Journal of the Egyptian Mathematical Society (2015) 23, 513–517

Egyptian Mathematical Society

Journal of the Egyptian Mathematical Society

www.etms-eg.org
www.elsevier.com/locate/joems

ORIGINAL ARTICLE

Translation L/W-surfaces in Euclidean 3-Space E^3

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Received 31 July 2013; revised 3 June 2014; accepted 13 July 2014
Available online 11 February 2015

KEYWORDS

Translation surfaces;
Weingarten surfaces

Abstract In this paper, we construct and obtain the necessary condition of Weingarten and linear Weingarten translation surfaces in E^3. Special cases of these types are investigated and plotted.

2000 MATHEMATICS SUBJECT CLASSIFICATION: 53A04; 53A05

1. Introduction

Translation surfaces properties have an important role in shaping the construction and for architectural design. Translation surfaces are used for shells in engineering. Also, In the design of structures such as membranes, domes, cables grids, vaults, foldable structures and so on [1].

Here, and in the sequel, we assume that the indices $\{i,j\}$ run over the range $\{1,2\}$. In the study of the differential geometry of submanifolds, it is common to examine different types of curvature conditions. More precisely, one is eager to determine all submanifolds satisfying such a condition. An interesting curvature property to study for a surface $M : \mathbf{X} = \mathbf{X}(s_i)$ in an Euclidean space E^3, is the one that requires the existence a functional relationship $\varphi(k_i) = 0$ between the principal curvatures is called Weingarten surfaces or W-surface. Using the Gaussian and mean curvatures (K, H)we can redefine W-surfaces, as surfaces satisfying $\varphi(K,H) = 0$, or, equivalently, the corresponding Jacobian determinant is identically zero, i.e.,

$$\varphi(K,H) = \left|\frac{\partial(K,H)}{\partial(s_i)}\right| \equiv 0. \tag{1.1}$$

Also, if the surfaces satisfy a linear equation with respect to K and H, that is, $aK + bH = c, (a,b,c \in \mathbb{R}, (a,b,c) \neq (0,0,0))$, are called linear Weingarten surfaces or LW-surfaces. When the constant $b = 0$, a linear Weingarten surface reduces to a surface with constant Gaussian curvature. When the constant $a = 0$ a linear Weingarten surface reduces to a surface with constant mean curvature. In such a sense, the linear Weingarten surfaces can be regarded as a natural generalization of surfaces with constant Gaussian curvature or with constant mean curvature [2–4].

In Euclidean 3-space E^3 the relations $\varphi(K,H) = 0$, $\varphi(K_{II},H) = 0$, and $aK_{II} + bH = c$, on ruled surfaces have been investigated in [3,5]. For surfaces with $K_{II} = H$, $K_{II} = \sqrt{K}$, $K_{II} = c$; we refer to [3,6–8] for the history and general results in this problem. Also, for non-developable ruled surface the linear relations $aK_{II} + bH + cK = const.$, $a^2 + b^2 \neq 0$ along each ruling, have been studied in [9].

2. Geometric preliminaries

Let $C_1 : \boldsymbol{\alpha} = \boldsymbol{\alpha}(s_1)$ and $C_2 : \boldsymbol{\beta} = \boldsymbol{\beta}(s_2)$ are two curves parametrized by the arc lengths s_i in E^3. Consider the Frenet frame $\{\mathbf{t}_i(s_i), \mathbf{n}_i(s_i), \mathbf{b}_i(s_i)\}$ associated with the curves C_i. The

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Peer review under responsibility of Egyptian Mathematical Society.

Production and hosting by Elsevier

http://dx.doi.org/10.1016/j.joems.2014.07.005
1110-256X

derivatives of the vectors $\mathbf{t}_i(s_i)$ and $\mathbf{b}_i(s_i)$, when expressed in the basis $\{\mathbf{t}_i, \mathbf{n}_i, \mathbf{b}_i\}$, yield geometrical entities, the natural curvatures $\kappa_i(s_i)$ and torsions $\tau_i(s_i)$, which give us information about the behavior of the curves $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ in a neighborhood of s_i, respectively. Then the Frenet formulas of the curves C_i are defined by [10]:

$$\frac{d}{ds_i}\begin{pmatrix}\mathbf{t}_i(s_i)\\ \mathbf{n}_i(s_i)\\ \mathbf{b}_i(s_i)\end{pmatrix} = \begin{pmatrix}0 & \kappa_i(s_i) & 0\\ -\kappa_i(s_i) & 0 & \tau_i(s_i)\\ 0 & -\tau_i(s_i) & 0\end{pmatrix}\begin{pmatrix}\mathbf{t}_i(s_i)\\ \mathbf{n}_i(s_i)\\ \mathbf{b}_i(s_i)\end{pmatrix}. \quad (2.1)$$

We denote a surface M in E^3 by

$$\mathbf{X}(s_i) = (x_k(s_i)), \quad k = 1, 2, 3$$

Let $\mathbf{N}$ be the standard unit normal vector field on a surface M defined by $\mathbf{N} = \frac{\mathbf{X}_{s_1} \times \mathbf{X}_{s_2}}{\|\mathbf{X}_{s_1} \times \mathbf{X}_{s_2}\|}$, where, $\mathbf{X}_{\mathbf{s_i}} = \frac{\partial \mathbf{X}}{\partial s_i}$. Then the 1-st and 2-nd fundamental forms of the surface M are defined respectively by

$$I = \sum_{i,j} g_{ij} ds_i\ ds_j,\ g_{ij} = < \mathbf{X}_{s_i}, \mathbf{X}_{s_j} > \text{ and}$$

$$II = \sum_{i,j} h_{ij}\ ds_i\ ds_j, h_{ij} = < \mathbf{X}_{s_i s_j}, \mathbf{N} > . \quad (2.2)$$

If the second fundamental form II is non-degenerate, then it can be regarded as a (pseudo-) Riemannian metric. Using classical notation, we define the second Gaussian curvature K_{II} by [11]

$$K_{II} = \frac{1}{h^2}\left(\begin{vmatrix} -\frac{h_{11,22}}{2} + h_{12,12} - \frac{h_{22,11}}{2} & \frac{h_{11,1}}{2} & h_{12,1} - \frac{h_{11,2}}{2}\\ h_{12,2} - \frac{h_{22,1}}{2} & h_{11} & h_{12}\\ \frac{h_{22,2}}{2} & h_{12} & h_{22}\end{vmatrix} - \begin{vmatrix} 0 & \frac{h_{11,2}}{2} & \frac{h_{22,1}}{2}\\ \frac{h_{11,2}}{2} & h_{11} & h_{12}\\ \frac{h_{22,1}}{2} & h_{12} & h_{22}\end{vmatrix}\right), \quad (2.3)$$

where, $h = \det(h_{ij}), h_{ij,l} = \frac{\partial h_{ij}}{\partial u^l}$, and $h_{ij,lm} = \frac{\partial^2 h_{ij}}{\partial u^l \partial u^m}$.

Since Brioschi's formulas in Euclidean 3-spaces, we are able to define H_{II} of M by replacing the components of the first fundamental form g_{ij} by the components of the second fundamental form h_{ij} respectively in Brioschi's formula. Consequently, the second mean curvature H_{II} is given by [12]:

$$H_{II} = H - \frac{1}{2}\Delta\left(\ln\sqrt{|K|}\right), \quad (2.4)$$

where Δ, is the Laplacian with respect to the second fundamental form of M, expressed as:

$$\Delta = -\frac{1}{\sqrt{|\ h\ |}}\frac{\partial}{\partial u^i}\left[\sqrt{|\ h\ |}h^{ij}\ \frac{\partial}{\partial u^j}\right], \quad \left(h^{ij}\right) = \left(h_{ij}\right)^{-1}. \quad (2.5)$$

3. Intrinsic geometry of translation surfaces in E^3

When a space curve is translated over another space curve, the resulting surface can be considered as the most general appearance of a translation surface. Consequently, this surface can be parameterized as the sum of two space curves. Quite often, the class of translation surfaces is restricted to those that can be parameterized as the sum of two plane curves. So it can be parameterized by a patch [13]:

$$M : \mathbf{X}(s_i) = \boldsymbol{\alpha}(s_1) + \boldsymbol{\beta}(s_2), \quad s_i \in I, \quad (3.1)$$

where s_i are the parameters of the arc lengths of the curves $\boldsymbol{\alpha}, \boldsymbol{\beta}$ respectively.

Using (2.1) and (2.2), It is easily checked that the metric of M is given by

$$(g_{ij}) = \begin{pmatrix}1 & cos\theta\\ cos\theta & 1\end{pmatrix},\ det\ (g_{ij}) = sin^2\theta,\ \theta \neq n\pi,\ n = 0, 1, 2, \ldots \quad (3.2)$$

The unit normal vector of the surface M is given by

$$\mathbf{N}(s_i) = (\mathbf{t}_1 \times \mathbf{t}_2)\ cosec\ \theta, \quad (3.3)$$

where $\mathbf{t}_i$ are denotes to the tangents of the curves $\boldsymbol{\alpha}, \boldsymbol{\beta}$ respectively.

This leads to the coefficients of the second fundamental form h_{ij} where

$$(h_{ij}) = \begin{pmatrix}\kappa_1[\mathbf{t}_1\mathbf{t}_2\mathbf{n}_1]cosec\ \theta & 0\\ 0 & \kappa_2[\mathbf{t}_1\mathbf{t}_2\mathbf{n}_2]cosec\ \theta\end{pmatrix}, \quad (3.4)$$

$$det\ (h_{ij}) = \prod_{i,j,\, i\neq j}\kappa_i[\mathbf{t}_1\mathbf{t}_2\mathbf{n}_j]cosec^2\ \theta, \quad (3.5)$$

where κ_i are denotes to the curvatures of the curves α, β respectively and $[t_1 t_2 n_j]$ denotes to the triple scalar product to these vectors.

From Eqs. (3.1) and (3.4), one can see that the Gaussian and mean curvature functions of M are given by

$$K = \prod_{i,j,\, i\neq j}\kappa_i[\mathbf{t}_1\mathbf{t}_2\mathbf{n}_j]cosec^4\ \theta, \quad (3.6)$$

and

$$H = \frac{1}{2}\sum_i \kappa_i[\mathbf{t}_1\mathbf{t}_2\mathbf{n}_i]cosec^3\ \theta, \quad (3.7)$$

respectively.

4. Translation L/W-surfaces in E^3

In this section, we study a translation L/W-surfaces in E^3, which satisfies nontrivial relation between elements of the set $\{K,\ K_{II},\ H,\ H_{II}\}$, where (K, H) and $(K_{II}\ ,\ H_{II})$ are the Gaussian and mean curvatures of the first and second fundamental forms, respectively. Following the Jacobian and the linear equations with respect to the set $\{K,\ K_{II},\ H,\ H_{II}\}$, an interesting geometric question is raised. Classify the translation surfaces in E^3 satisfying the conditions:

$$\varphi(\mu, \nu) = 0, \quad (4.1)$$

and

$$a\mu + b\nu = c, \quad (4.2)$$

where $\mu, \nu \in \{K,\ H,\ K_{II},\ H_{II}\}, \mu \neq \nu$ and $(a, b, c) \neq (0, 0, 0)$.Thus, we can write the Jacobian and the linear equations (4.1) and (4.2) as the following:

$$(K)_{s_1}(H)_{s_2} - (K)_{s_2}(H)_{s_1} = 0, \quad (4.3)$$

$$(K)_{s_1}(K_{II})_{s_2} - (K)_{s_2}\ (K_{II})_{s_1} = 0, \quad (4.4)$$

$$(H)_{s_1}(K_{II})_{s_2} - (H)_{s_2}\ (K_{II})_{s_1} = 0, \quad (4.5)$$

$$(H)_{s_1}(H_{II})_{s_2} - (H)_{s_2}\ (H_{II})_{s_1} = 0, \quad (4.6)$$

$$(K_{II})_{s_1}(H_{II})_{s_2} - (K_{II})_{s_2}(H_{II})_{s_1} = 0, \quad (4.7)$$

and

$$aK + bH = c, \tag{4.8}$$

$$aK + bK_{II} = c, \tag{4.9}$$

$$aH + bK_{II} = c, \tag{4.10}$$

$$aH + bH_{II} = c, \tag{4.11}$$

$$aH_{II} + bK_{II} = c, \tag{4.12}$$

respectively.

Differentiating K and H with respect to s_i, one can get

$$\left.\begin{array}{l}(K)_{s_i} = \kappa_j(\kappa_i\tau_i - \kappa_i^2 + \dot{\kappa}_i)f_1(\theta) + \kappa_j(2\kappa_i^2 + \dot{\kappa}_i)f_2(\theta) - \kappa_i\kappa_j(\kappa_i - \tau_i)f_3(\theta)\\ (H)_{s_i} = \frac{1}{2}\sum_i \kappa_i\tau_i[\mathbf{t}_1\mathbf{t}_2\mathbf{b}_i] + \dot{\kappa}_i[\mathbf{t}_1\mathbf{t}_2\mathbf{n}_i]cosec^3\theta,\end{array}\right\}, \tag{4.13}$$

where $\cdot = \frac{d}{ds_i}, f_1(\theta) = cos\theta\ cosec^2\theta, f_2(\theta) = sec\theta\ cosec^2\theta,\ f_3(\theta) = sec^2\theta\ cosec^2\theta$.

Thus, one can see that the Jacobian equation (4.3) is valid for

$$[\mathbf{t}_1\mathbf{t}_2\mathbf{b}_i] \neq 0, [\mathbf{t}_1\mathbf{t}_2\mathbf{n}_i] \neq 0, \tag{4.14}$$

and

$$(\kappa_i, \tau_i) = (c_i, c_j), \quad c_i, c_j = const. \neq 0, \tag{4.15}$$

which characterizes a circular helix curves. Therefore, we have the following

Theorem 1. The translation surface M is a W-surface in E^3 for a circular helix curves $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ with non-zero constant curvatures and torsions (see Fig. 1).

Theorem 2. The translation surface M is a W-surface in E^3 if the triple scalar products $[\mathbf{t}_1\mathbf{t}_2\mathbf{b}_i]$ and $[\mathbf{t}_1\mathbf{t}_2\mathbf{n}_i]$ are numerically equal to the volumes of parallelepipeds whose edges are determined by these vectors.

Next, to facilitate and simplify the calculations we used some geometric concepts of involutes and Bertrand curves of translation surfaces as a special cases. In the case αinvolute of β, we denote the surface by $\widetilde{M}$, so some of the previous results of the fundamental quantities take the symbol $\sim$ over them. Similarly, in the case $\alpha$and β are Bertrand curves, we take the symbol $\approx$.

4.1. α involute of β

When the tangents to a curve β are normals to another curve α, the latter is called an involute of the former. Hence the tangent

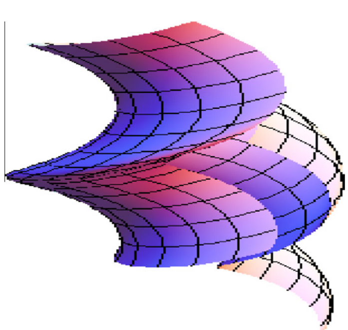

Figure 1 A translation W-surface for a circular helix curves $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

to the involute is parallel to the principal normal to the given curve. So, we have [14]:

$$\langle \mathbf{t}_1, \mathbf{t}_2 \rangle = 0, \quad \mathbf{t}_1 = \pm\ \mathbf{n}_2. \tag{4.16}$$

Thus and using (3.6) and (3.7), one can see that $det\ (h_{ij}) = 0$, so we have

Corollary 1. *The Gaussian curvature function of M is vanished* $\left(\widetilde{K} = 0\right)$ *and the second Gaussian curvature function is indefinite* $\left(\widetilde{K}_{II} = \infty\right)$.

Corollary 2. *The mean and second mean curvatures functions of* $\widetilde{M}$ *are given by*

$$\widetilde{H}_{II} = \widetilde{H} = \frac{1}{2}\kappa_1[\mathbf{n}_1\mathbf{n}_2\mathbf{t}_2]. \tag{4.17}$$

From the above results and taking into account of Jacobian Eqs. (4.4), (4.5) and (4.7) *we have*

Corollary 3. *The translation surface* $\widetilde{M}$ *is a flat surface and the curvature* κ_2 *of the curve* β *is vanished.*

Corollary 4. *There are no W-translation surfaces in* E^3*. Using Jacobian* Eqs. (4.3) and (4.6), *we find that they are vanished identically. So we get*

Corollary 5. *The translation surface* $\widetilde{M}$ *is a W-surface in* E^3*. According to the linear* Eqs. (4.9), (4.10) and (4.12) *we have*

Corollary 6. *There are no LW-translation surfaces in* E^3*. Based on the linear* Eqs. (4.8) and (4.11), *one can see that*

$$\widetilde{H} = constant \neq 0 \Rightarrow \kappa_1[\mathbf{n}_1\mathbf{n}_2\mathbf{t}_2] = constant \neq 0, \tag{4.18}$$

thus, for the translation surface $\widetilde{M}$ *of constant mean curvature (cmc), we have*

$$\kappa_1 = constant \neq 0, \quad [\mathbf{n}_1\mathbf{n}_2\mathbf{t}_2] = constant \neq 0, \tag{4.19}$$

which gives the following

Theorem 3. *The translation surface* $\widetilde{M}$ *of (cmc) is a LW-surface in* E^3 *for a circle curve* $\boldsymbol{\alpha}$ *with non-zero constant curvature.*

Theorem 4. *The translation surface* $\widetilde{M}$ *of (cmc) is a LW-surface in* E^3 *if the triple scalar product* $[\mathbf{n}_1\mathbf{n}_2\mathbf{t}_2]$ *is numerically equal to the volume of parallelepiped whose edges are determined by these vectors.*

4.2. α and β are Bertrand curves

Saint-Venant proposed and Bertrand solved the problem of finding the curves whose principal normals are also the principal normals of another curve. A pair of curves α and β having their principal normals in common, are said to be conjugate or associate Bertrand curves. Also the tangents to the two curves are inclined at a constant angle. So, we have [14]:

$$\mathbf{n}_1 = \mathbf{n}_2, \quad \langle \mathbf{t}_1, \mathbf{t}_2 \rangle = constant \neq 0, \quad \langle \mathbf{t}_i\ , \mathbf{n}_j \rangle = 0, \quad i \neq j. \tag{4.20}$$

Thus and using (2.3), (2.4), (3.6) and (3.7) we have

Corollary 7. *The Gaussian and second Gaussian curvature functions of* $\stackrel{\approx}{M}$ *are given by*

$$\stackrel{\approx}{K} = \kappa_1\kappa_2 cosec^2\ \theta, \quad \stackrel{\approx}{K}_{II} = 0. \tag{4.21}$$

Corollary 8. *The mean and second mean curvatures functions of* $\stackrel{\approx}{M}$ *are given by*

$$\stackrel{\approx}{H} = H, \quad \stackrel{\approx}{H}_{II} = \stackrel{\approx}{H} - \frac{1}{2}\Delta\left(\ln\sqrt{\left|\stackrel{\approx}{K}\right|}\right), \tag{4.22}$$

where,

$$\Delta\left(\ln\sqrt{\left|\stackrel{\approx}{K}\right|}\right) = \frac{1}{4}\sum_{i,j,\, i\neq j}\frac{1}{\kappa_i^3}\left(3\dot{\kappa}_i^2 - 2\kappa_i\ddot{\kappa}_i\right)\left[\mathbf{t}_1\mathbf{t}_2\mathbf{n}_j\right] cosec\ \theta.$$

Differentiating $\stackrel{\approx}{K}$, $\stackrel{\approx}{H}$ *and* $\stackrel{\approx}{H}_{II}$, *with respect to* s_i, *we get*

$$\left.\begin{aligned} &\left(\stackrel{\approx}{K}\right)_{s_i} = \dot{\kappa}_i\kappa_j cosec^2\theta, \quad \left(\stackrel{\approx}{H}\right)_{s_i} = (H)_{s_i}, \\ &\left(\stackrel{\approx}{H}_{II}\right)_{s_i} = \left(\stackrel{\approx}{H}\right)_{s_i} + \tfrac{1}{8\kappa_i^4}\left(9\dot{\kappa}_i^3 + 2\kappa_i^2\dddot{\kappa}_i - 10\kappa_i\dot{\kappa}_i\ddot{\kappa}_i\right) cosec\ \theta \\ &\qquad + \tfrac{\tau_i}{8\kappa_j^3}\left(3\dot{\kappa}_j^2 - 2\kappa_j\ddot{\kappa}_j\right)[\mathbf{t}_1\mathbf{t}_2\mathbf{b}_i] cosec\ \theta, \ i\neq j. \end{aligned}\right\} \tag{4.23}$$

From the results obtained previously, we find that the Jacobian equation (4.3), *is splited to following conditions*

$$[\mathbf{t}_1\mathbf{t}_2\mathbf{b}_i] = 0, [\mathbf{t}_1\mathbf{t}_2\mathbf{n}_i] = 0, \tag{4.24}$$

$$\kappa_i^2\dot{\kappa}_j\tau_i = 0, \quad \kappa_i\dot{\kappa}_i\dot{\kappa}_j = 0, \quad i\neq j, \tag{4.25}$$

which implies

$$(\kappa_1, \tau_1) = (0, 0), \quad (\kappa_2, \tau_2) = (c, 0), \tag{4.26}$$

Therefore, we have the following

Corollary 9. *The translation surface* $\stackrel{\approx}{M}$ *is a W-surface in* E^3 *for a straight line* $\boldsymbol{\alpha}$ *and for a circle curve* $\boldsymbol{\beta}$ *with non-zero constant curvature* (see Fig. 2).

Corollary 10. *The translation surface* $\stackrel{\approx}{M}$ *is a W-surface in* E^3 *if the vectors* $(\mathbf{t}_1, \mathbf{t}_2, \mathbf{b}_i)$ *and* $(\mathbf{t}_1, \mathbf{t}_2, \mathbf{n}_i)$ *are coplanar.*

Similarly, from the Jacobian equation (4.6), we get the same Corollary 9 in addition to

$$(\kappa_i, \tau_i) = (c_i, 0). \tag{4.27}$$

Therefore, we have the following

Corollary 11. *The translation surface* $\stackrel{\approx}{M}$ *is a W-surface in* E^3 *for a circle curves* $\boldsymbol{\alpha}$ *and* $\boldsymbol{\beta}$ *with non-zero constant curvature.* (see Fig. 3).*In view of Jacobian* Eqs. (4.4), (4.5) and (4.7), *one can find that they are vanished identically. so we get*

Theorem 5. *The translation surface* $\stackrel{\approx}{M}$ *is a W-surface in* E^3. *At the end, we want to shed light on the linear relations* (4.2) *in the case of* α*and* β *are Bertrand curves. We got convergent and almost similar results as in the Jacobian relations* (4.1). *It may be remarked that the linear relations* (4.8),(4.10), (4.11) and (4.12), *give the same* Theorem 4 and Corollary 10 *together,*

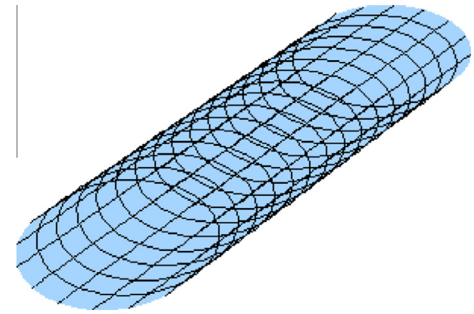

Figure 2 A translation W-surface for a straight line $\boldsymbol{\alpha}$ and for a circle curve $\boldsymbol{\beta}$.

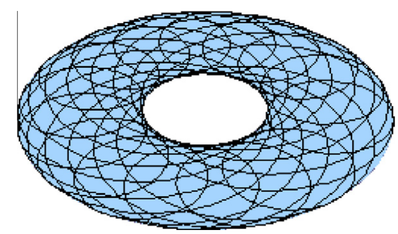

Figure 3 A translation W-surface for a circle curves $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$.

but the linear relation (4.9) *gives the same* Corollary 11 *only. Thus, we give the following theorem*

Theorem 6. *The translation surface* $\stackrel{\approx}{M}$ *is a LW-surface in* E^3.

Remark. It has been observed from the previous results that in the general case, the translation W-surfaces resulting are generated by space curves. But in the special cases, the output translation L/W-surfaces are generated by planar curves after solving the differential equations resulting from Weingarten's condition.

Acknowledgment

I greatly appreciate the help of my professor, Dr. Nassar H. Abdel All, Head of Mathematics Department, Assiut University, Egypt, who revised this paper carefully and provided some important comments on it. Also, the author thanks the editor and the referees for their comments and suggestions to improve the paper.

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