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Generalized involute and evolute curves of equiform spacelike curves with a timelike equiform principal normal in E_1^3

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Abstract

Equiform geometry is considered as a generalization of the other geometries. In this paper, involute and evolute curves are studied in the case of the curve α is an equiform spacelike with a timelike equiform principal normal vector N. Furthermore, the equiform frames of the involute and evolute curves are obtained. Also, the equiform curvatures of the involute and evolute curves are obtained in Minkowski 3-space.

Keywords: Minkowski 3-space, Involute, Evolute, Equiform geometry, Equiform curvatures

AMS Subject Classification: 53A35, 53C50

Introduction

In the last two decades, curves in Minkowski space have been studied by many mathematicians such as [1–3]. Specially, involute and evolute curves got an interest from alot of mathematicians in Minkowski 3-space. According to the usual Frenet frame, involute and evolute curves in Minkowski 3-space E_1^3 were defined and studied in [1, 2, 4, 5]. The equiform geometry was defined in different spaces such as Galilean space [6], pseudo-Galilean space [7], Euclidean space [8], isotropic space [9], and Minkowski space [3, 10, 11].

In this paper, we firstly introduce the equiform parameter, the equiform frame, and the equiform formulas in the case of equiform spacelike curves with a timelike equiform principal normal in Minkowski space E_1^3. Secondly, we introduce the involute and the evolute of the equiform spacelike curve with a timelike equiform principal normal. Further, the equiform frames for the involute and the evolute curves are obtained. Also, the equiform curvatures of the involute and the evolute curves are obtained.

Preliminaries

The three-dimensional Lorentzian space, or Minkowski 3-space E_1^3, is the space R^3 equipped with the metric g defined as:

$$g(U, V) = -u_1v_1 + u_2v_2 + u_3v_3,$$

where $U = (u_1, u_2, u_3)$ and $V = (v_1, v_2, v_3)$ are any two vectors in R^3. The vector U in E_1^3 may be lightlike if $g(U,U) = 0$ and $U \neq 0$ or spacelike if $g(U,U) > 0$ or $U = 0$ or timelike if $g(U,U) < 0$. The norm (length) of the vector U is defined by $|| U || = \sqrt{| g(U,U) |}$.

The Lorentzian cross product is given by:

$$U \wedge V = (u_3v_2 - u_2v_3, u_3v_1 - u_1v_3, u_1v_2 - u_2v_1),$$

where U and $V \in E_1^3$ [12, 13].

A differentiable map $\alpha : I \subset R \to E_1^3$ is called smooth curve in Minkowski 3-space, where I is an open interval. Suppose that $\{t(s), n(s), b(s)\}$ be the orthonormal Frenet frame along the curve $\alpha(s)$, where $t(s), n(s)$, and $b(s)$ are the tangent, the principal normal, and the binormal vectors of the curve α, respectively.

Any curve α in Minkowski 3-space can be one of the following cases and below the corresponding Frenet formulas:

(1) α is a spacelike curve with

(i) a spacelike principal normal, then Frenet formulas are given by:

$$\frac{dt}{ds} = \dot{t} = kn, \frac{dn}{ds} = \dot{n} = -kt + \tau b, \frac{db}{ds} = \dot{b} = \tau n.$$

$$g(t,t) = g(n,n) = 1, g(b,b) = -1,$$

$$g(t,n) = g(n,b) = g(b,t) = 0.$$

(ii) a timelike principal normal, then Frenet formulas are given by:

$$\dot{t} = kn, \dot{n} = kt + \tau b, \dot{b} = \tau n,$$

$$g(t,t) = g(b,b) = 1, g(n,n) = -1,$$

$$g(t,n) = g(n,b) = g(b,t) = 0.$$

(iii) a null (lightlike) principal normal, then Frenet formulas are given by:

$$\dot{t} = kn, \dot{n} = \tau n, \dot{b} = -kt - \tau b,$$

$$g(t,t) = 1, g(n,n) = g(b,b) = 0, g(n,b) = 1,$$

$$g(t,n) = g(t,b) = 0.$$

(2) α is a timelike curve, then Frenet formulas are given by:

$$\dot{t} = kn, \dot{n} = kt + \tau b, \dot{b} = -\tau n,$$

$$g(t,t) = -1, g(n,n) = g(b,b) = 1, g(n,b) = 1,$$

$$g(t,n) = g(t,b) = 0.$$

[13]

(3) α is a lightlike curve, then Frenet formulas are given by:

$$\dot{t} = kn, \dot{n} = \tau t - kb, \dot{b} = -\tau n,$$

$$g(n, n) = 1, g(t, t) = g(b, b) = 0, g(t, b) = 1,$$
$$g(t, n) = g(n, b) = 0.$$

[13]

The equiform geometry has minor importance related to usual one, and the curves that appear here in equiform geometry can be seen as a generalization of well-known curves from other geometries.

Let $\gamma(s) = t(s)$ be the spherical tangent indicatrix of the curve α and σ be an arc length parameter of γ. We can make a reparameterization of α by the parameter σ, $\alpha = \alpha(\sigma) : I \to E_1^3$, the parameter σ is called the equiform parameter, of the curve $\alpha(\sigma)$.

Let σ be the arc length parameter of spherical tangent indicatrix ζ, then we have:

$$|| \frac{d\zeta}{d\sigma} \frac{d\sigma}{ds} || = || \kappa n ||,$$
$$\frac{d\sigma}{ds} = \kappa = \frac{1}{\rho}.$$

By integration with respect to s, we have:

$$\sigma = \int \frac{ds}{\rho},$$

where ρ is the radius of curvature of α [11].

Let T, N, and B be the orthogonal equiform frame along the curve $\alpha(\sigma)$ in Minkowski 3-space, where T, N, and B are the equiform-tangent, the equiform-normal, and the equiform-binomial vectors of the curve $\alpha(\sigma)$, respectively. They are given by $T = \frac{d\alpha}{d\sigma} = \rho t, N = \rho n, b = \rho b$ [10, 11].

The function $K_1 : I \to R$ defined by $K_1 = \frac{d\rho}{ds}$ is called the first equiform curvature of $\alpha(\sigma)$, and the function $K_2 : I \to R$ defined by $K_2 = \frac{\tau}{\kappa}$ is the called second equiform curvature of $\alpha(\sigma)$.

Definition 1 *A curve $\alpha(\sigma)$ is an equiform spacelike if $g(T, T) = \rho^2 > 0$ or $T = 0$, equiform timelike if $g(T, T) = -\rho^2 < 0$, or equiform null if $g(T, T) = 0$ and $T \neq 0$.*

If $\alpha(\sigma)$ is an equiform spacelike with a timelike equiform principal normal vector, then the equiform formulas are given in [10] by:

$$\frac{dT}{d\sigma} = T' = K_1 T + N,$$

$$\frac{dN}{d\sigma} = N' = T + K_1 N + K_2 B,$$
$$\frac{dB}{d\sigma} = B' = K_2 N + K_1 B,$$

where

$$g(T, T) = -g(N, N) = g(B, B) = \rho^2,$$

$$g(T, N) = g(N, B) = g(B, T) = 0.$$

Lemma 1 *Suppose that a curve α is an equiform spacelike with a timelike equiform principal normal N. If $\alpha(\sigma)$ is parameterized by the equiform parameter σ, then:*

$$T = \frac{d\alpha}{d\sigma}, N = \frac{B \wedge T}{\rho}, B = -\frac{\alpha' \wedge \alpha''}{\rho}.$$

Lemma 2 *If a curve $\alpha(\sigma^*)$ is an equiform spacelike with a timelike equiform principal normal N and σ^* is not necessary the equiform parameter of the curve α, then:*

$$T = \frac{\rho \frac{d\alpha}{d\sigma^*}}{|| \frac{d\alpha}{d\sigma^*} ||}, N = \frac{B \wedge T}{\rho}, B = -\frac{\rho\left(\frac{d\alpha}{d\sigma^*} \wedge \frac{d^2\alpha}{d\sigma^{*2}}\right)}{|| \frac{d\alpha}{d\sigma^*} \wedge \frac{d^2\alpha}{d\sigma^{*2}} ||}$$

Lemma 3 *Suppose that a curve α is an equiform spacelike with a timelike equiform principal normal N. Then, the equiform curvatures are given by:*

$$K_1 = \frac{g(T', T)}{\rho^2} = \frac{-g(N', N)}{\rho^2} = \frac{g(B', B)}{\rho^2},$$

$$K_2 = \frac{g(N', B)}{\rho^2} = \frac{-g(B', N)}{\rho^2}.$$

Definition 2 *A curve $\alpha(\sigma)$ is an ordinary helix if the second equiform curvature $K_2 = 0$, and it is a general helix if K_2 is constant.*

The involute of an equiform spacelike curve with a timelike equiform principal normal

In this section, we study the involute curve of the equiform spacelike curve with a timelike equiform principal normal vector N in E_1^3. Also, the equiform frame of the involute curve is introduced. Furthermore, the equiform curvatures of the involute curve are obtained.

Definition 3 *Let $\alpha(\sigma)$ be an equiform spacelike curve with a timelike equiform principal normal and a curve $\beta(\sigma)$ be given, then the curve β is called an involute of the curve α, if the tangent at the point $\alpha(\sigma)$ to the curve α passes through the tangent at the point $\beta(\sigma)$ to the curve β. In the other words, $\beta(\sigma)$ is an involute of $\alpha(\sigma)$ if the equation $g(T, T^*) = 0$ is satisfied. $\beta(\sigma)$ can be written in terms of the curve α as:*

$$\beta(\sigma) = \alpha(\sigma) + \lambda(\sigma)T(\sigma)$$

Let the equiform frames of the curve $\alpha(\sigma)$ and $\beta(\sigma)$ be $\{T, N, B\}$ and $\{T^*, N^*, B^*\}$, respectively.

Theorem 1 *Let $\alpha(\sigma)$ be an equiform spacelike curve with a timelike equiform principal normal and suppose that a curve $\beta(\sigma)$ is the involute of the curve α. Then,*

$$\beta(\sigma) = \alpha(\sigma) + \frac{c - s}{\rho}T(\sigma),$$

where c is constant.

Proof Suppose that $\beta(\sigma)$ is the involute of $\alpha(\sigma)$. Then, we can write $\beta(\sigma)$ as:

$$\beta(\sigma) = \alpha(\sigma) + \lambda(\sigma)T(\sigma). \quad (1)$$

By taking the derivative of Eq. (1), with respect to σ, we have:

$$\frac{d\beta}{d\sigma} = (1 + \lambda(\sigma)K_1 + \lambda'(\sigma))T + \lambda(\sigma)N.$$

Since $g(T^*, T) = 0$, then we have the differential equation:

$$\lambda'(\sigma) + \lambda(\sigma)K_1 + 1 = 0$$

Hence,

$$\lambda = \frac{c-s}{\rho}. \tag{2}$$

From Eqs. (1) and (2), we obtain:

$$\beta(\sigma) = \alpha(\sigma) + \frac{1}{\rho}(c-s)T(\sigma). \tag{3}$$

□

Corollary 1 *The distance between the curve $\alpha(\sigma)$ and its involute $\beta(\sigma)$ is $|c-s|$.*

Theorem 2 *Let $\alpha(\sigma)$ be an equiform spacelike curve with a timelike equiform principal normal and suppose that a curve β is an involute of the curve α, then:*

$$\begin{bmatrix} T^* \\ N^* \\ B^* \end{bmatrix} = \frac{\rho^*}{\rho}\begin{bmatrix} 0 & 1 & 0 \\ \frac{-1}{\sqrt{K_2^2+1}} & 0 & \frac{-K_2}{\sqrt{K_2^2+1}} \\ \frac{-K_2}{\sqrt{K_2^2+1}} & 0 & \frac{1}{\sqrt{K_2^2+1}} \end{bmatrix}\begin{bmatrix} T \\ N \\ B \end{bmatrix}.$$

Proof By taking the derivative of Eq. (3) with respect to σ, we have:

$$\frac{d\beta}{d\sigma} = \frac{c-s}{\rho}N, \qquad \left\| \frac{d\beta}{d\sigma} \right\| = |c-s|. \tag{4}$$

Then,

$$T^*(\sigma) = \frac{\rho^* \frac{d\beta}{d\sigma}}{\| \frac{d\beta}{d\sigma} \|} = \pm\frac{\rho^*}{\rho}N.$$

Let us assume that:

$$T^* = \frac{\rho^*}{\rho}N. \tag{5}$$

By taking the derivative of Eq. (4) with respect to σ, we have:

$$\frac{d^2\beta}{d\sigma^2} = \frac{(c-s)}{\rho}T - N + \frac{(c-s)K_2}{\rho}B. \tag{6}$$

Hence, we have:

$$\frac{d\beta}{d\sigma} \wedge \frac{d^2\beta}{d\sigma^2} = \frac{(c-s)^2}{\rho}[-K_2T + B], \tag{7}$$

$$\left\| \frac{d\beta}{d\sigma} \wedge \frac{d^2\beta}{d\sigma^2} \right\| = (c-s)^2\sqrt{K_2^2+1}. \tag{8}$$

$$B^* = \frac{\rho^*\left(\frac{d\beta}{d\sigma} \wedge \frac{d^2\beta}{d\sigma^2}\right)}{\| \frac{d\beta}{d\sigma} \wedge \frac{d^2\beta}{d\sigma^2} \|} = \frac{\rho^*}{\rho\sqrt{K_2^2+1}}[-K_2T + B]. \tag{9}$$

Since $N^* = -\frac{B^* \wedge T^*}{\rho^*}$, then we obtain:

$$N^* = \frac{\rho^*}{\rho\sqrt{K_2^2+1}}[-T - K_2B]. \tag{10}$$

□

Corollary 2 *If $\alpha(\sigma)$ is an equiform spacelike curve with a timelike equiform principal normal vector, then its involutes are equiform timelike curves.*

Theorem 3 *Let $\beta(\sigma)$ be an involute of the curve $\alpha(\sigma)$, and K_1^*, K_2^* be the first and second equiform curvatures of the curve β, respectively. Then, K_1^* and K_2^* are given respectively by:*

$$K_1^* = \frac{-\rho}{|c-s|}\frac{d\rho^*}{ds}, \qquad K_2^* = \frac{-\rho^* K_2'}{|c-s|(K_2^2+1)}.$$

Proof Since $T^* = \frac{d\beta}{d\sigma^*} = \frac{d\beta}{d\sigma}\frac{d\sigma}{\sigma^*}$. Using Eq. (4), we obtain:

$$\frac{d\sigma}{d\sigma^*} = \frac{\rho^*}{|c-s|}. \tag{11}$$

By taking the derivative Eq. (5) and using Eq. (11), we get:

$$\frac{dT^*}{d\sigma^*} = T^{*\prime} = \left(\frac{\rho^*}{\rho}T + \frac{d\rho^*}{ds}N + \frac{K_2\rho^*}{\rho}B\right)\left(\frac{\rho^*}{|\,c-s\,|}\right),$$

$$g(T^{*\prime}, T^*) = \frac{\rho^{*2}\rho}{|\,c-s\,|}\frac{d\rho^*}{ds}.$$

Therefore, the first equiform curvature $K_1^* = \frac{-g(T^{*\prime},T^*)}{\rho^{*2}}$ is given by:

$$K_1^* = \frac{-\rho}{|c-s|}\frac{d\rho^*}{ds}.$$

Now, suppose that Eq. (10) is:

$$N^* = -aT - aK_2B,$$

where $a = \frac{\rho^*}{\rho\sqrt{K_2^2+1}}$. Taking the derivative of the above equation with respect to σ^*, we have:

$$N^{*\prime} = \left[-\left(aK_1 + \frac{da}{d\sigma}\right)T + (-a - aK_2^2)N\right.$$

$$\left.-\left(aK_1K_2 + a\frac{dK_2}{d\sigma} + \frac{da}{d\sigma}K_2\right)B\right]\frac{\rho^*}{|\,c-s\,|},$$

$$g(N^{*\prime}, B^*) = \frac{-\rho^2\rho^*}{|c-s|}a^2K_2'.$$

Thus, the second equiform curvature $K_2^* = \frac{g(N^{*\prime},B^*)}{\rho^{*2}}$ is given by:

$$K_2^* = \frac{-\rho^* K_2'}{|c-s|(K_2^2+1)}.$$

□

Corollary 3 *If $\alpha(\sigma)$ is an equiform spacelike curve with a timelike equiform principal normal N and $\beta(\sigma)$ is an involute of $\alpha(\sigma)$, then:*

1. *If $\alpha(\sigma)$ is a planar curve, then $\beta(\sigma)$ is also planar.*
2. *If $\alpha(\sigma)$ is an ordinary helix ($K_2 = 0$), then $\beta(\sigma)$ is planar.*
3. *If $\alpha(\sigma)$ is a general helix ($K_2 = c$), then $\beta(\sigma)$ is planar.*

Proof The proofs come forward from the equation of K_2^*. □

The evolute of equiform spacelike curve with a timelike equiform principal normal

In this section, the evolute curves of the equiform spacelike curve with a timelike equiform principal normal N are studied in E_1^3. Moreover, the equiform frame of the evolute curve is introduced. Furthermore, the equiform curvatures of the evolute are computed.

Definition 4 *Let $\alpha(\sigma)$ be an equiform spacelike with a timelike equiform principal normal and a curve γ with the same interval be given. For $\forall\sigma \in I$, if the tangent at the point $\gamma(\sigma)$ to the curve $\gamma(\sigma)$ passes through the point $\alpha(s)$ and*

$$g(T^*(\sigma), T(\sigma)) = 0,$$

then $\gamma(\sigma)$ is called an evolute of the curve $\alpha(\sigma)$.

Let the Frenet frame of the curve $\alpha(\sigma)$ and $\gamma(\sigma)$ be $\{T, N, B\}$ and $\{T^*, N^*, B^*\}$, respectively.

Theorem 4 *Let $\alpha(\sigma)$ be an equiform spacelike with a timelike equiform principal normal and a curve $\gamma(\sigma)$ be an evolute of α, then:*

$$\gamma(\sigma) = \alpha(\sigma) - N(\sigma) + \tanh\left(\int K_2 d\sigma + c\right) B(\sigma),$$

where $c \in R$
and

$$\frac{d\sigma}{d\sigma^*} = \frac{\rho^* \cosh\left(\int K_2 d\sigma + c\right)}{\rho \left| -K_1 + K_2 \tanh\left(\int K_2 d\sigma + c\right) \right|}.$$

Proof Suppose that a curve $\gamma(\sigma)$ be the evolute of the curve $\alpha(\sigma)$. Then, the vector $\gamma(\sigma) - \alpha(\sigma)$ is perpendicular to the vector $T(\sigma)$. Then,

$$\gamma(\sigma) - \alpha(\sigma) = \lambda(\sigma)N(\sigma) + \mu B(\sigma). \tag{12}$$

By taking the derivative of Eq. (12) with respect to σ, we have:

$$\frac{d\gamma}{d\sigma} = [1 + \lambda(\sigma)]\, T + [\lambda(\sigma)K_1 + \lambda'(\sigma) + \mu K_2]\, N$$

$$+ [\lambda(\sigma)K_2 + \mu K_1 + \mu']\, B.$$

Then, we get:

$$g\left(\frac{d\gamma}{d\sigma}, T\right) = [1 + \lambda(\sigma)]\, g(T, T)$$

$$[(\lambda(\sigma)K_1 + \lambda'(\sigma) + \mu K_2]\, g(N, T)$$

$$+ [\lambda(\sigma)K_2 + \mu K_1 + \mu']\, g(B, T).$$

Since $g\left(\frac{d\gamma}{d\sigma}, T\right) = 0$, then we have:

$$\lambda(\sigma) = -1, \tag{13}$$

and hence,

$$\frac{d\gamma}{d\sigma} = (-K_1 + \mu K_2)N + (\mu' + \mu K_1 - K_2)B. \tag{14}$$

From Eqs. (12) and (14), the vector $\frac{d\gamma}{d\sigma}$ is parallel to the vector $\gamma - \alpha$, and we have:

$$\frac{-K_1 + \mu K_2}{\lambda(\sigma)} = \frac{\mu' + \mu K_1 - K_2}{\mu}.$$

Also, we have:

$$K_2 = \frac{\mu'}{1 - \mu^2}.$$

By taking the integration of the last equation, we get:

$$\int K_2 d\sigma + c = \tanh^{-1}(\mu(\sigma)).$$

Hence, we find:

$$\mu(\sigma) = \tanh\left(\int K_2 d\sigma + c\right). \tag{15}$$

By substituting from Eqs. (13) and (15) into Eq. (12), we have:

$$\gamma(\sigma) = \alpha(\sigma) - N(\sigma) + \tanh\left(\int K_2 d\sigma + c\right)B(\sigma). \tag{16}$$

Since,

$$T^* = \gamma' = \left[-K_1 + K_2 \tanh\left(\int K_2 d\sigma + c\right)\right].$$

$$.\left[N - \tanh\left(\int K_2 d\sigma + c\right)B\right]\frac{d\sigma}{d\sigma^*} \tag{17}$$

and

$$g(T^*, T^*) = \left[-K_1 + K_2 \tanh\left(\int K_2 d\sigma + c\right)\right]^2$$

$$.\rho^2 sech^2\left(\int K_2 d\sigma + c\right)\left(\frac{d\sigma}{d\sigma^*}\right)^2, \tag{18}$$

moreover, we get:

$$\frac{d\sigma}{d\sigma^*} = \frac{\rho^* \ \cosh\left(\int K_2 d\sigma + c\right)}{\rho \ \left|-K_1 + K_2 \tanh\left(\int K_2 d\sigma + c\right)\right|}. \tag{19}$$

□

Theorem 5 *Let $\gamma : I \to E_1^3$ be the evolute curve of the equiform spacelike curve $\alpha : I \to E_1^3$. Then, the equiform frame of the curve γ is given by:*

$$\begin{bmatrix} T^* \\ N^* \\ B^* \end{bmatrix} = \frac{\rho^*}{\rho}\begin{bmatrix} 0 & \cosh z & -\sinh z \\ -1 & 0 & 0 \\ 0 & -\sinh z & \cosh z \end{bmatrix}\begin{bmatrix} T \\ N \\ B \end{bmatrix},$$

where $z = \left(\int K_2 d\sigma + c\right)$.

Proof By similar proof of Theorem 2, we obtain the required. □

Corollary 4 *If the curve* α *is a equiform spacelike curve with a timelike equiform principal normal, then its evolutes are equiform timelike curves.*

Proof The proof comes forward from Theorem 5. □

Theorem 6 *Let the curve* γ *be an evolute of the curve* α *and let* K_1^* *and* K_2^* *be the first and second equiform curvetures of the curve* γ*. Then,*

$$K_1^* = \frac{d\rho^*}{ds}\frac{\left|\cosh\left(\int K_2 d\sigma + c\right)\right|}{\left|-K_1 + K_2\tanh\left(\int K_2 d\sigma + c\right)\right|},$$

$$K_2^* = \frac{\rho^*\left|\sinh 2\left(\int K_2 d\sigma + c\right)\right|}{2\rho\left|-K_1 + K_2\tanh\left(\int K_2 d\sigma + c\right)\right|}.$$

Proof By taking the derivative of N^* with respect to σ^*, we have:

$$N^{*\prime} = \left[\frac{\rho^*\left|\cosh\left(\int K_2 d\sigma + c\right)\right|}{\rho\left|-K_1 + K_2\tanh\left(\int K_2 d\sigma + c\right)\right|}\right]$$

$$.\left[\frac{-d\rho^*}{ds}T - \frac{\rho^*}{\rho}N\right].$$

Then,

$$g(N^{*\prime}, N^*) = \frac{d\rho^*}{ds}\frac{\rho^{*2}\left|\cosh\left(\int K_2 d\sigma + c\right)\right|}{\left|-K_1 + K_2\tanh\left(\int K_2 d\sigma + c\right)\right|}.$$

Therefore,

$$K_1^* = \frac{d\rho^*}{ds}\frac{\left|\cosh\left(\int K_2 d\sigma + c\right)\right|}{\left|-K_1 + K_2\tanh\left(\int K_2 d\sigma + c\right)\right|}. \tag{20}$$

Also,

$$g(N^{*\prime}, B^*) = \frac{-\rho^{*3}\left|\cosh\left(\int K_2 d\sigma + c\right)\right|\sinh\left(\int K_2 d\sigma + c\right)}{\rho\left|K_1 - K_2\tanh\left(\int K_2 d\sigma + c\right)\right|}.$$

Thus,

$$K_2^* = \frac{g(N^{*\prime}, B^*)}{\rho^{*2}}$$

$$= \frac{-\rho^*\ \cosh\left(\int K_2 d\sigma + c\right)\left|\sinh\left(\int K_2 d\sigma + c\right)\right.}{\rho\left|-K_1 + K_2\tanh\left(\int K_2 d\sigma + c\right)\right|}.$$

Therefore,

$$K_2^* = \frac{-\rho^*\left|\sinh 2\left(\int K_2 d\sigma + c\right)\right|}{2\rho\left|-K_1 + K_2\tanh\left(\int K_2 d\sigma + c\right)\right|}. \tag{21}$$

□

Corollary 5 *If the curve* $\alpha(\sigma)$ *is planar, then its evolute curve* $\beta(\sigma)$ *is also planar.*

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Authors' contributions
The author collected the data, performed the calculations, and was a major contributor in writing the manuscript.

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Competing interests
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References

1. Bahaddin, B., Karacan, M.: On the involute and evolute curves of the timelike curve in Minkowski 3-space. Demonstratio Math. (2007). https://doi.org/10.1515/dema-2007-0320
2. Bahaddin, B., Karacan, M.: On involute and evolute curves of spacelike curve with a spacelike principal normal in Minkowski 3-space. Int. J. Math. Combin. **1**, 27–37 (2009)
3. Solouma, E.: Special equiform Smarandache curves in Minkowski space-time. J. Egy. Math. Soc. **25**(3), 319–325 (2017)
4. Bilici, M.: On the involutes of the spacelike curve with a timelike binormal in Minkowski 3-space. Int. Math. Forum. **4**(31), 1497–1509 (2009)
5. Ozturk, U., Ozturk, E., Ilarslan, K.: On the involute-evolute of the pseudonull curve in Minkowski 3-space. J. Appl. Math., 6 (2013). Article ID 651495. https://doi.org/10.1155/2013/651495
6. Pavkovic, B. J., Kamenarovic, I.: The equiform differential geometry of curves in the Galilean space G_3. Glas. Mat. **22**(42), 449–457 (1987)
7. Divjak, E. Z. B.: The equiform differential geometry of curves in the pseudo-Galilean space. Math. Commun. **13**(2), 321–332 (2008)
8. Nawratil, G.: Quaternionic approach to equiform kinematics and line-elements of Euclidean 4-space and 3-space. Comput. Aided Geom. Design. **47**, 150–162 (2016)
9. Pavkovic, B. J.: Equiform geometry of curves in the isotropic spaces I_1^3 and I_2^3. Rad JAZU, 39–44 (1986)
10. El-sayied, H. K., Elzawy, M., Elsharkawy, A.: Equiform spacelike normal curves according to equiform-Bishop frame in E_1^3. Math. Meth. Appl. Sci. **17** (2017). https://doi.org/10.1002/mma.4618
11. El-sayied, H. K., Elzawy, M., Elsharkawy, A.: Equiform timelike normal curves in Minkowski space E_1^3. Far East. J. Math. Sci. **101**, 1619–1629 (2017)
12. O'Neill, B.: Semi-Riemannian Geometry with Applications to Relativity. Academic Press, New York (1983)
13. Walrave, J.: Curves and surfaces in Minkowski space. Doctoral thesis. Leuven, K.U. Faculty of Science, Leuven (1995)

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