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μ-Lacunary $\chi^2_{A_{uv}}$-convergence of order α with p-metric defined by mn sequence of moduli Musielak

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Abstract We study some connections between $\mu-$ lacunary strong $\chi^2_{A_{uv}}-$convergence with respect to a mn sequence of moduli Musielak and $\mu-$ lacunary $\chi^2_{A_{uv}}-$ statistical convergence, where A is a sequence of four dimensional matrices $A(uv)=\left(a^{m_1,\ldots,m_r n_1,\ldots,n_s}_{k_1,\ldots,k_r \ell_1,\ldots,\ell_s}(uv)\right)$ of complex numbers.

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1. Introduction

Throughout w,χ and Λ denote the classes of all, gai and analytic scalar valued single sequences, respectively.

We write w^2 for the set of all complex sequences (x_{mn}), where $m,n\in\mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under the coordinate wise addition and scalar multiplication.

Some initial works on double sequence spaces are found in Bromwich [1]. Later on, they were investigated by Hardy [2], Moricz [3], Moricz and Rhoades [4], Basarir and Solankan [5], Tripathy [6], Turkmenoglu [7], and many others.

We procure the following sets of double sequences:

$$\mathcal{M}_u(t):=\left\{(x_{mn})\in w^2: sup_{m,n\in N}|x_{mn}|^{t_{mn}}<\infty\right\},$$
$$\mathcal{C}_p(t):=\left\{(x_{mn})\in w^2: p-lim_{m,n\to\infty}\left|x_{mn}-l\right|^{t_{mn}}=1\, for\, some\, l\in\mathbb{C}\right\},$$
$$\mathcal{C}_{0p}(t):=\left\{(x_{mn})\in w^2: p-lim_{m,n\to\infty}|x_{mn}|^{t_{mn}}=1\right\},$$
$$\mathcal{L}_u(t):=\left\{(x_{mn})\in w^2: \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}|x_{mn}|^{t_{mn}}<\infty\right\},$$
$$\mathcal{C}_{bp}(t):=\mathcal{C}_p(t)\bigcap\mathcal{M}_u(t)\text{ and }\mathcal{C}_{0bp}(t)=\mathcal{C}_{0p}(t)\bigcap\mathcal{M}_u(t);$$

where $t=(t_{mn})$ is the sequence of strictly positive reals t_{mn} for all $m,n\in\mathbb{N}$ and $p-lim_{m,n\to\infty}$ denotes the limit in the Pringsheim's sense. In the case $t_{mn}=1$ for all

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$m,n\in\mathbb{N}; \mathcal{M}_u(t), \mathcal{C}_p(t), \mathcal{C}_{0p}(t), \mathcal{L}_u(t), \mathcal{C}_{bp}(t)$ and $\mathcal{C}_{0bp}(t)$ reduce to the sets $\mathcal{M}_u, \mathcal{C}_p, \mathcal{C}_{0p}, \mathcal{L}_u, \mathcal{C}_{bp}$ and $\mathcal{C}_{0bp}$, respectively. Now, we may summarize the knowledge given in some document related to the double sequence spaces. Gökhan and Colak [8,9] have proved that $\mathcal{M}_u(t)$ and $\mathcal{C}_p(t), \mathcal{C}_{bp}(t)$ are complete paranormed spaces of double sequences and gave the $\alpha-, \beta-, \gamma-$ duals of the spaces $\mathcal{M}_u(t)$ and $\mathcal{C}_{bp}(t)$. Quite recently, in her PhD thesis, Zelter [10] has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [11] and Tripathy [12] have independently introduced the statistical convergence and Cauchy for double sequences and given the relation between statistical convergent and strongly Cesàro summable double sequences. Altay and Basar [13] have defined the spaces $\mathcal{BS}, \mathcal{BS}(t), \mathcal{CS}_p, \mathcal{CS}_{bp}, \mathcal{CS}_r$ and $\mathcal{BV}$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathcal{M}_u, \mathcal{M}_u(t), \mathcal{C}_p, \mathcal{C}_{bp}, \mathcal{C}_r$ and $\mathcal{L}_u$, respectively, and also examined some properties of those sequence spaces and determined the $\alpha-$ duals of the spaces $\mathcal{BS}, \mathcal{BV}, \mathcal{CS}_{bp}$ and the $\beta(\vartheta)-$ duals of the spaces $\mathcal{CS}_{bp}$ and $\mathcal{CS}_r$ of double series. Basar and Sever [14] have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well-known space ℓ_q of single sequences and examined some properties of the space $\mathcal{L}_q$. Quite recently Subramanian and Misra [15] have studied the space $\chi^2_M(p,q,u)$ of double sequences and gave some inclusion relations.

The class of sequences which are strongly Cesàro summable with respect to a modulus was introduced by Maddox [16] as an extension of the definition of strongly Cesàro summable sequences. Cannor [17] further extended this definition to a definition of strong $A-$ summability with respect to a modulus where $A=(a_{n,k})$ is a nonnegative regular matrix and established some connections between strong $A-$ summability, strong $A-$ summability with respect to a modulus, and $A-$ statistical convergence. In [18] the notion of convergence of double sequences was presented by A. Pringsheim. Also, in [19–21] the four dimensional matrix transformation $(Ax)_{k,\ell}=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}a_{k\ell}^{mn}x_{mn}$ was studied extensively by Robison and Hamilton.

We need the following inequality in the sequel of the paper. For $a,b,\geqslant 0$ and $0<p<1$, we have

$$(a+b)^p\leqslant a^p+b^p. \tag{1.1}$$

The double series $\sum_{m,n=1}^{\infty}x_{mn}$ is called convergent if and only if the double sequence (s_{mn}) is convergent, where $s_{mn}=\sum_{i,j=1}^{m,n}x_{ij}(m,n\in\mathbb{N})$.

A sequence $x=(x_{mn})$ is said to be double analytic if $sup_{mn}\mid x_{mn}|^{1/m+n}<\infty$. The vector space of all double analytic sequences will be denoted by Λ^2. A sequence $x=(x_{mn})$ is called double gai sequence if $((m+n)!\mid x_{mn}\mid)^{1/m+n}\to 0$ as $m,n\to\infty$. The double gai sequences will be denoted by χ^2. Let $\phi=\{all\ finite\ sequences\}$.

Consider a double sequence $x=(x_{ij})$. The (m,n)th section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]}=\sum_{i,j=0}^{m,n}x_{ij}\mathfrak{I}_{ij}$ for all $m,n\in\mathbb{N}$; where $\mathfrak{I}_{ij}$ denotes the double sequence whose only non zero term is a $\frac{1}{(i+j)!}$ in the $(i,j)^{th}$ place for each $i,j\in\mathbb{N}$.

An FK-space(or a metric space)X is said to have AK property if $(\mathfrak{I}_{mn})$ is a Schauder basis for X. Or equivalently $x^{[m,n]}\to x$.

An FDK-space is a double sequence space endowed with a complete metrizable; locally convex topology under which the coordinate mappings $x=(x_k)\to(x_{mn})(m,n\in\mathbb{N})$ are also continuous.

Let M and Φ are mutually complementary modulus functions. Then, we have:

(i) For all $u,y\geqslant 0$,

$$uy\leqslant M(u)+\Phi(y), (Young's inequality)\ \text{[See [22]]}. \tag{1.2}$$

(ii) For all $u\geqslant 0$,

$$u\eta(u)=M(u)+\Phi(\eta(u)). \tag{1.3}$$

(iii) For all $u\geqslant 0$, and $0<\lambda<1$,

$$M(\lambda u)\leqslant \lambda M(u). \tag{1.4}$$

Lindenstrauss and Tzafriri [23] used the idea of Orlicz function to construct Orlicz sequence space

$$\ell_M=\left\{x\in w: \sum_{k=1}^{\infty}M\left(\frac{|x_k|}{\rho}\right)<\infty,\ for\ some\ \rho>0\right\}.$$

The space ℓ_M with the norm

$$\|x\|=inf\left\{\rho>0: \sum_{k=1}^{\infty}M\left(\frac{|x_k|}{\rho}\right)\leqslant 1\right\},$$

becomes a Banach space which is called an Orlicz sequence space. For $M(t)=t^p(1\leqslant p<\infty)$, the spaces ℓ_M coincide with the classical sequence space ℓ_p.

A sequence $f=(f_{mn})$ of modulus function is called a Musielak-modulus function. A sequence $g=(g_{mn})$ defined by

$$g_{mn}(v)=sup\{|v|u-(f_{mn})(u): u\geqslant 0\}, m,n=1,2,\dots$$

is called the complementary function of a Musielak-modulus function f. For a given Musielak modulus function f, the Musielak-modulus sequence space t_f is defined as follows

$$t_f=\left\{x\in w^2: I_f(|x_{mn}|)^{1/m+n}\to 0\ as\ m,n\to\infty\right\},$$

where I_f is a convex modular defined by

$$I_f(x)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}f_{mn}(|x_{mn}|)^{1/m+n},\ x=(x_{mn})\in t_f.$$

We consider t_f equipped with the Luxemburg metric

$$d(x,y)=sup_{mn}\left\{inf\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}f_{mn}\left(\frac{|x_{mn}|^{1/m+n}}{mn}\right)\right)\leqslant 1\right\}.$$

If X is a sequence space, we give the following definitions:

(i) $X'=$ the continuous dual of X;

(ii) $X^{\alpha}=\left\{a=(a_{mn}): \sum_{m,n=1}^{\infty}|a_{mn}x_{mn}|<\infty, for\ each\ x\in X\right\}$;

(iii) $X^{\beta}=\left\{a=(a_{mn}): \sum_{m,n=1}^{\infty}a_{mn}x_{mn}\ is\ convergent,\ for\ each\ x\in X\right\}$;

(iv) $X^{\gamma}=\left\{a=(a_{mn}): sup_{mn}\geqslant 1\mid \sum_{m,n=1}^{M,N}a_{mn}x_{mn}|<\infty, for\ each\ x\in X\right\}$;

(v) *let X be an FK − space $\supset\phi$; then* $X^f=\{f(\mathfrak{I}_{mn}): f\in X'\}$;

(vi) $X^{\delta} = \left\{ a = (a_{mn}) : sup_{mn} \mid a_{mn}x_{mn}\right|^{1/m+n} < \infty, for\ each\ x \in X\};$

$X^{\alpha}, X^{\beta}, X^{\gamma}$ are called $\alpha-(or\ Köthe-Toeplitz)dual\ of\ X$, $\beta-(or\ generalized-Köthe-Toeplitz)dual\ of\ X, \gamma-dual\ of\ X$, $\delta-dual\ of\ Xrespectively$. X^{α} is defined by Gupta and Pradhan [24]. It is clear that $X^{\alpha} \subset X^{\beta}$ and $X^{\alpha} \subset X^{\gamma}$, but $X^{\beta} \subset X^{\gamma}$ does not hold, since the sequence of partial sums of a double convergent series need not to be bounded.

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz as follows

$$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\}$$

for $Z = c, c_0$ and ℓ_{∞}, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$.

Here c, c_0 and ℓ_{∞} denote the classes of convergent, null and bounded scalar valued single sequences respectively. The difference sequence space bv_p of the classical space ℓ_p is introduced and studied in the case $1 \leqslant p \leqslant \infty$ by Başar and Altay and in the case $0 < p < 1$ by Altay and Başar. The spaces $c(\Delta), c_0(\Delta), \ell_{\infty}(\Delta)$ and bv_p are Banach spaces normed by

$$\|x\| = |x_1| + sup_{k \geqslant 1}|\Delta x_k| \text{ and } \|x\|_{bv_p} = \left(\sum_{k=1}^{\infty}|x_k|^p\right)^{1/p}, \ (1 \leqslant p < \infty).$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \left\{x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z\right\}$$

where $Z = \Lambda^2, \chi^2$ and $\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$ for all $m, n \in \mathbb{N}$.

2. Definition and preliminaries

Let $mn(\geqslant 2)$ be an integer. A function $x : (M \times N) \times (M \times N) \times \cdots \times (M \times N)$. $(M \times N)(m \times n-factors) \to \mathbb{R}(\mathbb{C})$ is called a real complex $mn-$ sequence, where $\mathbb{N}, \mathbb{R}$ and $\mathbb{C}$ denote the sets of natural numbers and complex numbers respectively. Let $m_1, m_2, \ldots, m_r, n_1, n_2, \ldots, n_s \in \mathbb{N}$ and X be a real vector space of dimension w, where $m_1, m_2, \ldots, m_r, n_1, n_2, \ldots, n_s \leqslant w$. A real valued function $d_p(x_{11}, \ldots, x_{m_1,m_2,\ldots,m_r,n_1,n_2,\ldots,n_s}) = \|(d_1(x_{11}, 0), \ldots, d_n(x_{m_1,m_2,\ldots,m_r,n_1,n_2,\ldots,n_s}, 0))\|_p$ on X satisfying the following four conditions:

(i) $\|(d_1(x_{11}, 0), \ldots, d_n(x_{m_1,m_2,\ldots,m_r,n_1,n_2,\ldots,n_s}, 0))\|_p = 0$ if and only if $d_1(x_{11}, 0), \ldots, d_{m_1,m_2,\ldots,m_r,n_1,n_2,\ldots,n_s}(x_{m_1,m_2,\ldots,m_r,n_1,n_2,\ldots,n_s}, 0)$ are linearly dependent,

(ii) $\|(d_1(x_{11}, 0), \ldots, d_{m_1,m_2,\ldots,m_r,n_1,n_2,\ldots,n_s}(x_{m_1,m_2,\ldots,m_r,n_1,n_2,\ldots,n_s}, 0))\|_p$ is invariant under permutation,

(iii) $\|(\alpha d_1(x_{11}, 0), \ldots, d_{m_1,m_2,\ldots,m_p,n_1,n_2,\ldots,n_q}(x_{m_1,m_2,\ldots,m_r,n_1,n_2,\ldots,n_s}, 0))\|_p = |\alpha| \, \|(d_1(x_{11}, 0), \ldots, d_n(x_{m_1,m_2,\ldots,m_p,n_1,n_2,\ldots,n_q}, 0))\|_p, \alpha \in \mathbb{R}$

(iv) $d_p((x_{11}, y_{11}), (x_{12}, y_{12}), \ldots, (x_{m_1,m_2,\ldots,m_r,n_1,n_2,\ldots,n_s}, y_{m_1,m_2,\ldots,m_r,n_1,n_2,\ldots,n_s})) = (d_X(x_{11}, x_{12}, \ldots, x_{m_1,m_2,\ldots,m_r,n_1,n_2,\ldots,n_q})^p + d_Y(y_{11}, y_{12}, \ldots, y_{m_1,m_2,\ldots,m_p,n_1,n_2,\ldots,n_s})^p)^{1/p}$ $for\ 1 \leqslant p < \infty$; (or)

(v) $d((x_{11}, y_{11}), (x_{12}, y_{12}), \ldots, (x_{m_1,m_2,\ldots,m_r,n_1,n_2,\ldots,n_s}, y_{m_1,m_2,\ldots,m_r,n_1,n_2,\ldots,n_s})) := \sup\{d_X(x_{11}, x_{12}, \ldots, x_{m_1,m_2,\ldots,m_r,n_1,n_2,\ldots,n_s}), d_Y(y_{11}, y_{12}, \ldots, y_{m_1,m_2,\ldots,m_r,n_1,n_2,\ldots,n_s})\}$, for $x_{11}, x_{12}, \ldots, x_{m_1,m_2,\ldots,m_r,n_1,n_2,\ldots,n_s} \in X, y_{11}, y_{12}, \ldots, y_{m_1,m_2,\ldots,m_r,n_1,n_2,\ldots,n_s} \in Y$ is called the $p-$ product metric of the Cartesian product of $m_1, m_2, \ldots, m_r, n_1, n_2, \ldots, n_s$ metric spaces is the $p-$ norm of the $m \times n$-vector of the norms of the $m_1, m_2, \ldots, m_r, n_1, n_2, \ldots, n_s$ subspaces.

A trivial example of p product metric of $m_1, m_2, \ldots, m_r, n_1, n_2, \ldots, n_s$ metric space is the p norm space is $X = \mathbb{R}$ equipped with the following Euclidean metric in the product space is the p norm:

$$\|(d_1(x_{11}, 0), \ldots, d_n(x_{m_1,m_2,\ldots,m_r,n_1,n_2,\ldots,n_s}, 0))\|_E = sup(|det(d_{m_1,m_2,\ldots,m_r,n_1,n_2,\ldots,n_s}(x_{m_1,m_2,\ldots,m_r,n_1,n_2,\ldots,n_s}))|)$$

$$= sup\left(\begin{vmatrix} d_{11}(x_{11}, 0) & d_{12}(x_{12}, 0) & \cdots & d_{1n}(x_{1,n_1,n_2,\ldots,n_s}, 0) \\ d_{21}(x_{21}, 0) & d_{22}(x_{22}, 0) & \cdots & d_{2n}(x_{2,n_1,n_2,\ldots,n_s}, 0) \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ d_{m1n1}(x_{m1n1}, 0) & d_{m2n2}(x_{m2n2}, 0) & \ldots & d_{m_1,m_2,\ldots,m_r,n_1,n_2,\ldots,n_s}(x_{m_1,m_2,\ldots,m_r,n_1,n_2,\ldots,n_s}, 0) \end{vmatrix}\right)$$

where $x_i = (x_{i1}, \ldots, x_{i,n_1,n_2,\ldots,n_s}) \in \mathbb{R}^n$ for each $i = 1, 2, \ldots, m_1, m_2, \ldots, m_r$.

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the $p-$ metric. Any complete $p-$ metric space is said to be $p-$ Banach metric space.

By a lacunary sequence $\theta = (m_r n_s)$, where $m_0 n_0 = 0$, we shall mean an increasing sequence of non-negative integers with $h_{rs} = m_r n_s - m_{r-1} n_{s-1} \to \infty$ as $r, s \to \infty$. The intervals determined by θ will be denoted by $I_{rs} = (m_{r-1} n_{s-1}, m_r n_s]$.

Let $F = (f_{mn})$ be a $mn-$ sequence of moduli musielak such that $lim_{u \to 0^+} sup_{mn} f_{mn}(u) = 0$. Throughout this paper $\chi^2_{A_{uv}}-$ convergence of $p-$ metric of $mn-$ sequence of musielak modulus function determinated by F will be denoted by $f_{mn} \in F$ for every $m, n \in \mathbb{N}$.

The purpose of this paper was to introduce and study a concept of lacunary strong $\chi^2_{A_{uv}}-$ convergence of $p-$ metric with respect to a $mn-$ sequence of moduli musielak.

We now introduce the generalizations of lacunary strongly $\chi^2_{A_{uv}}-$ convergence of $p-$ metric with respect a $mn-$ sequence of musielak modulus function and investigate some inclusion relations.

Let A denote a sequence of the matrices $A^{uv} = \left(a^{m_1,\ldots,m_r n_1,\ldots,n_s}_{k_1,\ldots,k_r \ell_1,\ldots,\ell_s}(uv)\right)$ of complex numbers. We write for any sequence $x = (x_{mn}), y_{ij}(uv) = A^{uv}_{ij}(x) = \sum_{m_1,\dots,m_r}^{\infty} \sum_{n_1,\dots,n_s}^{\infty} \left(a^{m_1,\dots,m_r n_1,\dots,n_s}_{k_1,\dots,k_r \ell_1,\dots,\ell_s}(uv)\right)((m_1,\dots,m_r+n_1,\dots,n_s)! \mid x_{m_1,\dots,m_r n_1,\dots,n_s} \mid)^{1/m_1,\dots,m_r+n_1,\dots,n_s}$ if it exits for each i and uv. We $A^{uv}(x) = \left(A^{uv}_{ij}(x)\right)_{ij}, Ax = (A^{uv}(x))_{uv}$.

Definition 2.1. Let μ be a valued measure on $\mathbb{N}\times\mathbb{N}$ and $F = \left(f^{ij}_{m_1,\dots,m_r n_1,\dots,n_s}\right)$ be a $mn-$ sequence of moduli musielak, A denote the sequence of four dimensional infinite matrices of complex numbers and X be locally convex Hausdorff topological linear space whose topology is determined by a set of continuous semi norms η and $(X, \|(d(x_{11},0), d(x_{12},0), \dots, d(x_{m_1,m_2,\dots,m_{r-1}n_1,n_2,\dots,n_{s-1}},0))\|_p)$ be a $p-$metric space, $q = \left(q_{ij}\right)$ be double analytic sequence of strictly positive real numbers. By $w^2(p-X)$ we denote the space of all sequences defined over $\left(X, \|\left(d(x_{11},0), d(x_{12},0), \dots, d\left(x_{m_1,m_2,\dots,m_{r-1}n_1,n_2,\dots,n_{s-1}},0\right)\right)\|_p\right)^{\mu}$.
In the present paper we define the following sequence spaces:

$$\left[\chi^{2q\eta}_{AfN^{\alpha}_{\theta}}, \|(d(x_{11}), d(x_{12}), \dots, d(x_{m_1,m_2,\dots,m_{r-1}n_1,n_2,\dots,n_{s-1}}))\|_p\right]^{\mu}$$
$$= \mu\left(lim_{rs}\left\{\left[f_{ij}\left(\|N^{\alpha}_{\theta}(x), (d(x_{11},0), d(x_{12},0), \dots, d(x_{m_1,m_2,\dots,m_{r-1}n_1,n_2,\dots,n_{s-1}},0))\|_p\right)\right]^{q_{ij}} \geqslant \epsilon\right\}\right) = 0, \text{ where } N^{\alpha}_{\theta}(x)$$
$$= \frac{1}{h^{\alpha}_{rs}} \sum_{i\in I_{rs}} \sum_{j\in I_{rs}} \left(\eta\left(A^{uv}_{ij}\left(((m_1,\dots,m_r+n_1,\dots,n_s)! |x_{m_1,\dots,m_r n_1,\dots,n_s}|)^{1/m_1,\dots,m_r+n_1,\dots,n_s}\right)\right)\right),$$

uniformly in uv

$$\left[\Lambda^{2q\eta}_{AfN^{\alpha}_{\theta}}, \|(d(x_{11},0), d(x_{12},0), \dots, d(x_{m_1,m_2,\dots,m_{r-1}n_1,n_2,\dots,n_{s-1}},0))\|_p\right]^{\mu}$$
$$= \mu\left(sup_{rs}\left\{\left[f_{uv}\left(\|N^{\alpha}_{\theta}(x), (d(x_{11},0), d(x_{12},0), \dots, (x_{m_1,m_2,\dots,m_{r-1}n_1,n_2,\dots,n_{s-1}},0))\|_p\right)\right]^{q_{ij}} \geqslant k\right\}\right) = 0,$$

$$\text{where } e = \begin{pmatrix} 1 & 1 & \cdots 1 \\ 1 & 1 & \cdots 1 \\ . & & \\ . & & \\ . & & \\ 1 & 1 & \cdots 1 \end{pmatrix}.$$

The main aim of this paper was to introduce the idea of summability of double lacunary sequence spaces in $p-$ metric spaces using a two valued measure. We also make an effort to study $\mu-$of lacunary double sequences with respect to a sequence of moduli Musielak in $p-$ metric spaces and two valued measure μ. We also plan to study some topological properties and inclusion relation between these spaces.

3. Main results

Proposition 3.1. *Let μ be a two valued measure, $\left[\chi^{2q\eta}_{AfN^{\alpha}_{\theta}}, \|(d(x_{11},0), d(x_{12},0), \dots, d(x_{m_1,m_2,\dots,m_{r-1}n_1,n_2,\dots,n_{s-1}},0))\|_p\right]^{\mu}$ and $\left[\Lambda^{2q\eta}_{AfN^{\alpha}_{\theta}}, \|(d(x_{11},0), d(x_{12},0), \dots, d(x_{m_1}, m_2,\dots,m_{r-1}n_1,n_2,\dots,n_{s-1},0))\|_p\right]^{\mu}$ are linear spaces.*

Proof. It is routine verification. Therefore the proof is omitted. □

3.1. The inclusion relation between $\left[\chi^{2q\eta}_{AfN^{\alpha}_{\theta}}, \|(d(x_{11},0), d(x_{12},0), \dots, d(x_{m_1,m_2,\dots,m_{r-1}n_1,n_2,\dots,n_{s-1}},0))\|_p\right]^{\mu}$ *and* $\left[\Lambda^{2q\eta}_{AfN^{\alpha}_{\theta}}, \|(d(x_{11},0), d(x_{12},0), \dots, d(x_{m_1,m_2,\dots,m_{r-1}n_1,n_2,\dots,n_{s-1}},0))\|_p\right]^{\mu}$

Theorem 3.2. *Let μ be a two valued measure and A be a $mn-$ sequence the four dimensional infinite matrices $A^{uv} = \left(a^{m_1,\dots,m_r n_1,\dots,n_s}_{k_1,\dots,k_r \ell_1,\dots,\ell_s}(uv)\right)$ of complex numbers and $F = \left(f^{ij}_{mn}\right)$ be a $mn-$ sequence of moduli musielak. If $x = (x_{mn})$ lacunary strong $A_{uv}-$ convergent of order α to zero then $x = (x_{mn})$ lacunary strong $A_{uv}-$ convergent of order α to zero with respect to $mn-$ sequence of moduli musielak, (i.e.)*

$$\left[\chi^{2q\eta}_{AN^{\alpha}_{\theta}}, \|(d(x_{11},0), d(x_{12},0), \dots, d(x_{m_1,m_2,\dots,m_{r-1}n_1,n_2,\dots,n_{s-1}},0))\|_p\right]^{\mu}$$
$$\subset \left[\chi^{2q\eta}_{AfN^{\alpha}_{\theta}}, \|(d(x_{11},0), d(x_{12},0), \dots, d(x_{m_1,m_2,\dots,m_{r-1}n_1,n_2,\dots,n_{s-1}},0))\|_p\right]^{\mu}.$$

Proof. Let $F = \left(f^{ij}_{mn}\right)$ be a $mn-$ sequence of moduli musielak and put $supf^{ij}_{mn}(1) = T$. Let $x = (x_{mn}) \in \left[\chi^{2q\eta}_{AN^{\alpha}_{\theta}}, \|(d(x_{11},0), d(x_{12},0), \dots, d(x_{m_1,m_2,\dots,m_{r-1}n_1,n_2,\dots,n_{s-1}},0))\|_p\right]^{\mu}$ and $\epsilon > 0$. We choose $0 < \delta < 1$ such that $f^{ij}_{mn}(u) < \epsilon$ for every u with $0 \leqslant u \leqslant \delta (i,j \in \mathbb{N})$. We can write

$$\left[\chi^{2q\eta}_{AfN^{\alpha}_{\theta}}, \|(d(x_{11},0), d(x_{12},0), \dots, d(x_{m_1,m_2,\dots,m_{r-1}n_1,n_2,\dots,n_{s-1}},0))\|_p\right]^{\mu}$$
$$= \left[\chi^{2q\eta}_{AfN^{\alpha}_{\theta}}, \|(d(x_{11},0), d(x_{12},0), \dots, d(x_{m_1,m_2,\dots,m_{r-1}n_1,n_2,\dots,n_{s-1}},0))\|_p\right]^{\mu}$$
$$+ \left[\chi^{2q\eta}_{AfN^{\alpha}_{\theta}}, \|(d(x_{11},0), d(x_{12},0), \dots, d(x_{m_1,m_2,\dots,m_{r-1}n_1,n_2,\dots,n_{s-1}},0))\|_p\right]^{\mu},$$

where the first part is over $\leqslant \delta$ and second part is over $> \delta$. By definition of Musielak modulus f^{ij}_{mn} for every ij, we have

$$\left[\chi^{2q\eta}_{AfN^{\alpha}_{\theta}}, \|(d(x_{11},0), d(x_{12},0), \dots, d(x_{m_1,m_2,\dots,m_{r-1}n_1,n_2,\dots,n_{s-1}},0))\|_p\right]^{\mu}$$
$$\leqslant \epsilon^{H_2} + \left(2T\delta^{-1}\right)^{H_2}\left[\chi^{2q\eta}_{AfN^{\alpha}_{\theta}}, \|(d(x_{11},0), d(x_{12},0), \dots, d(x_{m_1,m_2,\dots,m_{r-1}n_1,n_2,\dots,n_{s-1}},0))\|_p\right]^{\mu}.$$

Therefore
$x = (x_{mn}) \in \left[\chi^{2q\eta}_{AfN^{\alpha}_{\theta}}, \|(d(x_{11},0), d(x_{12},0), \dots, d(x_{m_1}, m_2,\dots,m_{r-1}n_1,n_2,\dots,n_{s-1},0))\|_p\right]^{\mu}$. □

Theorem 3.3. *Let μ be a two valued measure and A be a $mn-$ sequence of the four dimensional infinite matrices $A^{uv} = \left(a^{m_1,\dots,m_r n_1,\dots,n_s}_{k_1,\dots,k_r \ell_1,\dots,\ell_s}(uv)\right)$ of complex numbers, $q = \left(q_{ij}\right)$ be a $mn-$ sequence of positive real numbers with $0 < infq_{ij} = H_1 \leqslant supq_{ij} = H_2 > \infty$ and $F = \left(f^{ij}_{mn}\right)$ be a $mn-$ sequence of moduli Musielak. If $lim_{u,v\to\infty} inf_{ij} \frac{f_{ij}(uv)}{uv} > 0$, then*

$$\left[\chi^{2q\eta}_{AfN^{\alpha}_{\theta}}, \|(d(x_{11},0),d(x_{12},0),\ldots,d(x_{m_1,m_2,\ldots,m_{r-1}n_1,n_2,\ldots,n_{s-1}},0))\|_p\right]^{\mu}$$
$$=\left[\chi^{2q\eta}_{AN^{\alpha}_{\theta}}, \|(d(x_{11},0),d(x_{12},0),\ldots,d(x_{m_1,m_2,\ldots,m_{r-1}n_1,n_2,\ldots,n_{s-1}},0))\|_p\right]^{\mu}.$$

Proof. If $lim_{u,v\to\infty} inf_{ij} \frac{f_{ij}(uv)}{uv} > 0$, then there exists a number $\beta > 0$ such that $f_{ij}(uv) \geqslant \beta u$ for all $u \geqslant 0$ and $i,j \in \mathbb{N}$. Let

$$x=(x_{m_1},\ldots,m_r n_1,\ldots,n_s) \in \left[\chi^{2q\eta}_{AfN^{\alpha}_{\theta}}, \|(d(x_{11},0),d(x_{12},0),\ldots,d(x_{m_1,m_2,\ldots,m_{r-1}n_1,n_2,\ldots,n_{s-1}},0))\|_p\right]^{\mu}.$$

Clearly

$$\left[\chi^{2q\eta}_{AfN^{\alpha}_{\theta}}, \|(d(x_{11},0),d(x_{12},0),\ldots,d(x_{m_1,m_2,\ldots,m_{r-1}n_1,n_2,\ldots,n_{s-1}},0))\|_p\right]^{\mu}$$
$$\geqslant \beta\left[\chi^{2q\eta}_{AN^{\alpha}_{\theta}}, \|(d(x_{11},0),d(x_{12},0),\ldots,d(x_{m_1,m_2,\ldots,m_{r-1}n_1,n_2,\ldots,n_{s-1}},0))\|_p\right]^{\mu}.$$

Therefore

$$x=(x_{m_1}\ldots,m_r n_1\ldots,n_s) \in \left[\chi^{2q\eta}_{AN^{\alpha}_{\theta}}, \|(d(x_{11},0),d(x_{12},0),\ldots,d(x_{m_1,m_2,\ldots,m_{r-1}n_1,n_2,\ldots,n_{s-1}},0))\|_p\right]^{\mu}.$$

Using Theorem 3.3, the proof is complete. □

We now give an example to show that

$$\left[\chi^{2q\eta}_{AfN^{\alpha}_{\theta}}, \|(d(x_{11},0),d(x_{12},0),\ldots,d(x_{m_1,m_2,\ldots,m_{r-1}n_1,n_2,\ldots,n_{s-1}},0))\|_p\right]^{\mu}$$
$$\neq\left[\chi^{2q\eta}_{AN^{\alpha}_{\theta}}, \|(d(x_{11},0),d(x_{12},0),\ldots,d(x_{m_1,m_2,\ldots,m_{r-1}n_1,n_2,\ldots,n_{s-1}},0))\|_p\right]^{\mu}$$

in the case when $\beta = 0$. Consider $A = I$, unit matrix, $\eta(x) = ((m_1,\ldots,m_r+n_1,\ldots,n_s)! \mid x_{m_1,\ldots,m_r n_1}, \quad \ldots, n_s \mid)^{1/m_1,\ldots,m_r+n_1,\ldots,n_s}$, $q_{ij} = 1$ for every $i,j \in \mathbb{N}$ and $f^{ij}_{mn}(x) = \frac{|x_{m_1,\ldots,m_r n_1,\ldots,n_s}|^{1/((m_1,\ldots,m_r+n_1,\ldots,n_s)(i+1)(j+1))}}{((m_1,\ldots,m_r+n_1,\ldots,n_s)!)^{1/m_1,\ldots,m_r+n_1,\ldots,n_s}} (i,j \geqslant 1, x > 0)$ in the case $\beta > 0$. Now we define $x_{ij} = h^{\alpha}_{rs}$ if $i,j = m_r n_s$ for some $r,s \geqslant 1$ and $x_{ij} = 0$ otherwise. Then we have,

$$\left[\chi^{2q\eta}_{AfN^{\alpha}_{\theta}}, \|(d(x_{11},0),d(x_{12},0),\ldots,d(x_{m_1,m_2,\ldots,m_{r-1}n_1,n_2,\ldots,n_{s-1}},0))\|_p\right]^{\mu} \to 1 \text{ as } r,s\to\infty$$

and so

$$x=(x_{m_1,\ldots,m_r n_1,\ldots,n_s}) \notin \left[\chi^{2q\eta}_{AN^{\alpha}_{\theta}}, \|(d(x_{11},0),d(x_{12},0),\ldots,d(x_{m_1,m_2,\ldots,m_{r-1}n_1,n_2,\ldots,n_{s-1}},0))\|_p\right]^{\mu}$$

3.2. The inclusion relation between $\left[\chi^{2q\eta}_{AfN^{\alpha}_{\theta}}, \|(d(x_{11},0),d(x_{12},0),\ldots,d(x_{m_1,m_2,\ldots,m_{r-1}n_1,n_2,\ldots,n_{s-1}},0))\|_p\right]^{\mu}$ *and* $\left[\chi^{2\eta}_{AS^{\alpha}_{\theta}}, \|(d(x_{11},0),d(x_{12},0),\ldots,d(x_{m_1,m_2,\ldots,m_{r-1}n_1,n_2,\ldots,n_{s-1}},0))\|_p\right]^{\mu}$

In this section we introduce natural relationship between μ be a two valued measure lacunary $A^{uv}-$ statistical convergence of order α and μ be a two valued measure lacunary strong $A^{uv}-$ convergence of order α with respect to $mn-$ sequence of moduli Musielak.

Definition 3.4. Let μ be a two valued measure and θ be a lacunary $mn-$ sequence. Then a $mn-$ sequence $x = \left(x_{m_1,\ldots,m_r n_1,\ldots,n_s}\right)$ is said to be $\mu-$ lacunary statistically convergent of order α to a number zero if for every $\epsilon > 0, \mu\left(lim_{rs\to\infty} h^{-\alpha}_{rs} \mid K_{\theta}(\epsilon) \mid\right) = 0$, where $\mid K_{\theta}(\epsilon) \mid$ denotes the number of elements in

$$K_{\theta}(\epsilon) = \mu(\{i,j\in I_{rs} : ((m_1,\ldots,m_r+n_1,\ldots,n_s)! \mid x_{m_1,\ldots,m_r n_1,\ldots,n_s} - 0 \mid)^{1/m_1,\ldots,m_r+n_1,\ldots,n_s} \geqslant \epsilon\Big\}\Big) = 0.$$

The set of all lacunary statistical convergent of order α of $mn-$ sequences is denoted by $\left(S^{\alpha}_{\theta}\right)^{\mu}$.

Let μ be a two valued measure and $A^{uv} = \left(a^{m_1,\ldots,m_r n_1,\ldots,n_s}_{k_1,\ldots,k_r\ell_1,\ldots,\ell_s}(uv)\right)$ be an four dimensional infinite matrix of complex numbers. Then a $mn-$ sequence $x = \left(x_{m_1,\ldots,m_r n_1,\ldots,n_s}\right)$ is said to be $\mu-$ lacunary $A-$ statistically convergent of order α to a number zero if for every $\epsilon > 0, \mu\left(lim_{rs\to\infty} h^{-\alpha}_{rs} \mid KA_{\theta}(\epsilon) \mid\right) = 0$, where $\mid KA_{\theta}(\epsilon) \mid$ denotes the number of elements in

$KA_{\theta}(\epsilon) = \mu\left(\{i,j\in I_{rs} : \left((m_1,\ldots,m_r+n_1,\ldots,n_s)! \mid x_{m_1,\ldots,m_r} n_1,\ldots,\ n_s - 0 \mid\right)^{1/m_1,\ldots,m_r+n_1,\ldots,n_s} \geqslant \epsilon\}\right) = 0$. The set of all lacunary $A-$ statistical convergent of order α of $mn-$ sequences is denoted by $\left(S^{\alpha}_{\theta}(A)\right)^{\mu}$.

Definition 3.5. Let μ be a two valued measure and A be a $mn-$ sequence of the four dimensional infinite matrices $A^{uv} = \left(a^{m_1,\ldots,m_r n_1,\ldots,n_s}_{k_1,\ldots,k_r\ell_1,\ldots,\ell_s}(uv)\right)$ of complex numbers and let $q = \left(q_{ij}\right)$ be a $mn-$ sequence of positive real numbers with $0 < infq_{ij} = H_1 \leqslant supq_{ij} = H_2 < \infty$. Then a $mn-$ sequence $x = (x_{m_1,\ldots,m_r n_1,\ldots,n_s})$ is said to be $\mu-$ lacunary $A^{uv}-$ statistically convergent of order α to a number zero if for every $\epsilon > 0, \mu\left(lim_{rs\to\infty} h^{-\alpha}_{rs} \mid KA_{\theta\eta}(\epsilon) \mid\right) = 0$, where $\mid KA_{\theta\eta}(\epsilon) \mid$ denotes the number of elements in

$$KA_{\theta\eta}(\epsilon) = \mu(\{i,j\in I_{rs} : ((m_1,\ldots,m_r+n_1,\ldots,n_s)! \mid x_{m_1,\ldots,m_r n_1,\ldots,n_s} - 0 \mid)^{1/m_1,\ldots,m_r+n_1,\ldots,n_s} \geqslant \epsilon\Big\}\Big) = 0.$$

The set of all $\mu-$ lacunary $A_{\eta}-$ statistical convergent of order α of $mn-$ sequences is denoted by $\left(S^{\alpha}_{\theta}(A,\eta)\right)^{\mu}$.

The following theorems give the relations between $\mu-$ lacunary $A^{uv}-$ statistical convergence of order α and $\mu-$ lacunary strong $A^{uv}-$ convergence of order α with respect to a $mn-$ sequence of moduli Musielak.

Theorem 3.6. *Let μ be a two valued measure and $F = \left(f_{ij}\right)$ be a $mn-$ sequence of moduli Musielak. Then*

$$\left[\chi^{2q\eta}_{AfN^{\alpha}_{\theta}}, \|(d(x_{11},0),d(x_{12},0),\ldots,d(x_{m_1,m_2,\ldots,m_{r-1}n_1,n_2,\ldots,n_{s-1}},0))\|_p\right]^{\mu}$$
$$\subseteq\left[\chi^{2\eta}_{AS^{\alpha}_{\theta}}, \|(d(x_{11},0),d(x_{12},0),\ldots,d(x_{m_1,m_2,\ldots,m_{r-1}n_1,n_2,\ldots,n_{s-1}},0))\|_p\right]^{\mu}$$

if and only if $\mu(lim_{ij\to\infty}f_{ij}(u)) > 0, (u > 0)$.

Proof. Let $\epsilon > 0$ and

$$x = (x_{m_1,\dots,m_r n_1,\dots,n_s}) \in \left[\chi^{2q\eta}_{AfN^\alpha_\theta}, \|(d(x_{11},0), d(x_{12},0), \dots, d(x_{m_1,m_2,\dots,m_{r-1}n_1,n_2,\dots,n_{s-1}},0))\|_p\right]^\mu.$$

If $\mu(lim_{ij\to\infty}f_{ij}(u)) > 0, (u > 0)$, then there exists a number $d > 0$ such that $f_{ij}(\epsilon) > d$ for $u > \epsilon$ and $i,j \in \mathbb{N}$. Let

$$\left[\chi^{2q\eta}_{AfN^\alpha_\theta}, \|(d(x_{11},0), d(x_{12},0), \dots, d(x_{m_1,m_2,\dots,m_{r-1}n_1,n_2,\dots,n_{s-1}},0))\|_p\right]^\mu \geqslant h^{-\alpha}_{rs} d^{H_1} K A_{0\eta}(\epsilon).$$

It follows that

$$\left[\chi^{2\eta}_{AfS^\alpha_\theta}, \|(d(x_{11},0), d(x_{12},0), \dots, d(x_{m_1,m_2,\dots,m_{r-1}n_1,n_2,\dots,n_{s-1}},0))\|_p\right]^\mu.$$

Conversely, suppose that $\mu(lim_{ij\to\infty}f_{ij}(u)) > 0$ does not hold, then there is a number $t > 0$ such that $\mu(lim_{ij\to\infty}f_{ij}(t)) = 0$. We can select a lacunary $mn-$ sequence $\theta = (m_1, \dots, m_r n_1, \dots, n_s)$ such that $f_{ij}(t) < 2^{-rs}$ for any $i > m_1, \dots, m_r, j > n_1, \dots, n_s$. Let $A = I$, unit matrix, define the $mn-$ sequence x by putting $x_{ij} = t$ if

$$m_1, m_2, \dots, m_{r-1}n_1, n_2, \dots, n_{s-1} < i,j < \frac{m_1, m_2, \dots, m_r n_1, n_2, \dots, n_s + m_1, m_2, \dots, m_{r-1}n_1, n_2, \dots, n_{s-1}}{2}$$

and $x_{ij} = 0$ if

$$\frac{m_1, m_2, \dots, m_r n_1, n_2, \dots, n_s + m_1, m_2, \dots, m_{r-1}n_1, n_2, \dots, n_{s-1}}{2} \leqslant i,j \leqslant m_1, m_2, \dots, m_r n_1, n_2, \dots, n_s.$$

We have

$$x = (x_{m_1,\dots,m_r n_1,\dots,n_s}) \in \left[\chi^{2q\eta}_{AfN^\alpha_\theta}, \|(d(x_{11},0), d(x_{12},0), \dots, d(x_{m_1,m_2,\dots,m_{r-1}n_1,n_2,\dots,n_{s-1}},0))\|_p\right]^\mu$$

but

$$x \notin \left[\chi^{2\eta}_{AS^\alpha_\theta}, \|(d(x_{11},0), d(x_{12},0), \dots, d(x_{m_1,m_2,\dots,m_{r-1}n_1,n_2,\dots,n_{s-1}},0))\|_p\right]^\mu. \quad \square$$

Theorem 3.7. *Let* μ *be a two valued measure and* $F = (f_{ij})$ *be a* $mn-$ *sequence of moduli Musielak. Then*

$$\left[\chi^{2q\eta}_{AfN^\alpha_\theta}, \|(d(x_{11},0), d(x_{12},0), \dots, d(x_{m_1,m_2,\dots,m_{r-1}n_1,n_2,\dots,n_{s-1}},0))\|_p\right]^\mu \supseteq \left[\chi^{2\eta}_{AS^\alpha_\theta}, \|(d(x_{11},0), d(x_{12},0), \dots, d(x_{m_1,m_2,\dots,m_{r-1}n_1,n_2,\dots,n_{s-1}},0))\|_p\right]^\mu$$

if and only if $\mu(sup_u sup_{ij} f_{ij}(u)) < \infty$.

Proof. Let

$$x \in \left[\chi^{2\eta}_{AS^\alpha_\theta}, \|(d(x_{11},0), d(x_{12},0), \dots, d(x_{m_1,m_2,\dots,m_{r-1}n_1,n_2,\dots,n_{s-1}},0))\|_p\right]^\mu.$$

Suppose that $h(u) = sup_{ij} f_{ij}(u)$ and $h = sup_u h(u)$. Since $f_{ij}(u) \leqslant h$ for all i,j and $u > 0$, we have for all u, v,

$$\left[\chi^{2\eta}_{AS^\alpha_\theta}, \|(d(x_{11},0), d(x_{12},0), \dots, d(x_{m_1,m_2,\dots,m_{r-1}n_1,n_2,\dots,n_{s-1}},0))\|_p\right]^\mu \leqslant h^{H_2} h^{-\alpha}_{rs} \mid K A_{0\eta}(\epsilon) \mid + \mid h(\epsilon)\mid^{H_2}.$$

It follows from $\epsilon \to 0$ that

$$x \in \left[\chi^{2q\eta}_{AfN^\alpha_\theta}, \|(d(x_{11},0), d(x_{12},0), \dots, d(x_{m_1,m_2,\dots,m_{r-1}n_1,n_2,\dots,n_{s-1}},0))\|_p\right]^\mu.$$

Conversely, suppose that $\mu(sup_u sup_{ij} f_{ij}(u)) = \infty$. Then we have $0 < u_{11} < \cdots < u_{r-1s-1} < u_{rs} < \cdots$, such that $f_{m_r n_s}(u_{rs}) \geqslant h^\alpha_{rs}$ for $r,s \geqslant 1$. Let $A = I$, unit matrix, define the $mn-$ sequence x by putting $x_{ij} = u_{rs}$ if $i,j = m_1 m_2, \dots, m_r n_1 n_2, \dots, n_s$ for some $r,s = 1,2,\dots$ and $x_{ij} = 0$ otherwise. Then we have

$$x \in \left[\chi^{2\eta}_{AS^\alpha_\theta}, \|(d(x_{11},0), d(x_{12},0), \dots, d(x_{m_1,m_2,\dots,m_{r-1}n_1,n_2,\dots,n_{s-1}},0))\|_p\right]^\mu$$

but

$$x \notin \left[\chi^{2q\eta}_{AfN^\alpha_\theta}, \|(d(x_{11},0), d(x_{12},0), \dots, d(x_{m_1,m_2,\dots,m_{r-1}n_1,n_2,\dots,n_{s-1}},0))\|_p\right]^\mu. \quad \square$$

References

[1] T.J.I'A. Bromwich, An introduction to the theory of infinite series, Macmillan and Co.Ltd., New York, 1965.

[2] G.H. Hardy, On the convergence of certain multiple series, Proc. Camb. Phil. Soc. 19 (1917) 86–95.

[3] F. Moricz, Extentions of the spaces c and c_0 from single to double sequences, Acta. Math. Hung. 57 (1–2) (1991) 129–136.

[4] F. Moricz, B.E. Rhoades, Almost convergence of double sequences and strong regularity of summability matrices, Math. Proc. Camb. Phil. Soc. 104 (1988) 283–294.

[5] M. Basarir, O. Solancan, On some double sequence spaces, J. Indian Acad. Math. 21 (2) (1999) 193–200.

[6] B.C. Tripathy, On statistically convergent double sequences, Tamkang J. Math. 34 (3) (2003) 231–237.

[7] A. Turkmenoglu, Matrix transformation between some classes of double sequences, J. Inst. Math. Comp. Sci. Math. Ser. 12 (1) (1999) 23–31.

[8] A. Gökhan, R. Çolak, The double sequence spaces $c^P_2(p)$ and $c^{PB}_2(p)$, Appl. Math. Comput. 157 (2) (2004) 491–501.

[9] A. Gökhan, R. Çolak, Double sequence spaces ℓ^∞_2, Appl. Math. Comput. 160 (1) (2005) 147–153.

[10] M. Zeltser, Investigation of double sequence spaces by soft and hard analitical methods, Dissertationes Mathematicae Universitatis Tartuensis 25, Tartu University Press, Univ. of Tartu, Faculty of Mathematics and Computer Science, Tartu, 2001.

[11] M. Mursaleen, O.H.H. Edely, Statistical convergence of double sequences, J. Math. Anal. Appl. 288 (1) (2003) 223–231.

[12] B.C. Tripathy, B. Sarma, Statistically convergent difference double sequence spaces, Acta Math. Sinica 24 (5) (2008) 737–742.

[13] B. Altay, F. Başar, Some new spaces of double sequences, J. Math. Anal. Appl. 309 (1) (2005) 70–90.

[14] F. Başar, Y. Sever, The space $\mathcal{L}_p$ of double sequences, Math. J. Okayama Univ. 51 (2009) 149–157.

[15] N. Subramanian, U.K. Misra, The semi normed space defined by a double gai sequence of modulus function, Fasciculi Math. 46 (2010).
[16] I.J. Maddox, Sequence spaces defined by a modulus, Math. Proc. Cambridge Philos. Soc. 100 (1) (1986) 161–166.
[17] J. Cannor, On strong matrix summability with respect to a modulus and statistical convergence, Canad. Math. Bull. 32 (2) (1989) 194–198.
[18] A. Pringsheim, Zurtheorie derzweifach unendlichen zahlenfolgen, Math. Ann. 53 (1900) 289–321.
[19] H.J. Hamilton, Transformations of multiple sequences, Duke Math. J. 2 (1936) 29–60.
[20] H.J. Hamilton, A generalization of multiple sequences transformation, Duke Math. J. 4 (1938) 343–358.
[21] H.J. Hamilton, Preservation of partial limits in multiple sequence transformations, Duke Math. J. 4 (1939) 293–297.
[22] P.K. Kamthan, M. Gupta, Sequence Spaces and Series, Lecture Notes, Pure and Applied Mathematics, vol. 65, Marcel Dekker, Inc., New York, 1981.
[23] J. Lindenstrauss, L. Tzafriri, On Orlicz sequence spaces, Israel J. Math. 10 (1971) 379–390.
[24] M. Gupta, S. Pradhan, On certain type of modular sequence space, Turk J. Math. 32 (2008) 293–303.