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ORIGINAL ARTICLE

On Akbar-Zadeh's theorem on a Finsler space of constant curvature[☆]

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Abstract The aim of the present paper is to give two *intrinsic* generalizations of Akbar-Zadeh's theorem on a Finsler space of constant curvature. Some consequences, of these generalizations, are drown.

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0. Introduction

Using the tools of the traditional tensor calculus, in his paper [1], Akbar-Zadeh proved that if the h-curvature R^r_{ijk} of the Cartan connection $C\Gamma$ associated with a Finsler manifold (M, L), $dim M \geqslant 3$, satisfies

$$R^r_{ijk} = k(g_{ij}\delta^r_k - g_{ik}\delta^r_j),$$

where k is a scalar function on $\mathcal{T}\, M$, positively homogeneous of degree zero ((0) p-homogeneous), then

- (a) k is constant,
- (b) if $k \neq 0$, then
  - (1) the v-curvature of $C\Gamma$ vanishes: $S^r_{ijk} = 0$,
  - (2) the hv-curvature of $C\Gamma$ is symmetric with respect to the last two indices: $P^r_{ijk} = P^r_{ikj}$.

In his paper [2], Hōjō showed, also by local calculations, that if the h-curvature R^r_{ijk} of the generalized Cartan connection $C\Gamma$, $dim M \geqslant 3$, satisfies[1]

$$R^r_{ijk} = k\mathfrak{A}_{j,k}\left\{q\, g_{ij}\delta^r_k + (q-2)(g_{ij}\ell_k\ell^r - \delta^r_j\ell_i\ell_k\right\},$$

where k is a (0) p-homogeneous scalar function and $1 \neq q \in \mathbb{R}$, then

- (a) k is constant,
- (b) if $k \neq 0$, then
  - (1) the v-curvature of $C\Gamma$ satisfies $S^r_{ijk} = \frac{q-2}{2(1-q)}\mathfrak{A}_{j,k}\{\hbar_{ij}\hbar^r_k\}$,
  - (2) the hv-curvature of $C\Gamma$ is symmetric with respect to the last two indices.

The aim of the present paper is to provide *intrinsic* proofs of Akbar-Zadeh's and Hōjō's theorems. As a by-product, some consequences concerning S_3-like and S_4-like spaces are deduced.

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[1] $\mathfrak{A}_{ij}$ indicates interchanges of indices j and k, and subtraction: $\mathfrak{A}_{ij}\{F_{ij}\} = F_{ij} - F_{ji}$.

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Thus the present work is formulated in a coordinate-free form, without being trapped into the complications of indices. Naturally, the coordinate expressions of the obtained results coincide with the starting local formulations.

1. Notation and preliminaries

In this section, we give a brief account of the basic concepts of the pullback approach to intrinsic Finsler geometry necessary for this work. For more details, we refer to [3–5]. We shall use the same notations as in [5].

In what follows, we denote by $\pi : \mathcal{T}M \to M$ the subbundle of nonzero vectors tangent to M and by $\mathfrak{X}(\pi(M))$ the $\mathfrak{F}(\mathcal{T}M)$-module of differentiable sections of the pullback bundle $\pi^{-1}(TM)$. The elements of $\mathfrak{X}(\pi(M))$ will be called π-vector fields and will be denoted by barred letters $\overline{X}$. The tensor fields on $\pi^{-1}(TM)$ will be called π-tensor fields. The fundamental π-vector field is the π-vector field $\overline{\eta}$ defined by $\overline{\eta}(u) = (u, u)$ for all $u \in \mathcal{T}M$.

We have the following short exact sequence of vector bundles

$$0 \to \pi^{-1}(TM) \xrightarrow{\gamma} T(\mathcal{T}M) \xrightarrow{\rho} \pi^{-1}(TM) \to 0,$$

with the well known definitions of the bundle morphisms ρ and γ. The vector space $V_u(\mathcal{T}M) = \{X \in T_u(\mathcal{T}M) : d\pi(X) = 0\}$ is called the vertical space to M at u.

Let D be a linear connection (or simply a connection) on the pullback bundle $\pi^{-1}(TM)$. We associate with D the map $K : T\mathcal{T}M \to \pi^{-1}(TM) : X \mapsto D_X\overline{\eta}$, called the connection map of D. The vector space $H_u(\mathcal{T}M) = \{X \in T_u(\mathcal{T}M) : K(X) = 0\}$ is called the horizontal space to M at u. The connection D is said to be regular if $T_u(\mathcal{T}M) = V_u(\mathcal{T}M) \oplus H_u(\mathcal{T}M) \forall u \in \mathcal{T}M$.

If M is endowed with a regular connection, then the vector bundle maps $\gamma, \rho|_{H(\mathcal{T}M)}$ and $K|_{V(\mathcal{T}M)}$ are vector bundle isomorphisms. The map $\beta := (\rho|_{H(\mathcal{T}M)})^{-1}$ will be called the horizontal map of the connection D.

The horizontal ((*h*)*h*-) and mixed ((*h*)*hv*-) torsion tensors of D, denoted by Q and T respectively, are defined by

$$Q(\overline{X},\overline{Y}) = \mathbf{T}(\beta\overline{X}\beta\overline{Y}),\ T(\overline{X},\overline{Y}) = \mathbf{T}(\gamma\overline{X},\beta\overline{Y}) \quad \forall \overline{X},\overline{Y} \in \mathfrak{X}(\pi(M)),$$

where $\mathbf{T}$ is the torsion tensor field of D defined by

$$\mathbf{T}(X, Y) = D_X\rho Y - D_Y\rho X - \rho[X, Y] \quad \forall X, Y \in \mathfrak{X}(\mathcal{T}M).$$

The horizontal (*h*-), mixed (*hv*-) and vertical (*v*-) curvature tensors of D, denoted by R, P and S respectively, are defined by

$$R(\overline{X},\overline{Y})\overline{Z} = \mathbf{K}(\beta\overline{X}\beta\overline{Y})\overline{Z}, \quad P(\overline{X},\overline{Y})\overline{Z} = \mathbf{K}(\beta\overline{X},\gamma\overline{Y})\overline{Z},$$
$$S(\overline{X},\overline{Y})\overline{Z} = \mathbf{K}(\gamma\overline{X},\gamma\overline{Y})\overline{Z},$$

where $\mathbf{K}$ is the (classical) curvature tensor field associated with D.

The contracted curvature tensors of D, denoted by $\widehat{R}$, $\widehat{P}$ and $\widehat{S}$ respectively, known also as the (*v*)*h*-, (*v*)*hv*- and (*v*)*v*-torsion tensors, are defined by

$$\widehat{R}(\overline{X},\overline{Y}) = R(\overline{X},\overline{Y})\overline{\eta}, \quad \widehat{P}(\overline{X},\overline{Y}) = P(\overline{X},\overline{Y})\overline{\eta},$$
$$\widehat{S}(\overline{X},\overline{Y}) = S(\overline{X},\overline{Y})\overline{\eta}.$$

If M is endowed with a metric g on $\pi^{-1}(TM)$, we write

$$\begin{aligned} R(\overline{X},\overline{Y},\overline{Z},\overline{W}) &:= g(R(\overline{X},\overline{Y})\overline{Z},\overline{W}), \ldots, S(\overline{X},\overline{Y},\overline{Z},\overline{W}) \\ &:= g(S(\overline{X},\overline{Y})\overline{Z},\overline{W}). \end{aligned} \tag{1.1}$$

The following result is of extreme importance.

Theorem 1.1 [6]. *Let (M, L) be a Finsler manifold and g the Finsler metric defined by L. There exists a unique regular connection $\$$ on $\pi^{-1}(TM)$ such that*

(a) *$\$$ is metric : $\nabla g = 0$,*
(b) *The (h)h-torsion of $\$$ vanishes: $Q = 0$,*
(c) *The (h)hv-torsion T of $\$$ satisfies: $g(T(\overline{X},\overline{Y}),\overline{Z}) = g(T(\overline{X},\overline{Z}),\overline{Y})$.*

This connection is called the Cartan connection of the Finsler manifold (M, L).

2. First generalization of Akbar-Zadeh theorem

In this section, we investigate an intrinsic generalization of Akbar-Zadeh theorem. We begin first with the following two lemmas which will be useful for subsequent use.

Lemma 2.1. *Let $\$$ be the Cartan connection of a Finsler manifold (M, L). For a π-tensor field ω of type $(1, 1)$, we have the following commutation formulae:*

(a) $\left(\overset{2}{\nabla}\overset{2}{\nabla}\omega\right)(\overline{X},\overline{Y},\overline{Z}) - \left(\overset{2}{\nabla}\overset{2}{\nabla}\omega\right)(\overline{Y},\overline{X},\overline{Z}) = \omega(S(\overline{X},\overline{Y})\overline{Z}) - S(\overline{X},\overline{Y})\omega(\overline{Z})$,
(b) $\left(\overset{2}{\nabla}\overset{1}{\nabla}\omega\right)(\overline{X},\overline{Y},\overline{Z}) - \left(\overset{1}{\nabla}\overset{2}{\nabla}\omega\right)(\overline{Y},\overline{X},\overline{Z}) = \omega(P(\overline{X},\overline{Y})\overline{Z}) - P(\overline{X},\overline{Y})\omega(\overline{Z}) + \left(\overset{2}{\nabla}\omega\right)(\widehat{P}(\overline{X},\overline{Y}),\overline{Z}) + \left(\overset{1}{\nabla}\omega\right)(T(\overline{Y},\overline{X}),\overline{Z})$,
(c) $\left(\overset{1}{\nabla}\overset{1}{\nabla}\omega\right)(\overline{X},\overline{Y},\overline{Z}) - \left(\overset{1}{\nabla}\overset{1}{\nabla}\omega\right)(\overline{Y},\overline{X},\overline{Z}) = \omega(R(\overline{X},\overline{Y})\overline{Z}) - R(\overline{X}, \overline{Y})\omega(\overline{Z}) + \left(\overset{2}{\nabla}\omega\right)(\widehat{R}(\overline{X},\overline{Y}),\overline{Z})$,

where $\overset{1}{\nabla}$ and $\overset{2}{\nabla}$ are the h- and v-covariant derivatives associated with $\$$.

Lemma 2.2. *Let (M, L) be a Finsler manifold, g the Finsler metric defined by $L, \ell := L^{-1}i_{\overline{\eta}}g$ and $\hbar := g - \ell \circ \ell$ the angular metric tensor. Then we have:*

(a) $\overset{1}{\nabla} L = 0, \quad \overset{2}{\nabla} L = \ell.$
(b) $\underset{1}{\nabla}\ell = 0, \quad \underset{2}{\nabla}\ell = L^{-1}\hbar.$
(c) $i_{\overline{\eta}}\ell = L, \quad i_{\overline{\eta}}\hbar = 0.$

Proof. The assertions follow the facts that $\nabla g = 0$ and $g(\overline{\eta}, \overline{\eta}) = L^2$. □

Now, we have

Theorem 2.3. *Let (M, L) be a Finsler manifold of dimension n and g the Finsler metric defined by L. If the (v)h-torsion tensor $\widehat{R}$ of the Cartan connection is of the form*

$$\widehat{R}(\overline{X},\overline{Y}) = kL(\ell(\overline{X})\overline{Y} - \ell(\overline{Y})\overline{X}), \tag{2.1}$$

where k is a positive homogeneous function of degree 0 on TM, then:

(a) $\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}}R(\overline{X},\overline{Y})\overline{Z}=0.$ [2]
(b) *k is constant if $\dim M\geqslant 3$.*

Proof.

(a) We have [7]:

$$\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}}R(\overline{X},\overline{Y})\overline{Z}=\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}}T(\widehat{R}(\overline{X},\overline{Y}),\overline{Z}). \tag{2.2}$$

From (2.1), noting that the $(h)hv$-torsion T is symmetric, we obtain

$$\begin{aligned}\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}}T(\widehat{R}(\overline{X},\overline{Y}),\overline{Z})&=kLT(\ell(\overline{X})\overline{Y}-\ell(\overline{Y})\overline{X},\overline{Z})+kLT(\ell(\overline{Y})\overline{Z}\\&\quad-\ell(\overline{Z})\overline{Y},\overline{X})+kLT(\ell(\overline{Z})\overline{X}-\ell(\overline{X})\overline{Z},\overline{Y})\\&=kL\{\ell(\overline{X})T(\overline{Y},\overline{Z})-\ell(\overline{Y})T(\overline{X},\overline{Z})\}\\&\quad+kL\{\ell(\overline{Y})T(\overline{Z},\overline{X})-\ell(\overline{Z})T(\overline{Y},\overline{X})\}\\&\quad+kL\{\ell(\overline{Z})T(\overline{X},\overline{Y})-\ell(\overline{X})T(\overline{Z},\overline{Y})\}=0.\end{aligned} \tag{2.3}$$

Hence, the result follows from (2.2) and (2.3).

(b) We have [7]:

$$\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}}\{(\nabla_{\beta\overline{X}}R)(\overline{Y},\overline{Z},\overline{W})+P(\widehat{R}(\overline{X},\overline{Y}),\overline{Z})\overline{W}\}=0. \tag{2.4}$$

From (2.1), noting that the $(v)hv$-torsion $\widehat{P}$ is symmetric [7], we get

$$\begin{aligned}\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}}\widehat{P}(\widehat{R}(\overline{X},\overline{Y}),\overline{Z})&=kL\Big\{\ell(\overline{X})\widehat{P}(\overline{Y},\overline{Z})-\ell(\overline{Y})\widehat{P}(\overline{X},\overline{Z})\Big\}\\&\quad+kL\Big\{\ell(\overline{Y})\widehat{P}(\overline{Z},\overline{X})-\ell(\overline{Z})\widehat{P}(\overline{Y},\overline{X})\Big\}\\&\quad+kL\Big\{\ell(\overline{Z})\widehat{P}(\overline{X},\overline{Y})-\ell(\overline{X})\widehat{P}(\overline{Z},\overline{Y})\Big\}=0.\end{aligned}$$

From this and (2.4) it follows that

$$\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}}(\nabla_{\beta\overline{X}}\widehat{R})(\overline{Y},\overline{Z})=0. \tag{2.5}$$

Again from (2.1), noting that $\nabla_{\beta\overline{X}}\ell=0$ (Lemma 2.2(b)), (2.5) reads

$$\begin{aligned}&L(\nabla_{\beta\overline{X}}k)\{\ell(\overline{Y})\overline{Z}-\ell(\overline{Z})\overline{Y}\}+L(\nabla_{\beta\overline{Y}}k)\{\ell(\overline{Z})\overline{X}-\ell(\overline{X})\overline{Z}\}\\&\quad+L(\nabla_{\beta\overline{Z}}k)\{\ell(\overline{X})\overline{Y}-\ell(\overline{Y})\overline{X}\}=0.\end{aligned}$$

Setting $\overline{Z}=\overline{\eta}$ into the above equation, noting that $\ell(\overline{\eta})=L$ (Lemma 2.2(c)), we obtain

$$\begin{aligned}&L(\nabla_{\beta\overline{X}}k)\{\ell(\overline{Y})\overline{\eta}-L\overline{Y}\}+L(\nabla_{\beta\overline{Y}}k)\{L\overline{X}-\ell(\overline{X})\overline{\eta}\}\\&\quad+L(\nabla_{\beta\overline{\eta}}k)\{\ell(\overline{X})\overline{Y}-\ell(\overline{Y})\overline{X}\}=0.\end{aligned}$$

Taking the trace of both sides with respect to $\overline{Y}$, it follows that

$$\nabla_{\beta\overline{X}}k=L^{-1}(\nabla_{\beta\overline{\eta}}k)\ell(\overline{X}). \tag{2.6}$$

On the other hand, we have [7]

$$\begin{aligned}&(\nabla_{\gamma\overline{X}}R)(\overline{Y},\overline{Z},\overline{W})+(\nabla_{\beta\overline{Y}}P)(\overline{Z},\overline{X},\overline{W})-(\nabla_{\beta\overline{Z}}P)(\overline{Y},\overline{X},\overline{W})\\&\quad-P(\overline{Z},\widehat{P}(\overline{Y},\overline{X}))\overline{W}+R(T(\overline{X},\overline{Y}),\overline{Z})\overline{W}-S(\widehat{R}(\overline{Y},\overline{Z}),\overline{X})\overline{W}\\&\quad+P(\overline{Y},\widehat{P}(\overline{Z},\overline{X}))\overline{W}-R(T(\overline{X},\overline{Z}),\overline{Y})\overline{W}=0.\end{aligned} \tag{2.7}$$

Setting $\overline{W}=\overline{\eta}$ into the above relation, noting that $K\circ\gamma=id_{\mathfrak{X}(\pi(M))}, K\circ\beta=0$ and $\widehat{S}=0$, it follows that

$$\begin{aligned}&(\nabla_{\gamma\overline{X}}\widehat{R})(\overline{Y},\overline{Z})-R(\overline{Y},\overline{Z})\overline{X}+(\nabla_{\beta\overline{Y}}\widehat{P})(\overline{Z},\overline{X})-(\nabla_{\beta\overline{Z}}\widehat{P})(\overline{Y},\overline{X})\\&\quad-\widehat{P}(\overline{Z},\widehat{P}(\overline{Y},\overline{X}))+\widehat{R}(T(\overline{X},\overline{Y}),\overline{Z})+\widehat{P}(\overline{Y},\widehat{P}(\overline{Z},\overline{X}))\\&\quad-\widehat{R}(T(\overline{X},\overline{Z}),\overline{Y})=0.\end{aligned}$$

Applying the cyclic sum $\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}}$ on the above equation, taking **(a)** into account, we get

$$\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}}(\nabla_{\gamma\overline{X}}\widehat{R})(\overline{Y},\overline{Z})=0. \tag{2.8}$$

Substituting (2.1) into (2.8), using $(\nabla_{\gamma\overline{X}}\ell)(\overline{Y})=L^{-1}\hbar(\overline{X},\overline{Y})$ (Lemma 2.2(b)), we have

$$\begin{aligned}&L(\nabla_{\gamma\overline{Z}}k)\{(\ell(\overline{X})\overline{Y}-\ell(\overline{Y})\overline{X})\}+L(\nabla_{\gamma\overline{Y}}k)\{(\ell(\overline{Z})\overline{X}-\ell(\overline{X})\overline{Z})\}\\&\quad+L(\nabla_{\gamma\overline{X}}k)\{(\ell(\overline{Y})\overline{Z}-\ell(\overline{Z})\overline{Y})\}+k\ell(\overline{Z})\{(\ell(\overline{X})\overline{Y}-\ell(\overline{Y})\overline{X})\}\\&\quad+k\ell(\overline{Y})\{(\ell(\overline{Z})\overline{X}-\ell(\overline{X})\overline{Z})\}+k\ell(\overline{X})\{(\ell(\overline{Y})\overline{Z}-\ell(\overline{Z})\overline{Y})\}\\&\quad+kL\{(\hbar(\overline{X},\overline{Z})\overline{Y}-\hbar(\overline{Y},\overline{Z})\overline{X})\}+kL\{(\hbar(\overline{Z},\overline{Y})\overline{X}-\hbar(\overline{X},\overline{Y})\overline{Z})\}\\&\quad+kL\{(\hbar(\overline{Y},\overline{X})\overline{Z}-\hbar(\overline{Z},\overline{X})\overline{Y})\}=0.\end{aligned}$$

Setting $\overline{Z}=\overline{\eta}$ into the above relation, noting that $\ell(\overline{\eta})=L$,$\hbar(\overline{\eta},.)=0$ (Lemma 2.2(c)) and $\nabla_{\gamma\overline{\eta}}k=0$, we conclude that

$$L^2\Big\{\nabla_{\gamma\overline{X}}k\,\phi(\overline{Y})-\nabla_{\gamma\overline{Y}}k\,\phi(\overline{X})\Big\}=0, \tag{2.9}$$

where ϕ is a vector π-form defined by

$$g(\phi(\overline{X}),\overline{Y}):=\hbar(\overline{X},\overline{Y}). \tag{2.10}$$

Taking the trace of both sides of (2.9) with respect to $\overline{Y}$, noting that $Tr(\phi)=n-1$ [8], it follows that

$$(n-2)\nabla_{\gamma\overline{X}}k=0.$$

Consequently,

$$\nabla_{\gamma\overline{X}}k=0\quad\text{for all }\overline{X}\in\mathfrak{X}(\pi(M)),\ \text{if }n\geqslant 3. \tag{2.11}$$

Now, applying the v-covariant derivative with respect to $\overline{Y}$ on both sides of (2.6), yields

$$\begin{aligned}\ell(\overline{Y})\nabla_{\beta\overline{X}}k+L\left(\overset{2}{\nabla}\overset{1}{\nabla}k\right)(\overline{X},\overline{Y})&=L^{-1}\hbar(\overline{X},\overline{Y})(\nabla_{\beta\overline{\eta}}k)\\&\quad+\ell(\overline{X})(\overset{2}{\nabla}\overset{1}{\nabla}k)(\overline{\eta},\overline{Y}).\end{aligned}$$

Since, $\overset{2}{\nabla}\overset{1}{\nabla}k=\overset{1}{\nabla}\overset{2}{\nabla}k=0$ (Lemma 2.1 and (2.11)), the above relation reduces to

$$\ell(\overline{Y})\nabla_{\beta\overline{X}}k=L^{-1}\hbar(\overline{X},\overline{Y})(\nabla_{\beta\overline{\eta}}k),$$

whenever $n\geqslant 3$. Setting $\overline{Y}=\overline{\eta}$ into the above equation, noting that $\ell(\overline{\eta})=L$ and $\hbar(.,\overline{\eta})=0$, it follows that $\nabla_{\beta\overline{\eta}}k=0$. Consequently, again by (2.6),

$$\nabla_{\beta\overline{X}}k=0\quad\text{for all}\quad\overline{X}\in\mathfrak{X}(\pi(M)),\quad\text{if }n\geqslant 3. \tag{2.12}$$

Now, Eqs. (2.11) and (2.12) imply that k is a constant if $n\geqslant 3$.

This completes the proof. □

Theorem 2.4. *Let (M,L) be a Finsler manifold with dimension $n\geqslant 3$ and let $q\neq 1$ be an arbitrary real number. If the h-curvature tensor R satisfies*

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[2] $\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}}$ *denotes the cyclic sum over $\overline{X},\overline{Y}$ and $\overline{Z}$.*

$$\begin{aligned}R(\overline{X},\overline{Y})\overline{Z} = k\,\mathfrak{A}_{\overline{X},\overline{Y}}\{q\,g(\overline{X},\overline{Z})\overline{Y} + (q-2)\{L^{-1}g(\overline{X},\overline{Z})\ell(\overline{Y})\bar{\eta} \\ -\ell(\overline{Y})\ell(\overline{Z})\overline{X}\}\},\end{aligned} \tag{2.13}$$

where k is an $h(0)$- homogeneous function, then

(a) *k is a constant.*
(b) *If $k \neq 0$, we have:*
  (1) $P(\overline{X},\overline{Y})\overline{Z} = P(\overline{Y},\overline{X})\overline{Z}$ (i.e., (M,L) is symmetric),
  (2) $S(\overline{X},\overline{Y})\overline{Z} = \frac{2-q}{2(q-1)L^2}\{\hbar(\overline{X},\overline{Z})\phi(\overline{Y}) - \hbar(\overline{Y},\overline{Z})\phi(\overline{X})\}$.

Proof.

(a) Setting $\overline{Z} = \bar{\eta}$ into (2.13), we get

$$\widehat{R}(\overline{X},\overline{Y}) = 2(q-1)kL\{\ell(\overline{X})\overline{Y} - \ell(\overline{Y})\overline{X}\}. \tag{2.14}$$

From this and Theorem 2.3, the result follows.

(b) (1) Applying the v-covariant derivative with respect to $\overline{W}$ on both sides of (2.13), we get

$$(\nabla_{\beta\overline{W}}R)(\overline{X},\overline{Y},\overline{Z}) = 0.$$

From this and (2.4) it follows that

$$\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}}\, P(\widehat{R}(\overline{X},\overline{Y}),\overline{Z})\overline{W} = 0. \tag{2.15}$$

In view of (2.14), noting that $k \neq 0$, (2.15) implies that

$$\begin{aligned}&2(q-1)L\{P(\ell(\overline{X})\overline{Y} - \ell(\overline{Y})\overline{X},\overline{Z})\overline{W}\} + 2(q-1) \\ &\quad L\{P(\ell(\overline{Y})\overline{Z} - \ell(\overline{Z})\overline{Y},\overline{X})\overline{W}\} + 2(q-1) \\ &\quad L\{P(\ell(\overline{Z})\overline{X} - \ell(\overline{X})\overline{Z},\overline{Y})\overline{W}\} = 0.\end{aligned}$$

Setting $\overline{Z} = \bar{\eta}$ into the above equation, taking into account the fact that $\ell(\bar{\eta}) = L$ and $P(.,\bar{\eta}). = P(\bar{\eta},.). = 0$ [7], we get

$$2(q-1)L\{P(\overline{X},\overline{Y})\overline{W} - P(\overline{Y},\overline{X})\overline{W}\} = 0.$$

Hence, the result follows.

(b) (2) Taking the cyclic sum $\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}}$ of (2.7) and using (b) (1), we obtain

$$\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}}\Big\{(\nabla_{\gamma\overline{X}}R)(\overline{Y},\overline{Z},\overline{W}) - S(\widehat{R}(\overline{Y},\overline{Z}),\overline{X})\overline{W}\Big\} = 0. \tag{2.16}$$

On the other hand, by taking the v-covariant derivative of both sides of (2.13), using $(\nabla_{\gamma\overline{X}}L) = \ell(\overline{X})$, $(\nabla_{\gamma\overline{X}}\ell)(\overline{Y}) = L^{-1}\hbar(\overline{X},\overline{Y})$ and $\nabla_{\gamma\overline{X}}g = 0$, we get

$$\begin{aligned}(\nabla_{\gamma\overline{X}}R)(\overline{Y},\overline{Z},\overline{W}) = k(q-2)\mathfrak{A}_{\overline{X},\overline{Y}}\{g(\overline{X},\overline{W})\hbar(\overline{Z},\overline{Y})\frac{\bar{\eta}}{L^2} \\ + g(\overline{X},\overline{W})\ell(\overline{Y})\frac{\phi(\overline{Z})}{L} - \hbar(\overline{Z},\overline{Y})\ell(\overline{W})\frac{\overline{X}}{L} \\ - \hbar(\overline{Z},\overline{W})\ell(\overline{Y})\frac{\overline{X}}{L}\}.\end{aligned}$$

Taking the cyclic sum $\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}}$ of both sides of the above equation and then setting $\overline{Z} = \bar{\eta}$, it follows that

$$\begin{aligned}\mathfrak{S}_{\overline{X},\overline{Y},\bar{\eta}}(\nabla_{\gamma\overline{X}}R)(\overline{Y},\bar{\eta},\overline{W}) = 2k(q-2)\{\hbar(\overline{Y},\overline{W})\phi(\overline{X}) \\ -\hbar(\overline{X},\overline{W})\phi(\overline{Y})\}.\end{aligned} \tag{2.17}$$

In view of (2.14), noting that $S(.,\bar{\eta}). = 0$ and S is antisymmetric [7], we obtain

$$\mathfrak{S}_{\overline{X},\overline{Y},\bar{\eta}}S(\widehat{R}(\overline{Y},\bar{\eta}),\overline{X})\overline{W} = 4kL^2(q-1)S(\overline{X},\overline{Y})\overline{W}. \tag{2.18}$$

Therefore, by setting $\overline{Z} = \bar{\eta}$ into (2.16), taking (2.17) and (2.18) into account, the result follows. □

Corollary 2.5. *Akbar-Zadeh's theorem [1] is a special case of Theorem 2.4, for which $q = 2$.*

Corollary 2.6. *If the h-curvature tensor R of (M,L), where $\dim M \geqslant 3$, satisfies*

$$R(\overline{X},\overline{Y})\overline{Z} = k\mathfrak{A}_{\overline{X},\overline{Y}}\left\{\{g(\overline{X},\overline{Z})\frac{\bar{\eta}}{L} - \ell(\overline{Z})\overline{X}\}\ell(\overline{Y})\right\},$$

then k is a constant and, moreover, if $k \neq 0$, we have:

(a) *(M,L) is symmetric.*
(b) *$S(\overline{X},\overline{Y})\overline{Z} = \frac{-1}{L^2}\{\hbar(\overline{X},\overline{Z})\phi(\overline{Y}) - \hbar(\overline{Y},\overline{Z})\phi(\overline{X})\}$.*

3. Second generalization of Akbar-Zadeh's theorem

In this section, we give a second, intrinsically formulated generalization of Akbar-Zadeh's theorem.

Theorem 3.1. *If the h-curvature tensor R of (M,L), $\dim M \geqslant 3$, satisfies*

$$R(\overline{X},\overline{Y})\overline{Z} = k\{g(\overline{X},\overline{Z})\overline{Y} - g(\overline{Y},\overline{Z})\overline{X} + \omega(\overline{X},\overline{Y})\overline{Z}\}, \tag{3.1}$$

where ω is an indicatory antisymmetric h(0) π-tensor field of type $(1,3)$ and k is an h(0)-function on TM, then

(a) *k is a constant.*
(b) *If $k \neq 0$, we have:*

$$(1) P(\overline{X},\overline{Y})\overline{Z} - P(\overline{X},\overline{Y})\overline{Z} = L^{-2}(\nabla_{\beta\bar{\eta}}\omega)(\overline{X},\overline{Y},\overline{Z}).$$

$$(2) S(\overline{X},\overline{Y})\overline{Z} = \frac{1}{L^2}\left\{\frac{1}{2kL^2}(\overset{1}{\nabla}\,\overset{1}{\nabla}\,\omega)(\bar{\eta},\bar{\eta},\overline{X},\overline{Y},\overline{Z}) + \omega(\overline{X},\overline{Y})\overline{Z}\right\}.$$

Proof.

(a) Follows from Theorem 2.3 by setting $\overline{Z} = \bar{\eta}$ into (3.1).
(b) (1) By (3.1), we have

$$\widehat{R}(\overline{X},\overline{Y}) = kL\{\ell(\overline{X})\overline{Y} - \ell(\overline{Y})\overline{X}\}, \tag{3.2}$$

and by (2.4), we have

$$\mathfrak{S}_{\overline{X},\overline{Y},\bar{\eta}}\{(\nabla_{\beta\overline{X}}R)(\overline{Y},\bar{\eta},\overline{W}) + P(\widehat{R}(\overline{X},\overline{Y}),\bar{\eta})\overline{W}\} = 0. \tag{3.3}$$

Now, substituting (3.1) and (3.2) into (3.3), we obtain

$$k\{(\nabla_{\beta\bar{\eta}}\omega)(\overline{X},\overline{Y},\overline{W}) - L^2\{P(\overline{X},\overline{Y})\overline{Z} - P(\overline{Y},\overline{X})\overline{Z}\}\} = 0$$

from this, since $k \neq 0$, the result follows.

(b) (2) Taking the cyclic sum $\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}}$ of (2.7), we obtain

$$\begin{aligned}\mathfrak{S}_{\overline{X},\overline{Y},\overline{Z}}\Big\{(\nabla_{\gamma\overline{X}}R)(\overline{Y},\overline{Z},\overline{W}) + (\nabla_{\beta\overline{Y}}P)(\overline{Z},\overline{X},\overline{W}) \\ -(\nabla_{\beta\overline{Z}}P)(\overline{Y},\overline{X},\overline{W}) - S(\widehat{R}(\overline{Y},\overline{Z}),\overline{X})\overline{W}\Big\} = 0.\end{aligned} \tag{3.4}$$

In view of (1) above, it follows that

$$\begin{aligned}(\nabla_{\beta\overline{W}}P)(\overline{X},\overline{Y},\overline{Z}) - (\nabla_{\beta\overline{W}}P)(\overline{Y},\overline{X},\overline{Z}) \\ = L^{-2}\Big(\overset{1}{\nabla}\,\overset{1}{\nabla}\,\omega\Big)(\overline{W},\bar{\eta},\overline{X},\overline{Y},\overline{Z}).\end{aligned}$$

From this we get

$$\mathfrak{S}_{\overline{X},\overline{Y},\bar{\eta}}\{(\nabla_{\beta\bar{\eta}}P)(\overline{X},\overline{Y},\overline{Z})-(\nabla_{\beta\bar{\eta}}P)(\overline{Y},\overline{X},\overline{Z})\}$$
$$=L^{-2}(\overset{1}{\nabla}\overset{1}{\nabla}\omega)(\bar{\eta},\bar{\eta},\overline{X},\overline{Y},\overline{Z}). \tag{3.5}$$

On the other hand, noting that ω is homogeneous of degree zero, we obtain

$$\begin{aligned}\mathfrak{S}_{\overline{X},\overline{Y},\bar{\eta}}(\nabla_{\gamma\overline{X}}R)(\overline{Y},\bar{\eta},\overline{W}) &= (\nabla_{\gamma\overline{X}}R)(\overline{Y},\bar{\eta},\overline{W})+(\nabla_{\gamma\overline{Y}}R)\\ &\quad\times(\bar{\eta},\overline{X},\overline{W})+(\nabla_{\gamma\bar{\eta}}R)\\ &\quad\times(\overline{X},\overline{Y},\overline{W})\\ &=2k\omega(\overline{X},\overline{Y})\overline{W}.\end{aligned} \tag{3.6}$$

$$\mathfrak{S}_{\overline{X},\overline{Y},\bar{\eta}}S(\widehat{R}(\overline{X},\overline{Y}),\bar{\eta})\overline{W}=2kL^2S(\overline{X},\overline{Y})\overline{W}. \tag{3.7}$$

Setting $\overline{Z}=\bar{\eta}$ into (3.4), taking into account (3.5)–(3.7), the result follows. ☐

Corollary 3.2. *Akbar-Zadeh's theorem [1] is obtained from the above Theorem by letting* $\omega=0$.

Corollary 3.3. *A Finsler manifold* (M,L) *is* S_3*-like if* ω *in Theorem 3.1 is given by*

$$\omega(\overline{X},\overline{Y})\overline{Z}=s\{\hbar(\overline{X},\overline{Z})\phi(\overline{Y})-\hbar(\overline{Y},\overline{Z})\phi(\overline{X})\}, \tag{3.8}$$

where ϕ *is given by (2.10) and* s *is a scalar function depending only on the position.*

Proof. From Theorem 3.1(b) and (3.8), the v-curvature tensor S takes the form:

$$S(\overline{X},\overline{Y})\overline{Z}=\frac{1}{L^2}\left\{s+\frac{(\overset{1}{\nabla}\overset{1}{\nabla}s)(\bar{\eta},\bar{\eta})}{2kL^2}\right\}\{\hbar(\overline{X},\overline{Z})\phi(\overline{Y})-\hbar(\overline{Y},\overline{Z})\phi(\overline{X})\}.$$

As the v-curvature tensor S is written in the above form, then the term

$$\left\{s+\frac{(\overset{1}{\nabla}\overset{1}{\nabla}s)(\bar{\eta},\bar{\eta})}{2kL^2}\right\}$$

depends on the position only [9], and so (M,L) is S_3-like. ☐

Corollary 3.4. *If the scalar function* s *in (3.8) is constant, we have:*

(a) $P(\overline{X},\overline{Y})\overline{Z}=P(\overline{Y},\overline{X})\overline{Z}$.
(b) $S(\overline{X},\overline{Y})\overline{Z}=\frac{s}{L^2}\{\hbar(\overline{X},\overline{Z})\phi(\overline{Y})-\hbar(\overline{Y},\overline{Z})\phi(\overline{X})\}$.

Corollary 3.5. *If the tensor field* ω *in Theorem 3.1 is given by*

$$\omega(\overline{X},\overline{Y})\overline{Z}=\mathfrak{A}_{\overline{X},\overline{Y}}\{H(\overline{X},\overline{Z})\phi(\overline{Y})+\hbar(\overline{X},\overline{Z})H_o(\overline{Y})\},$$

where H *is a symmetric indicatory* $h(0)$ *2-scalar* π*-form and* $H(\overline{X},\overline{Y})\ =:\ g(H_o(\overline{X}),\overline{Y})$, *then* (M,L) *is* S_4*-like, that is,*

$$S(\overline{X},\overline{Y})\overline{Z}=\frac{1}{L^2}\mathfrak{A}_{\overline{X},\overline{Y}}\{\mu(\overline{X},\overline{Z})\phi(\overline{Y})+\hbar(\overline{X},\overline{Z})\mu_o(\overline{Y})\},$$

where $\mu(\overline{X},\overline{Y})=\left\{H(\overline{X},\overline{Y})+\frac{\left(\overset{1}{\nabla}\overset{1}{\nabla}H\right)(\bar{\eta},\bar{\eta},\overline{X},\overline{Y})}{2kL^2}\right\}.$

Proof. The proof is clear and we omit it. ☐

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