	$\{a,b\}$	U	$\{a,b\}$	U	$\{a,b\}$	U
$\{a,c,d\}$	$\{b,c,d\}$	U	$\{a,c,d\}$	U	$\{a,c,d\}$	$\{a,c,d\}$	$\{a,c,d\}$	U	$\{a,c,d\}$	$\{a,c,d\}$
$\{b,c,d\}$	$\{b\}$	$\{a,b\}$	$\{c,d\}$	$\{b,c,d\}$	$\{b,c,d\}$	$\{b,c,d\}$	$\emptyset$	$\{b,c,d\}$	$\{b,c,d\}$	$\{b,c,d\}$
U	U	U	U	U	U	U	U	U	U	U
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

(i) $\underline{\mathcal{R}}_u(A) \subseteq \underline{\mathcal{R}}_r(A) \subseteq \underline{\mathcal{R}}_i(A)$.
(ii) $\underline{\mathcal{R}}_u(A) \subseteq \underline{\mathcal{R}}_l(A) \subseteq \underline{\mathcal{R}}_i(A)$.
(iii) $\overline{\mathcal{R}}_i(A) \subseteq \overline{\mathcal{R}}_r(A) \subseteq \overline{\mathcal{R}}_u(A)$.
(iv) $\overline{\mathcal{R}}_i(A) \subseteq \overline{\mathcal{R}}_l(A) \subseteq \overline{\mathcal{R}}_u(A)$.

Proof. (i) Let $x \in \underline{\mathcal{R}}_u(A)$, then $x \in A$ and $N_u(x) \subseteq A$. Thus $x \in A$ and $N_r(x) \subseteq A$ and this implies

$x \in \underline{\mathcal{R}}_r(A)$. Hence, $\underline{\mathcal{R}}_u(A) \subseteq \underline{\mathcal{R}}_r(A)$. Also, if $x \in \underline{\mathcal{R}}_r(A)$ then $x \in A$ and $N_r(x) \subseteq A$ which means that $x \in A$ and $N_i(x) \subseteq A$. Hence, $x \in \underline{\mathcal{R}}_i(A)$ and then $\underline{\mathcal{R}}_r(A) \subseteq \underline{\mathcal{R}}_i(A)$.

(ii) By similar way as in (i).

(iii) & (iv) By the duality of approximations. □

The converse of the above results is not true in general as the following example illustrates.

Example 4.1. Let the triple $\langle U, \mathcal{R}, \mathcal{C}_n \rangle$ be $\mathcal{G}_n - \boldsymbol{CAS}$, where $U = \{a, b, c, d\}$ and

$\mathcal{R} = \{(a,d), (b,b), (b,c), (c,b), (d,a), (d,c)\}$. Then we can get:

$N_r(a) = \{a,c\}, N_r(b) = \{b\}, N_r(c) = \{c\}, N_r(d) = \{d\}, N_l(a) = \{a\}, N_l(b) = \{b\}, N_l(c) = \{b,c\}, N_l(d) = \{d\}, N_u(a) = \{a,c\}, N_u(b) = \{b\}, N_u(c) = \{b,c\}, N_u(d) = \{d\}$ and $N_i(a) = \{a\}, N_i(b) = \{b\}, N_i(c) = \{c\}, N_i(d) = \{d\}$. Now, let $A = \{c,d\}$. Then we get $\underline{\mathcal{R}}_u(A) = \{d\}$, but $\underline{\mathcal{R}}_r(A) = \{c,d\}$. Also, if $B = \{a,d\}$ then $\underline{\mathcal{R}}_r(B) = \{d\}$, but $\underline{\mathcal{R}}_i(B) = \{a,d\}$. Similarly, if $D = \{a\}$ then $\underline{\mathcal{R}}_u(D) = \emptyset$, but $\underline{\mathcal{R}}_l(D) = \{a\}$.

Also, $\overline{\mathcal{R}}_l(A) = \{b,c,d\}$ and $\overline{\mathcal{R}}_u(A) = \{b,c,d\}$, but $\overline{\mathcal{R}}_i(A) = \{c,d\}$.

Similarly, $\overline{\mathcal{R}}_i(B) = \{a,d\}$ but $\overline{\mathcal{R}}_r(B) = \{a,c,d\}$ and $\overline{\mathcal{R}}_u(B) = \{a,c,d\}$.

Corollary 4.1. *Let the triple $\langle U, \mathcal{R}, \mathcal{C}_n \rangle$ be $\mathcal{G}_n - \boldsymbol{CAS}$ and $A \subseteq U$. Then(i) $\mathcal{B}_i(A) \subseteq \mathcal{B}_r(A) \subseteq \mathcal{B}_u(A)$.(ii) $\mathcal{B}_i(A) \subseteq \mathcal{B}_l(A) \subseteq \mathcal{B}_u(A)$.*

Proof. By using Proposition 4.1, the proof is obvious. □

Corollary 4.2. *Let the triple $\langle U, \mathcal{R}, \mathcal{C}_n \rangle$ be $\mathcal{G}_n - \boldsymbol{CAS}$ and $A \subseteq U$. Then(i) $\delta_u(A) \leqslant \delta_r(A) \leqslant \delta_i(A)$.(ii) $\delta_u(A) \leqslant \delta_l(A) \leqslant \delta_i(A)$.*

Proof. By using Propositions 4.1 and 4.2, the proof is obvious. □

The converse of the above results is not true in general as shown in Example 4.1.

Proposition 4.2. *Let the triple* $\langle U, \mathcal{R}, \mathcal{C}_n\rangle$ *be* $\mathcal{G}_n - \boldsymbol{CAS}$ *and* $A \subseteq U$. *Then the following statements are true in general:*

(i) A is *u-exact* $\Rightarrow A$ is *r-exact* $\Rightarrow A$ is *i-exact*.
(ii) A is *u-exact* $\Rightarrow A$ is *l-exact* $\Rightarrow A$ is *i-exact*.

Proof. (i) Let A is u-exact, then $\mathcal{B}_u(A) = \emptyset$. By using Corollary 4.1, we get $\mathcal{B}_r(A) = \emptyset$ and this implies A is r-exact. Also $\mathcal{B}_i(A) = \emptyset$, which means that A is i-exact.

(ii) By similar way, as in (i). □

The converse of the above proposition is not true in general, as the following example illustrates.

Example 4.2. In Example 4.1, the subset $\{a\}$ is i-exact but it is r-rough, l-rough and u-rough.

Remark 4.1. From the above results, we can notice that:

There are four different methods to approximate the sets. The best of these methods is that given by using $j = i$ in constructing the approximations of sets, since the boundary regions in this case are decreased or canceled by increasing the lower approximation and decreasing the upper approximation, that is for each $A \subseteq U, B_i(A) \subseteq B_j(A), \forall j = r, l, u$. Accordingly, this will play an important role for removing the vagueness (uncertainty) of rough sets. Moreover, this method is more accurate than others types, since for any subset $A \subseteq U, \delta_j(A) \leqslant \delta_i(A)$ and $\forall j = r, l, u$. Thus, this approach will help to extract and discover the hidden information in data that were collected from real-life applications and hence it is very useful in decision making.

5. Topologies induced from neighborhoods

Recently, the general topology has become an appropriate frame for every collection connected by relations. It should be noted that generating of topology by relations and the representation of topological concepts via relations will narrow the gap between topologists and those who are interested in applications of topology in their fields. In the present section, we introduce new method to generate different topologies by using the notion of neighborhood. By using this technique, we generate different topologies from binary relation and then we generate different four topologies from $\mathcal{G}_n - \boldsymbol{CAS}$. The relationships between these topologies are discussed. Many examples and counter examples to indicate the connections between these topologies are provided.

The following theorem is very interesting since it gives new method to generate general topology using the concept of neighborhood. Moreover, the used technique does not depend on the form of neighborhood. This technique opens the way for more topological applications on covering-based rough models.

Theorem 5.1. *Suppose that* $U \neq \emptyset$ *be any finite set, if for each* $p \in U$, *there exists a neighborhood* $\mathcal{N}(p)$ *such that* $\mathcal{N}(p) \subseteq U$. *Then the collection*

$\tau = \{A \subseteq U | \forall p \in A, \mathcal{N}(p) \subseteq A\}$ is a topology on U.

Proof. (T1) Clearly, U and $\emptyset$ belong to τ.

(T2) Let $\{A_k | k \in K\}$ be a family of elements in τ and let $p \in \cup_{k\in K} A_k$. Then there exists $k_0 \in K$ such that $p \in A_{k_0}$. Thus $\mathcal{N}(p) \subseteq A_{k_0}$ and this implies $\mathcal{N}(p) \subseteq \cup_{k\in K} A_k$. Hence $\cup_{k\in K} A_k \in \tau$.
(T3) Let $A_1, A_2 \in \tau$ and $p \in A_1 \cap A_2$. Then $p \in A_1$ and $p \in A_2$ which implies $\mathcal{N}(p) \subseteq A_1$ and $\mathcal{N}(p) \subseteq A_2$. Thus $\mathcal{N}(p) \subseteq A_1 \cap A_2$ and then $A_1 \cap A_2 \in \tau_j$. Accordingly τ is a topology on U. □

By using the above theorem, we can generate four different topologies from $\mathcal{G}_n - CAS$ as the following corollary illustrates.

Corollary 5.1. *Let the triple* $\langle U, \mathcal{R}, \mathcal{C}_n\rangle$ *be* $\mathcal{G}_n - \boldsymbol{CAS}$. *Then the topologies associated with* $\mathcal{G}_n - \boldsymbol{CAS}$ *are given by the following families:*$\tau_j = \{A \subseteq U | \forall p \in A, N_j(p) \subseteq A\}$, *for each*$j \in \{r, l, u, i\}$.

The following example is given to generate general topology from covering approximation space as follows.

Example 5.1. Let $\langle U, \mathcal{C}\rangle$ be a covering approximation space, where $U = \{a, b, c, d\}$ and $\mathcal{C} = \{\{a\}, \{a, b\}, \{b, c\}, \{d\}\}$. Then the neighborhoods of the elements of U are given, (by using Definition 2.8 [1]), as follows: $N(a) = \{a\}, N(b) = \{b\}$, $N(c) = \{b, c\}$ and $N(d) = \{d\}$. By using Proposition 5.1, the associated topology of $\langle U, \mathcal{C}\rangle$ is given by the class:

$$\tau = \{U, \emptyset, \{a\}, \{b\}, \{d\}, \{a, b\}, \{a, d\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}\}.$$

The following example is given to generate different topologies from the $\mathcal{G}_n - \boldsymbol{CAS}$ as follows.

Example 5.2. Let the triple $\langle U, \mathcal{R}, \mathcal{C}_n\rangle$ be $\mathcal{G}_n - \boldsymbol{CAS}$, where $U = \{a, b, c, d\}$ and

$\mathcal{R} = \{(a, a), (a, b), (b, c), (b, d), (c, a), (d, a)\}$. Then we can get:

$$N_r(a) = \{a\}, N_r(b) = \{a, b\}, N_r(c) = \{c, d\}, N_r(d) = \{c, d\},$$

$$N_l(a) = \{a\}, N_l(b) = \{b\}, N_l(c) = \{a, c, d\}, N_l(d) = \{a, c, d\},$$

$$N_u(a) = \{a\}, N_u(b) = \{a, b\}, N_u(c) = \{a, c, d\}, N_u(d) = \{a, c, d\}$$

and

$$N_i(a) = \{a\}, N_i(b) = \{b\}, N_i(c) = \{c, d\}, N_i(d) = \{c, d\}.$$

Then, the topologies associated with $\langle U, \mathcal{R}, \mathcal{C}_j\rangle$ are given by the classes:

$\tau_r = \{U, \emptyset, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}\}, \tau_l = \{U, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c, d\}\}$, $\tau_u = \{U, \emptyset, \{a\}, \{a, b\}, \{a, c, d\}\}$ and $\tau_i = \{U, \emptyset, \{a\}, \{b\}, \{a, b\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\}$.

Definition 5.1. Let the triple $\langle U, \mathcal{R}, \mathcal{C}_n \rangle$ be $\mathcal{G}_n - \boldsymbol{CAS}$ associated with topologies τ_j. Then for each $j \in \{r, l, u, i\}$, the subset $A \subseteq U$ is said to be j-open set if $A \in \tau_j$, and the complement of j-open set is called j-closed set. The family Γ_j of all j-closed sets of $\mathcal{G}_n - \boldsymbol{CAS}$ is defined by

$$\Gamma_j = \{F \subseteq U | F^c \in \tau_j\}.$$

Definition 5.2. Let the triple $\langle U, \mathcal{R}, \mathcal{C}_n \rangle$ be $\mathcal{G}_n - \boldsymbol{CAS}$ associated with topologies τ_j. Then for each $j \in \{r, l, u, i\}$, we define the j-interior and the j-closure of any subset $A \subseteq U$ in the topologies τ_j respectively as follows:

$int_j(A) = \cup\{G \in \tau_j | G \subseteq A\}$ and

$cl_j(A) = \cap\{H \in \Gamma_j | A \subseteq H\}$.

It is clear that, $int_j(A) \subseteq A \subseteq cl_j(A)$ for any $A \subseteq U$. In addition, $int_j(A)$(resp. $cl_j(A)$) is the largest j-open set contained in A (resp. the smallest j-closed set contains A).

Proposition 5.1. *Let the triple $\langle U, \mathcal{R}, \mathcal{C}_n \rangle$ be $\mathcal{G}_n - \boldsymbol{CAS}$ associated with topologies τ_j. Then the j-lower approximation (resp. the j-upper approximation) represents the j-interior (resp. the j-closure) of τ_j, that is: $\underline{\mathcal{R}}_j(A) = \{x \in A | N_j(x) \subseteq A\} = int_j(A)$ and*

$$\overline{\mathcal{R}}_j(A) = \{x \in U | N_j(x) \cap A \neq \emptyset\} = cl_j(A)$$

Proof. We shall prove the first statement and the second by duality of the approximations:

First, let $x \in \underline{\mathcal{R}}_j(A)$. Then $x \in A, N_j(x) \subseteq A$ and this implies $A \in \tau_j$ such that $x \in A$. Thus, from Definition 5.2, $A \subseteq int_j(A)$ and then $x \in int_j(A)$. Hence $\underline{\mathcal{R}}_j(A) \subseteq int_j(A)$.

Conversely, from Definition 5.2, since $int_j(A)$ is the largest j-open set contained in A then $\forall x \in int_j(A), N_j(x) \subseteq int_j(A)$ which means that $x \in A, N_j(x) \subseteq A$ and then $x \in \underline{\mathcal{R}}_j(A)$. Hence $int_j(A) \subseteq \underline{\mathcal{R}}_j(A)$ and accordingly $\underline{\mathcal{R}}_j(A) = int_j(A)$. □

Corollary 5.2. *Let the triple $\langle U, \mathcal{R}, \mathcal{C}_n \rangle$ be $\mathcal{G}_n - \boldsymbol{CAS}$ associated with topologies τ_j. Then for each $j \in \{r, l, u, i\}$, the subset $A \subseteq U$ is j-open set (resp. j-closed set) if $\underline{\mathcal{R}}_j(A) = A$ (resp. $\overline{\mathcal{R}}_j(A) = A$).*

Remark 5.1. According to the above results, we can introduce another method to generate different topologies induced from relation as follows: For each $j \in \{r, l, u, i\}$, the classes

$\tau_j = \{A \subseteq U | \underline{\mathcal{R}}_j(A) = A\}$ are topologies on U.

The following propositions introduce the relationships between the different topologies τ_j.

Proposition 5.2. *Let the triple $\langle U, \mathcal{R}, \mathcal{C}_n \rangle$ be $\mathcal{G}_n - \boldsymbol{CAS}$ associated with topologies τ_j. Then: (i) $\tau_r \subseteq \tau_i$. (ii) $\tau_l \subseteq \tau_i$.*

Proof. Let $A \in \tau_r$, then $\forall p \in A, N_r(p) \subseteq A$. Thus, $\forall p \in A, N_i(p) \subseteq A$ which implies $A \in \tau_i$. Hence $\tau_r \subseteq \tau_i$. Similarly, we can prove that $\tau_l \subseteq \tau_i$. □

Remark 5.2. Let the triple $\langle U, \mathcal{R}, \mathcal{C}_n \rangle$ be $\mathcal{G}_n - \boldsymbol{CAS}$ associated with topologies τ_j. Then the following statements are not necessarily true in general.

(*i*) $\tau_r = \tau_i$.

(*ii*) $\tau_l = \tau_i$.

Remark 5.3. Let the triple $\langle U, \mathcal{R}, \mathcal{C}_n \rangle$ be $\mathcal{G}_n - \boldsymbol{CAS}$ associated with topologies τ_j. Then τ_r and τ_l are not necessarily comparable.

The following example shows Remarks 5.2 and 5.3.

Example 5.3. Let the triple $\langle U, \mathcal{R}, \mathcal{C}_n \rangle$ be $\mathcal{G}_n - \boldsymbol{CAS}$, where $U = \{a, b, c, d\}$ and

$\mathcal{R} = \{(a, d), (b, b), (b, c), (c, b), (d, a), (d, c)\}$. Then we can get:

$\tau_r = \{U, \emptyset, \{b\}, \{c\}, \{d\}, \{a, c\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{b, c, d\}, \{a, c, d\}\}$,

$\tau_l = \{U, \emptyset, \{a\}, \{b\}, \{d\}, \{a, b\}, \{a, d\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}\}$, and $\tau_i = P(U)$.

Proposition 5.3. *Let the triple $\langle U, \mathcal{R}, \mathcal{C}_n \rangle$ be $\mathcal{G}_n - \boldsymbol{CAS}$ associated with topologies τ_j. Then: (i) $\tau_u \subseteq \tau_r$. (ii) $\tau_u \subseteq \tau_l$. (iii) $\tau_u \subseteq \tau_i$.*

Proof. Suppose $A \in \tau_u$, then $\forall p \in A, N_u(p) \subseteq A$. Thus, $\forall p \in A, N_r(p) \subseteq A$ and $N_l(p) \subseteq A$ which implies $A \in \tau_r$ and $A \in \tau_l$. Hence $\tau_u \subseteq \tau_r$ and $\tau_u \subseteq \tau_l$.

By using Proposition 5.2, we can get $\tau_u \subseteq \tau_i$. □

Remark 5.4. Let the triple $\langle U, \mathcal{R}, \mathcal{C}_n \rangle$ be $\mathcal{G}_n - \boldsymbol{CAS}$ associated with topologies τ_j. Then the following statements are not true in general.

(*i*) $\tau_u = \tau_r$.

(*ii*) $\tau_u = \tau_l$.

(*iii*) $\tau_u = \tau_i$.

The following example shows Remark 5.4.

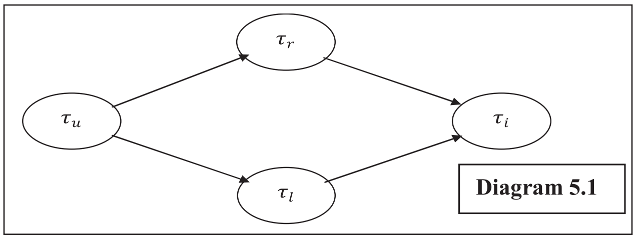
Example 5.4. Let the triple $\langle U, \mathcal{R}, \mathcal{C}_n \rangle$ be $\mathcal{G}_n - \boldsymbol{CAS}$, where $U = \{a, b, c, d\}$ and

$\mathcal{R} = \{(a, a), (a, b), (b, c), (b, d), (c, a), (d, a)\}$. Then we can get:

$\tau_r = \{U, \emptyset, \{a\}, \{a, b\}, \{c, d\}, \{a, c, d\}\}, \tau_l = \{U, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c, d\}\}$,

$\tau_u = \{U, \emptyset, \{a\}, \{a, b\}, \{a, c, d\}\}$ and $\tau_i = \{U, \emptyset, \{a\}, \{b\}, \{a, b\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}\}$.

Remark 5.5. Let the triple $\langle U, \mathcal{R}, \mathcal{C}_n \rangle$ be $\mathcal{G}_n - \boldsymbol{CAS}$ associated with topologies τ_j. Then the implications between different topologies τ_j can be given in the following diagram (where the arrow $\rightarrow$ means $\subseteq$).



6. Conclusion and future works

In this paper, we have introduced $\mathcal{G}_n-$ covering approximation space $\mathcal{G}_n - \boldsymbol{CAS}$ as a generalization to classical rough set theory and covering-based rough set theory using general binary relation. Accordingly, four different pairs of dual approximation operators have been defined and their properties have been discussed. The relationships among these operators were investigated. The best approximations of $\mathcal{G}_n - \boldsymbol{CAS}$ are in the case of $j = i$, since the approximations in this case are more accurate than the other cases $j \in \{u, r, l\}$. In addition, the boundary region is decreased by increasing the lower approximation and decreasing the upper approximation. Moreover, we have introduced comparisons between our approaches and some of the other approaches.

Finally, considering the notion of neighborhood, we have introduced a new method to generate general topological spaces. Using this technique, we have generated different topologies from any binary relation (directly from relations without using subbase or base) which will narrow the gap between topologists and applications.

In the future works, we will introduce many topological applications in rough context and also many real life applications by using the suggested structures in this paper.

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