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d_L-Filters of principal *MS*-algebras

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Abstract In this paper the notion of d_L-filters is introduced and characterized in principal *MS*-algebras. Also many properties of principal d_L-filters of a principal *MS*-algebra are observed and a characterization of the class of all principal d_L-filters is given. Finally, a relationship between d_L-filters and congruences on a principal *MS*-algebra is investigated.

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1. Introduction

T.S. Blyth and J.C. Varlet [1] introduced the class **MS** of all *MS*-algebras which is a common abstraction of de Morgan algebras and Stone algebras. T.S. Blyth and J.C. Varlet [2] characterized the subvarieties of **MS**. The class **MS** contains the well-known classes for examples Boolean algebras, de Morgan algebras, Kleene algebras and Stone algebras. Recently A. Badawy, D. Guffova and M. Haviar [3] introduced and characterized the class of principal *MS*-algebras by means of triples. A. Badawy [4] introduced de Morgan filters of decomposable *MS*-algebras. S. El-Assar and A. Badawy [5] introduced Homomorphisms and Subalgebras of *MS*-algebras. C. Luo and Y. Zeng [6] characterized the *MS*-algebras on which all congruences are in a one-to-one correspondence with the kernel ideals. In [7] M. Sambasiva Rao introduced the concepts of boosters and β-filters of *MS*-algebras. Also M. Sambasiva Rao [8] introduced the notion of e-filters of *MS*-algebras.

In this paper, the concept of d_L-filters is introduced in principal *MS*-algebras and then many properties of d_L-filters are studied. Various examples of d-filters are introduced. A characterization of d_L-filters of a principal *MS*-algebra is obtained. Also a principal d_L-filter of the form K_a on a principal *MS*-algebra L, for every $a \in L$ is introduced. Every principal d_L-filter can be expressed as K_a for some $a \in L$. It is proved that the class $K(L)$ of all principal d_L-filters forms a de Morgan algebra on its own. A one-to one correspondence between the set of all principal d_L-filters of a principal *MS*-algebra L and the set of all principal filters of $L^{\circ\circ}$ is obtained. Finally, a relationship between d_L-filters and congruences on a principal *MS*-algebra is investigated.

2. Preliminaries

In this section, some certain definitions and results which were introduced in the papers [1–3,9,10] are given.

A de Morgan algebra is an algebra $(L; \vee, \wedge, ^-, 0, 1)$ of type (2, 2, 1, 0, 0) where $(L; \vee, \wedge, 0, 1)$ is a bounded distributive lattice and $^-$ the unary operation of involution satisfies:

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$\overline{\overline{x}} = x, \overline{(x \vee y)} = \overline{x} \wedge \overline{y}, \overline{(x \wedge y)} = \overline{x} \vee \overline{y}$.

An MS-algebra is an algebra $(L; \vee, \wedge, ^\circ, 0, 1)$ of type (2, 2, 1, 0, 0) where $(L; \vee, \wedge, 0, 1)$ is a bounded distributive lattice and $^\circ$ the unary operation of involution satisfies:

$x \leqslant x^{\circ\circ}, (x \wedge y)^\circ = x^\circ \vee y^\circ, 1^\circ = 0$.

The class **MS** of all MS-algebras is equational. A de Morgan algebra is an MS-algebra satisfying the identity, $x = x^{\circ\circ}$. A K_2-algebra is an MS-algebra satisfying the additional two identities

$x \wedge x^\circ = x^\circ \wedge x^{\circ\circ}, (x \wedge x^\circ) \vee (y \vee y^\circ) = y \vee y^\circ$.

The class **S** of all Stone algebras is a subclass of **MS** and is characterized by the identity $x \wedge x^\circ = 0$. A Boolean algebra is an MS-algebra satisfying the identity $x \vee x^\circ = 1$.

Some of the basic properties of MS-algebras which were proved in [1,10] are given in the following Theorem.

Theorem 2.1. *For any two elements a, b of an MS-algebra L, we have*

(1) $0^\circ = 1$,
(2) $a \leqslant b \Rightarrow b^\circ \leqslant a^\circ$,
(3) $a^{\circ\circ\circ} = a^\circ$,
(4) $(a \vee b)^\circ = a^\circ \wedge b^\circ$,
(5) $(a \vee b)^{\circ\circ} = a^{\circ\circ} \vee b^{\circ\circ}$,
(6) $(a \wedge b)^{\circ\circ} = a^{\circ\circ} \wedge b^{\circ\circ}$.

Theorem 2.2. *Let L be an MS-algebra. Then*

(1) $L^{\circ\circ} = \{x \in L : x = x^{\circ\circ}\}$ *is a de Morgan algebra and a subalgebra of L,*
(2) $D(L) = \{x \in L : x^\circ = 0\}$ *is a filter of dense elements of L,*
(3) $B(L) = \{x \in L^{\circ\circ} : x \vee x^\circ = 1\}$ *is a Boolean algebra and a subalgebra of* $L^{\circ\circ}$.

For any MS-algebra L, let $F(L)$ denote to the set of all filters of L. It is known that $(F(L); \wedge, \vee)$ is a distributive lattice, where $F \wedge G = F \cap G$ and $F \vee G = \{f \wedge g : f \in F, g \in G\}$. Also, $[a) = \{x \in L : x \geqslant a\}$ is a principal filter of L generated by a.

By a congruence on an MS-algebra $(L; \vee, \wedge, \circ)$ we shall mean a lattice congruence θ such that

$(x, y) \in \theta$ implies$(x^\circ, y^\circ) \in \theta$

Through what follows, for an MS-algebra L we shall denote by ∇ the universal congruence on L. The Cokernel of the lattice congruence θ on a lattice L is defined as

$Coker\theta = \{x \in L : (x, 1) \in \theta\}$.

The following definition of a principal MS-algebra was introduced in [3].

Definition 2.3 (*Definition 2.1, 3*). An MS-algebra $(L; \vee, \wedge, \circ, 0, 1)$ is called a principal MS-algebra if it satisfies the following conditions

(1) the filter $D(L)$ is principal, i.e., there exists an element $d_L \in L$ such that $D(L) = [d_L)$;
(2) $x = x^{\circ\circ} \wedge (x \vee d_L)$ for any $x \in L$.

3. Properties of d_L-filters

In this Section, the concept of d_L-filters is introduced in a principal MS-algebra. Many properties and examples of d_L-filters are investigated. Also, a set of equivalent conditions is derived for a filter of a principal MS-algebra to become a d_L-filter.

Definition 3.1. Let L be a principal MS-algebra with the smallest dense element d_L. A filter F of L is called a d_L-filter if $d_L \in F$.

Clearly the filter $[d_L)$ is a d_L-filter of L. It is observed that $[d_L)$ is the smallest d_L-filter of L and L is the greatest d_L-filter of L.

Example 3.2.

(1) Every filter of a de Morgan algebra M is a d_L-filter as $d_M = 1$ belongs to any filter.
(2) Let $L = \{0, x, y, z, 1 : 0 < x < y < z < 1\}$ be a five element chain and $x^\circ = x, y^\circ = z^\circ = 0$. Clearly L is a principal MS-algebra with the smallest dense element y. We observe that the filters $\{y, z, 1\}, \{x, y, z, 1\}$ and L are d_L-filters of L but the filters $\{z, 1\}$ and $\{1\}$ are not.

Now, for every filter F of a principal MS-algebra L with the smallest dense element d_L, consider the set $L(F)$ as follows:

$L(F) = \{x \in L : x^{\circ\circ} \in F\}$.

We first state the following Lemma.

Lemma 3.3. *Let F be a filter of a principal MS-algebra L with the smallest dense element d_L. Then $L(F)$ is a d_L-filter of L containing F.*

Proof. Firstly we prove that $L(F)$ is a filter of L. Clearly $1 \in L(F)$. Let $x, y \in L(F)$. Then $x^{\circ\circ}, y^{\circ\circ} \in F$. It follows that $(x \wedge y)^{\circ\circ} = x^{\circ\circ} \wedge y^{\circ\circ} \in F$. Then $x \wedge y \in L(F)$. Again, let $x \in L(F)$ and $z \in L$ such that $z \geqslant x$. Hence $z^{\circ\circ} \geqslant x^{\circ\circ} \in F$. Then $z^{\circ\circ} \in F$ implies $z \in L(F)$. Therefore $L(F)$ is a filter of L. Since $d_L^{\circ\circ} = 1 \in F$, then $d_L \in L(F)$. So $L(F)$ is a d_L-filter of L. Since $x^{\circ\circ} \geqslant x$ for all $x \in F$, then $x^{\circ\circ} \in F$. Hence $x \in L(F)$. Therefore $F \subseteq L(F)$. □

A characterization of d_L-filters of a principal MS-algebra L is given in the following Theorem.

Theorem 3.4. *Let F be a filter of a principal MS-algebra L with the smallest dense element d_L. Then F is a d_L-filter if and only if $L(F) = F$.*

Proof. Let F be a d_L-filter. Then $d_L \in F$. Since $x \vee d_L \geqslant d_L$, then $x \vee d_L \in F$. Let $x \in L(F)$. Then $x^{\circ\circ} \in F$. Now by Definition 2.3 (2) we get

$x = x^{\circ\circ} \wedge (x \vee d_L) \in F$.

Then $L(F) \subseteq F$. By Lemma 3.3, $F \subseteq L(F)$. Therefore $L(F) = F$. Conversely, let $L(F) = F$. By the above Lemma 3.3, F is a d_L-filter of L. □

Let $F_{d_L}(L)$ be the class of all d_L-filters of a principal MS-algebra L. It is observed that the intersection and the supremum of two d_L-filters of L are again d_L-filters of L. Then we can formulate the following.

Theorem 3.5. *For any principal MS-algebra L with the smallest dense element d_L, the class $F_{d_L}(L)$ is a sublattice of $F(L)$ with unit.*

Now more examples of d_L-filters of a principal MS-algebra L are given in the following Lemma 3.6.

Lemma 3.6. *Let F be a filter of a principal MS-algebra L with the smallest dense element d_L. Then*

(1) *every maximal filter L is a d_L-filter,*
(2) *for any prime filter P of L the set $\ell(P) = \{x \in L : x^\circ \notin P\}$ is a d_L-filter.*

Proof.

(1) Let M be a maximal filter of L. Suppose $d_L \notin M$. Then $M \vee [d_L) = L$. Hence $a \wedge b = 0$ for some $a \in M, b \in [d_L)$. Then $0 = a \wedge b \geqslant a \wedge d_L$ implies $a \wedge d_L = 0$. It follows that $a \leqslant a^{\circ\circ} = a^{\circ\circ} \wedge d_L^{\circ\circ} = 0^{\circ\circ} = 0$, where $d_L^{\circ\circ} = 1$. Then $0 = a \in M$ which is a contradiction. Hence $d_L \in M$. Therefore, M is a d_L-filter of L.
(2) Since $0 = 1^\circ \notin P$, then $1 \in \ell(P)$. Let $x, y \in \ell(P)$. Then $x^\circ \notin P$ and $y^\circ \notin P$. Since P is prime, then we get $(x \wedge y)^\circ = x^\circ \vee y^\circ \notin P$. Hence $x \wedge y \in \ell(P)$. Let $x \in \ell(P)$ and $z \in L$ such that $z \geqslant x$. Thus $y^\circ \leqslant x^\circ$. Then $x^\circ \notin P$ implies $y^\circ \notin P$. Hence $y \in P$. Then $\ell(P)$ is a filter of L. Since $d_L^\circ = 0 \notin P$ then $d_L \in \ell(P)$. So, $\ell(P)$ is a d_L-filter of L. □

It is not true that every d_L-filter is a maximal filter. For, in Example 3.2 (2), the filter $\{y, z, 1\}$ is a d_L-filter but not a maximal filter.

Lemma 3.7. *Let L be a principal K_2-algebra with the smallest dense element d_L. Then*

(1) *The filter $L^\vee = \{x \vee x^\circ : x \in L\}$ is a d_L-filter.*
(2) *Any proper filter of L which contains either x or x° for all $x \in L$ is a d_L-filter.*

Proof.

(1) Since $d_L = d_L \vee d_L^\circ$, then $d_L \in L^\vee$ and $L^\vee$ is a d_L-filter of L.
(2) Let F be a proper filter contains either x or x° for all $x \in L$. Let $y \in L^\vee$. Then $y = x \vee x^\circ$ for some $x \in L$. By the hypotheses we get $y = x \vee x^\circ \in F$. Then $L^\vee \subseteq F$. From (1), $d_L \in L^\vee$. It follows that $d_L \in F$. Therefore F is a d_L-filter of L. □

A characterization of d_L-filters of a principal MS-algebra L is studied in the following Theorem.

Theorem 3.8. *Let F be a proper filter of a principal MS-algebra L with the smallest dense element d_L. Then the following conditions are equivalent.*

(1) *F is a d_L-filter,*
(2) *$x \vee d_L \in F$ for each $x \in L$,*
(3) *$x^{\circ\circ} \in F$ implies $x \in F$,*
(4) *For $x, y \in L, x^\circ = y^\circ$ and $x \in F$ imply $y \in F$.*

Proof.

(1) ⇒ (2) Let F be a d_L-filter of L. Then $d_L \in F$. Since $x \vee d_L \geqslant d_L \in F$ for all $x \in L$, then $x \vee d_L \in F$ and the condition (2) holds.
(2) ⇒ (3) Let $x \vee d_L \in F$ for all $x \in L$. Suppose $x^{\circ\circ} \in F$. Since L is principal, then $x = x^{\circ\circ} \wedge (x \vee d_L) \in F$ and the condition (3) holds.
(3) ⇒ (4) Let $x, y \in L$ and $x^\circ = y^\circ$. Suppose $x \in F$. Then $y^{\circ\circ} = x^{\circ\circ} \in F$. So by the condition (3), we get $y \in F$.
(4) ⇒ (1) Since $d_L^\circ = 0 = 1^\circ$ and $1 \in F$, by condition (4), we have $d_L \in F$. Therefore F is a d_L-filter of L. □

4. Principal d_L-filters

In this section, the concept of principal d_L-filters in the class of all principal MS-algebras is studied and characterized. Also a representation of any d_L-filter of a principal MS-algebra as a union of certain principal d_L-filters is given.

For any element a of a principal MS-algebra L with the smallest dense element d_L, consider the set K_a as follows:

$$K_a = \{x \in L : x^\circ \leqslant a^\circ\}$$

In the following Theorem 4.1, some of the basic properties of the set K_a are observed.

Theorem 4.1. *Let L be a principal MS-algebra with the smallest dense element d_L. Then for any two elements a, b of L we have*

(1) *K_a is a d_L-filter of L containing a,*
(2) *$K_a = [a^{\circ\circ} \wedge d_L)$,*
(3) *$K_a = K_{a^{\circ\circ}}$,*
(4) *$K_a = K_b$ if and only if $a^\circ = b^\circ$.*

Proof.

(1) Clearly $1 \in K_a$. Let $x, y \in K_a$. Then $x^\circ \leqslant a^\circ$ and $y^\circ \leqslant a^\circ$. Hence $(x \wedge y)^\circ = x^\circ \vee y^\circ \leqslant a^\circ$. It follows that $x \wedge y \in K_a$. Now, let $x \in L$ and $z \geqslant x$ for some $z \in L$. Then $z^\circ \leqslant x^\circ \leqslant a^\circ$. Thus $z^\circ \in K_a$. Therefore K_a is a d_L-filter of L. Since $d_L^\circ = 0 \leqslant a^\circ$, then K_a is a d_L-filter of L.
(2) Let $x \in K_a$. Then $x^\circ \leqslant a^\circ$ implies $x^{\circ\circ} \geqslant a^{\circ\circ}$. Hence $x = x^{\circ\circ} \wedge (x \vee d_L) \geqslant a^{\circ\circ} \wedge (x \vee d_L) \geqslant a^{\circ\circ} \wedge d_L$. Then $x \in [a^{\circ\circ} \wedge d_L)$. Then $K_a \subseteq [a^{\circ\circ} \wedge d_L)$ Conversely, let $x \in [a^{\circ\circ} \wedge d_L)$. Then $x \geqslant a^{\circ\circ} \wedge d_L$. It follows that $x^\circ \leqslant a^{\circ\circ\circ} \vee d_L^\circ = a^\circ$ as $a^\circ = a^{\circ\circ\circ}$ and $d_L^\circ = 0$. Hence $a \in K_a$ and $[a^{\circ\circ} \wedge d_L) \subseteq K_a$. Therefore $K_a = [a^{\circ\circ} \wedge d_L)$.
(3) From the fact that $x^\circ = x^{\circ\circ\circ}$ for all $x \in L$, we get

$$K_a = \{x \in L : x^\circ \leqslant a^\circ = a^{\circ\circ\circ}\} = K_{a^{\circ\circ}}.$$

(4) Let $K_a = K_b$. Since $a, b \in K_a = K_b$, then $a^\circ \leqslant b^\circ$ and $b^\circ \leqslant a^\circ$. It follows that $a^\circ = b^\circ$. Conversely, let $a^\circ = b^\circ$. Then $K_a = \{x \in L : x^\circ \leqslant a^\circ\} = \{x \in L : x^\circ \leqslant b^\circ\} = K_b$. □

For the principal MS-algebra, we have the following crucial lemma.

Theorem 4.2. *Let L be a principal MS-algebra with the smallest dense element d_L. Then every principal d_L-filter can be expressed as K_a for some $a \in L$.*

Proof. Let $F = [a)$ be a d_L-filter of L. We claim that $F = K_a$. Let $x \in F$. Then $x \geqslant a$. Then $x^{\circ\circ} \geqslant a^{\circ\circ} \geqslant a^{\circ\circ} \wedge d_L$ implies $x^{\circ\circ} \in [a^{\circ\circ} \wedge d_L) = K_a$. Since K_a is a d_L-filter and $x^{\circ\circ} \in K_a$, then by Theorem 3.8(3), $x \in K_a$, it follows that $F \subseteq K_a$. Conversely, since $a \leqslant d_L$ and $a \leqslant a^{\circ\circ}$, then $a \leqslant a^{\circ\circ} \wedge d_L$. Hence $K_a = [a^{\circ\circ} \wedge d_L) \subseteq [a) = F$. Therefore $F = K_a$. □

Consider $K(L) = \{K_a : a \in L\}$ the class of all principal d_L-filters of a principal MS-algebra L. More properties of principal d_L-filters are studied in Theorem 4.3.

Theorem 4.3. *Let L be a principal MS-algebra with the smallest dense element d_L. Then for any two elements a, b of L, the following statements are hold.*

(1) *$a \leqslant b$ in L implies $K_b \subseteq K_a$ in $K(L)$,*
(2) *$K_{a\wedge b} = K_a \vee K_b$,*
(3) *$K_{a\vee b} = K_a \cap K_b$,*
(4) *$K(L)$ is a bounded distributive lattice,*
(5) *$a \to K_{a^\circ}$ is an epimorphism of L into $K(L)$.*

Proof.

(1) Let $a \leqslant b$ in L. Assume $x \in K_b$, then $x^\circ \leqslant b^\circ \leqslant a^\circ$. Hence $x \in K_a$ and $F_b \subseteq K_a$.
(2) By Theorem 4.1 (2), we get

$$\begin{aligned} K_{a\wedge b} &= [(a \wedge b)^{\circ\circ} \wedge d_L) = [a^{\circ\circ} \wedge b^{\circ\circ} \wedge d_L) \\ &= [(a^{\circ\circ} \wedge d_L) \wedge (b^{\circ\circ} \wedge d_L)) = [a^{\circ\circ} \wedge d_L) \vee [b^{\circ\circ} \wedge d_L) \\ &= K_a \vee K_b. \end{aligned}$$

(3) Using Lemma (2) and by distributivity of L we get

$$\begin{aligned} K_{a\vee b} &= [(a \vee b)^{\circ\circ} \wedge d_L) = [(a^{\circ\circ} \vee b^{\circ\circ}) \wedge d_L) \\ &= [(a^{\circ\circ} \wedge d_L) \vee (b^{\circ\circ} \wedge d_L)) = [a^{\circ\circ} \wedge d_L) \cap [b^{\circ\circ} \wedge d_L) \\ &= K_a \cap K_b. \end{aligned}$$

(4) Clearly $K_1 = [d_L)$ and $K_0 = L$ are the smallest and the greatest elements of $K(L)$ respectively. Then by (2) and (3) we observe that $(K(L), \vee, \cap, [d_L), L)$ is a bounded lattice. Using the distributivity of L, we can get $K_a \vee (K_b \cap K_c) = (K_a \vee K_b) \cap (K_a \vee K_c)$. Therefore $K(L)$ is a bounded distributive.
(5) Define the mapping $f : L \to K(L)$ by $f(a) = K_{a^\circ}$. Let $a, b \in L$. Then by (2) and (3) above and Theorem 4.1(3) we get the following equalities.

$$\begin{aligned} &f(a \wedge b) = K_{(a\wedge b)^\circ} = K_{a^\circ \vee b^\circ} = K_{a^\circ} \cap K_{b^\circ} = f(a) \cap f(b), \\ &f(a \vee b) = K_{(a\vee b)^\circ} = K_{a^\circ \wedge b^\circ} = K_{a^\circ} \vee K_{b^\circ} = f(a) \vee f(b), \\ &f(a^\circ) = K_{a^{\circ\circ}} = K_a \\ &f(0) = [d_L) \quad \text{and} \quad f(1) = L. \end{aligned}$$

Then f is a $(0,1)$-lattice homomorphism. Now, for every $K_a \in K(L)$, by Theorem 4.1(3) we get $f(a^\circ) = K_{a^{\circ\circ}} = K_a$. Therefore f is an epimorphism. □

Consider the subset $I = \{a^{\circ\circ} \wedge d_L : a \in L\}$ of a principal MS-algebra L with the smallest dense element d_L. Now some properties of I are given in the following.

Theorem 4.4. *Let L be a principal MS-algebra with the smallest dense element d_L. Then the following statements hold*

(1) *I is an ideal of L,*
(2) *I is a de Morgan algebra on its own,*
(3) *$I \cong L^{\circ\circ}$,*
(4) *$K(L) \cong I$.*

Proof.

(1) Clearly $0 \in I$. Let $x, y \in I$. Then $x = a^{\circ\circ} \wedge d_L$ and $y = b^{\circ\circ} \wedge d_L$ for some $a, b \in L$. Hence $x \vee y = (a^{\circ\circ} \wedge d_L) \vee (b^{\circ\circ} \wedge d_L) = (a \vee b)^{\circ\circ} \wedge d_L$. It follows that $x \vee y \in I$. Again, let $x \in I$ and $z \leqslant x, z \in L$. Then $x = a^{\circ\circ} \wedge d_L$ for some $a \in L$. Since $z = z^{\circ\circ} \wedge (z \vee d_L)$, then $z = z \wedge x = z^{\circ\circ} \wedge (z \vee d_L) \wedge a^{\circ\circ} \wedge d_L = (z \wedge a)^{\circ\circ} \wedge d_L$. Therefore $z \in I$ and I is an ideal of L.
(2) We observe that 0 and d_L are the smallest and the greatest elements of I respectively. From (1), I is a bounded distributive lattice. Define the operation $^-$ on I by $\bar{x} = x^\circ \wedge d_L$. Then for any $x, y \in M$ we get

$$\begin{aligned} &\overline{x \wedge y} = (x \wedge y)^\circ \wedge d_L = (x^\circ \vee y^\circ) \wedge d_L = (x^\circ \wedge d_L) \vee (y^\circ \wedge d_L) \\ &\quad = \bar{x} \vee \bar{y}, \\ &\overline{x \vee y} = (x \vee y)^\circ \wedge d_L = x^\circ \wedge y^\circ \wedge d_L = (x^\circ \wedge d_L) \wedge (y^\circ \wedge d_L) \\ &\quad = \bar{x} \wedge \bar{y}, \\ &\bar{\bar{x}} = \overline{x^\circ \wedge d_L} = (x^\circ \wedge d_L)^\circ \wedge d_L = x^{\circ\circ} \wedge d_L \text{ as } d_L^\circ = x^{\circ\circ} \wedge (x \vee d_L) \\ &\quad \text{as } x \leqslant d_L = x. \end{aligned}$$

Therefore I is a de Morgan algebra.
(3) Define a mapping $f : L^{\circ\circ} \to I$ by $f(a) = a \wedge d_L$. Clearly $f(0) = 0$ and $f(1) = d_L$. For all $a, b \in L^{\circ\circ}$ we have $a = a^{\circ\circ}$ and $b = b^{\circ\circ}$. Now

$$\begin{aligned} &f(a \vee b) = (a \vee b) \wedge d_L = (a \wedge d_L) \vee (b \wedge d_L) = f(a) \vee f(b), \\ &f(a \wedge b) = (a \wedge b) \wedge d_L = (a \wedge d_L) \wedge (b \wedge d_L) = f(a) \wedge f(b), \\ &f(a^\circ) = a^\circ \wedge d_L = a^{\circ\circ\circ} \wedge d_L = \overline{a^{\circ\circ} \wedge d_L} = \overline{a \wedge d_L} = \overline{f(a)} \end{aligned}$$

Then f is a homomorphism. Let $f(a) = f(b)$. Then $a \wedge d_L = b \wedge d_L$ implies $a = a^{\circ\circ} \wedge d_L^{\circ\circ} = b^{\circ\circ} \wedge d_L^{\circ\circ} = b$ as $a = a^{\circ\circ}$ and $d_L^{\circ\circ} = 1$. Hence f is an injective map. Now we prove that f is a surjective map. Let $x \in M$. Then $x = a^{\circ\circ} \wedge d_L$ for some $a \in L$. Thus $f(a^{\circ\circ}) = a^{\circ\circ} \wedge d_L = x$. Therefore f is an isomorphism between two de Morgan algebras $L^{\circ\circ}$ and I.

(4) Define $g : I \to K(L)$ by $g(a^{\circ\circ} \wedge d_L) = K_{a^\circ}$. Let $x, y \in M$. Then $x = a^{\circ\circ} \wedge d_L$ for some $a \in L$ and $y = b^{\circ\circ} \wedge d_L$ for some $b \in L$. Now

$$g(x \vee y) = g((a \vee b)^{\circ\circ} \wedge d_L) = K_{(a\vee b)^\circ} = K_{a^\circ \wedge b^\circ} = K_{a^\circ} \vee K_{b^\circ} = g(x) \vee g(y),$$

$$g(x \wedge y) = g((a \wedge b)^{\circ\circ} \wedge d_L) = K_{(a\wedge b)^\circ} = K_{a^\circ \vee b^\circ} = K_{a^\circ} \cap K_{b^\circ} = g(x) \cap g(y),$$

$$g(\overline{x}) = g(a^\circ \wedge d_L) = K_{a^{\circ\circ}} = \overline{K_{a^\circ}} = \overline{g(x)}$$

Then g is a homomorphism. For any $K_a \in K(L)$, there exists $x = a^\circ \wedge d_L \in I$ such that $g(a^\circ \wedge d_L) = K_{a^{\circ\circ}} = K_a$. Hence g is a surjective. Suppose that $g(a^\circ \wedge d_L) = g(a^\circ \wedge d_L)$. Then $K_{a^\circ} = K_{b^\circ}$. By Theorem 4.1 (2), $[a^\circ \wedge d_L) = [b^\circ \wedge d_L)$. It follows that $a^\circ \wedge d_L = b^\circ \wedge d_L$. Then g is an injective mapping. Therefore g is an isomorphism. □

A one-to-one correspondence between the class of all principal d_L-filters of L and the class of all principal filters of $L^{\circ\circ}$ is obtained in the following Theorem 4.5.

Theorem 4.5. *Let L be a principal MS-algebra with the smallest dense element d_L. Then there exists a one-to-one correspondence between the class of all principal d_L-filters of L and the class of all principal filters of $L^{\circ\circ}$.*

Proof. Let F be a principal d_L-filter generated by the element a. One can easily prove that $[a) \cap L^{\circ\circ}$ is a filter of $L^{\circ\circ}$. Now let $x \in [a) \cap L^{\circ\circ}$. Then $x \geqslant a$ and $x \in L^{\circ\circ}$. Hence $x = x^{\circ\circ} \geqslant a^{\circ\circ}$. Therefore $a^{\circ\circ}$ is the smallest element of $[a) \cap L^{\circ\circ}$. Then $[a) \cap L^{\circ\circ}$ is a principal filter of $L^{\circ\circ}$ generated by $a^{\circ\circ}$. Conversely, let $A = [a)$ be a principal filter of $L^{\circ\circ}$. Then by Theorem 4.1(1) and (2), K_a is a principal d_L-filter of L. □

It is known that any filter of a finite MS-algebra is a principal filter. From the above Theorem, the following corollary is an immediate consequence.

Corollary 4.6. *Let L be a finite principal MS-algebra. Then we have*

(1) *Every d_L-filter can be expressed as K_a for some $a \in L$,*
(2) $F_{d_L}(L) = K(L)$.

Now, we can represent any d_L-filter of a principal MS-algebra L as a union of certain principal d_L-filters.

Theorem 4.7. *Let F be a d_L-filter of a principal MS-algebra L with the smallest dense d_L. Then $F = \bigcup_{x\in F} K_x$.*

Proof. Let $y \in F$. Since L is principal MS-algebra, then $y = y^{\circ\circ} \wedge (y \vee d_L) \geqslant y^{\circ\circ} \wedge d_L$. Thus $y \in [y^{\circ\circ} \wedge d_L) = K_y \subseteq \bigcup_{x\in F} K_x$. Then $F \subseteq \bigcup_{x\in F} K_x$. Conversely, let $y \in \bigcup_{x\in F} K_x$. Then $y \in K_z$ for some $z \in F$. Then $y^\circ \leqslant z^\circ$ implies $y^{\circ\circ} \geqslant z^{\circ\circ} \in F$. Then $y^{\circ\circ} \in F$ implies $y \in F$ as F is a d_L-filter. Thus $\bigcup_{x\in F} K_x \subseteq F$. Therefore $F = \bigcup_{x\in F} K_x$. □

5. Congruences via d_L-filters

In this section, the relationship between d_L-filters and congruences of a principal MS-algebra L is investigated.

Let L be a principal MS-algebra with the smallest dense element d_L. Define a binary relation θ_{d_L} on L as follows:

$(x, y) \in \theta_{d_L}$ if and only if $x \wedge d_L = y \wedge d_L$.

Some properties of θ_{d_L} are studied in the following Theorem 5.1.

Theorem 5.1. *Let L be a principal MS-algebra with the smallest dense element d_L. Then the following statements hold*

(1) *θ_{d_L} is a congruence on L with $Coker\theta_{d_L} = [d_L)$,*
(2) *$[x]\theta_{d_L} = [x^{\circ\circ}]\theta_{d_L}$ for all $x \in L$,*
(3) *L/θ_{d_L} is a de Morgan algebra on its own.*

Proof.

(1) It is clear that θ_{d_L} is a lattice congruence on L. Let $(x, y) \in \theta_{d_L}$. Then $x \wedge d_L = y \wedge d_L$. Hence $x^\circ = x^\circ \vee d_L^\circ = (x \wedge d_L)^\circ = (y \wedge d_L)^\circ = y^\circ \vee d_L^\circ = y^\circ$ as $d_{d_L}^\circ = 0$. Hence $x^\circ \wedge d_L = y^\circ \wedge d_L$ and $(x^\circ, y^\circ) \in \theta_{d_L}$. Therefore θ_{d_L} is a congruence on L. Now we have

$$Coker\theta_{d_L} = \{x \in L : (x, 1) \in \theta_{d_L}\} = \{x \in L : x \wedge d_L = d_L\} = \{x \in L : x \geqslant d_L\} = [d_L).$$

(2) Since $x = x^{\circ\circ} \wedge (x \vee d_L)$ for all $x \in L$, then we get

$$x \wedge d_L = x^{\circ\circ} \wedge (x \vee d_L) \wedge d_L = x^{\circ\circ} \wedge d_L$$

Then $(x, x^{\circ\circ}) \in \theta_{d_L}$ implies $[x]\theta_{d_L} = [x^{\circ\circ}]\theta_{d_L}$.

(3) It is known that the quotient set L/θ_{d_L} is $\{[x]\theta_{d_L} : x \in L\}$. Clearly $(L/\theta_{d_L}, \vee, \wedge)$ is a bounded distributive lattice with bounds $[0]\theta_{d_L}$ and $[1]\theta_{d_L} = [d_L)$, where $[x]\theta_{d_L} \wedge [y]\theta_{d_L} = [x \wedge y]\theta_{d_L}$ and $[x]\theta_{d_L} \vee [y]\theta_{d_L} = [x \vee y]\theta_{d_L}$. We can define a unary operation $^-$ on L/θ_{d_L} by $\overline{[x]}\theta_{d_L} = [x^\circ]\theta_{d_L}$. We observe $\overline{[0]}\theta_{d_L} = [1]\theta_{d_L}$ and $\overline{[1]}\theta_{d_L} = [0]\theta_{d_L}$. We have the following equalities

$$\overline{\overline{[x]}}\theta_{d_L} = [x^{\circ\circ}]\theta_{d_L} = [x]\theta_{d_L},$$

$$\overline{[x]\theta_{d_L} \wedge [y]\theta_{d_L}} = \overline{[x \wedge y]}\theta_{d_L} = [(x \wedge y)^\circ]\theta_{d_L} = [x^\circ \vee y^\circ]\theta_{d_L} = [x^\circ]\theta_{d_L} \vee [y^\circ]\theta_{d_L} = \overline{[x]}\theta_{d_L} \vee \overline{[y]}\theta_{d_L}.$$

Similarly we can prove that $\overline{[x]\theta_{d_L} \vee [x]\theta_{d_L}} = \overline{[x]}\theta_{d_L} \wedge \overline{[y]}\theta_{d_L}$. Then L/θ_{d_L} is a de Morgan algebra. □

The following Lemma characterizes the Cokernel d_L-filters.

Lemma 5.2. *Let θ be a congruence relation on a principal MS-algebra L with the smallest dense element d_L such that $\theta \geqslant \theta_{d_L}$. Then $Coker\theta$ is a d_L-filter of L.*

Proof. It is known that $Coker\theta$ is a filter of L. Since $d_L \wedge d_L = d_L = 1 \wedge d_L$, then $(d_L, 1) \in \theta_{d_L} \subseteq \theta$. Hence $(d_L, 1) \in \theta$ and $d_L \in Coker\theta$. Therefore $Coker\theta$ is a d_L-filter of L. □

For every element a of a principal MS-algebra L, define the relation θ_{K_a} on L as follows

$(x,y) \in \theta_{K_a}$ if and only if $x^{\circ\circ} \wedge a = y^{\circ\circ} \wedge a$

The following Theorem reveals many basic properties of θ_{K_a}.

Theorem 5.3. *Let L be a principal MS-algebra with the smallest dense element d_L. Then for any $a \in L$ we have the following*

(1) *θ_{K_a} is a lattice congruence on L with $Coker\theta_{K_a} = K_a$,*
(2) *$(x, x^{\circ\circ}) \in \theta_{K_a}$ for all $x \in L$,*
(3) *$\theta_{K_1} = \theta_{d_L}$ and $\theta_{K_0} = \nabla$,*
(4) *$\theta_{K_a} = \theta_{K_{a^{\circ\circ}}}$,*
(5) *L/θ_{K_a} is a de Morgan algebra on its own.*

Proof.

(1) Obviously θ_{K_a} is an equivalent relation on L. Let $(x,y),(c,d) \in \theta_{K_a}$. Then $x^{\circ\circ} \wedge a = y^{\circ\circ} \wedge a$ and $c^{\circ\circ} \wedge a = d^{\circ\circ} \wedge a$. Then we have the following equalities

$$\begin{aligned}(x \vee c)^{\circ\circ} \wedge a &= (x^{\circ\circ} \vee c^{\circ\circ}) \wedge a = (x^{\circ\circ} \wedge a) \vee (c^{\circ\circ} \wedge a)\\ &= (y^{\circ\circ} \wedge a) \vee (d^{\circ\circ} \wedge a) = (y^{\circ\circ} \vee d^{\circ\circ}) \wedge a\\ &= (y \vee d)^{\circ\circ} \wedge a\end{aligned}$$

and

$$\begin{aligned}(x \wedge c)^{\circ\circ} \wedge a &= (x^{\circ\circ} \wedge c^{\circ\circ}) \wedge a = (x^{\circ\circ} \wedge a) \wedge (c^{\circ\circ} \wedge a)\\ &= (y^{\circ\circ} \wedge a) \wedge (d^{\circ\circ} \wedge a) = (y^{\circ\circ} \wedge d^{\circ\circ}) \wedge a\\ &= (y \wedge d)^{\circ\circ} \wedge a\end{aligned}$$

Consequently $(x \vee c, y \vee d), (x \wedge c, y \wedge d) \in \theta_{K_a}$. Therefore θ_{K_a} is a lattice congruence relation on L. Now we show that $Coker\theta_{K_a} = K_a$.

$$\begin{aligned}Coker\theta_{K_a} &= \{x \in L : (x,1) \in \theta_{K_a}\} = \{x \in L : x^{\circ\circ} \wedge a\\ &= 1^{\circ\circ} \wedge a\} = \{x \in L : x^{\circ\circ} \wedge a = a\} = \{x \in L : x^{\circ\circ}\\ &\geqslant a\} = \{x \in L : x^{\circ} \leqslant a^{\circ}\} = K_a\end{aligned}$$

(2) Since $x^{\circ\circ} = x^{\circ\circ\circ\circ}$ for all $x \in L$, then $x^{\circ\circ} \wedge a = x^{\circ\circ\circ\circ} \wedge a$. It follows that $(x, x^{\circ\circ}) \in \theta_{K_a}$.

(3) Let $(x,y) \in \theta_{d_L}$. Thus $x \wedge d_L = y \wedge d_L$. Then $x^{\circ\circ} \wedge d_L^{\circ\circ} = y^{\circ\circ} \wedge d_L^{\circ\circ}$ implies $x^{\circ\circ} \wedge 1 = y^{\circ\circ} \wedge 1$. Hence $(x,y) \in K_1$ and $\theta_{d_L} \subseteq \theta_{K_1}$. Conversely, let $(x,y) \in K_1$. Then $x^{\circ\circ} \wedge 1 = y^{\circ\circ} \wedge 1$ implies $x^{\circ\circ} \wedge d_L = y^{\circ\circ} \wedge d_L$. Then $(x^{\circ\circ}, y^{\circ\circ}) \in \theta_{d_L}$. By Theorem 5.1(2), $(x, x^{\circ\circ}), (y^{\circ\circ}, y) \in \theta_{d_L}$. Then by transitivity of θ_{d_L} we get $(x,y) \in \theta_{d_L}$. Then $\theta_{K_1} \subseteq \theta_{d_L}$ and $\theta_{K_1} = \theta_{d_L}$. Now we observe that

$$\theta_{K_0} = \{(x,y) \in L \times L : x^{\circ\circ} \wedge 0 = 0 = y^{\circ\circ} \wedge 0\} = L \times L = \nabla.$$

(4) Now we proceed to show that $\theta_{K_a} = \theta_{K_{a^{\circ\circ}}}$

$$\begin{aligned}(x,y) \in \theta_{K_a} &\iff x^{\circ\circ} \wedge a = y^{\circ\circ} \wedge a \iff (x^{\circ\circ} \wedge a)^{\circ\circ}\\ &= (y^{\circ\circ} \wedge a)^{\circ\circ} \iff x^{\circ\circ} \wedge a^{\circ\circ} = y^{\circ\circ} \wedge a^{\circ\circ} \text{ as } a^{\circ\circ\circ\circ}\\ &= a^{\circ\circ} \iff (x,y) \in K_{a^{\circ\circ}}.\end{aligned}$$

(5) Since L is a bounded distributive lattice, then $(L/\theta_{K_a}, \vee, \wedge, [0]\theta_{K_a}, K_a)$ is a bounded distributive lattice. Define the unary operation $^-$ on L/θ_{K_a} by $\overline{[x]}\theta_{K_a} = [x^{\circ}]\theta_{K_a}$. Then we observe that

$$\begin{aligned}&\overline{\overline{[x]}}\theta_{K_a} = [x^{\circ\circ}]\theta_{K_a} = [x]\theta_{K_a},\\ &\overline{[x]\theta_{K_a} \wedge [y]\theta_{K_a}} = \overline{[x]}\theta_{K_a} \vee \overline{[y]}\theta_{K_a},\\ &\overline{[x]\theta_{K_a} \vee [y]\theta_{K_a}} = \overline{[x]}\theta_{K_a} \wedge \overline{[y]}\theta_{K_a}.\end{aligned}$$

Therefore $(L/\theta_{K_a}, \vee, \wedge, -, [0]\theta_{K_a}, K_a)$ is a de Morgan algebra. □

Let $Con_M(L) = \{\theta_{K_a} : a \in L^{\circ\circ}\}$. Now we prove the following.

Theorem 5.4. *Let L be a principal MS-algebra with the smallest dense element d_L. Then for any $a, b \in L^{\circ\circ}$ we have the following*

(1) *$a \leqslant b$ in $L^{\circ\circ}$ if and only if $\theta_{K_b} \leqslant \theta_{K_a}$ in $Con_M(L)$,*
(2) *$\theta_{K_a} \cap \theta_{K_b} = \theta_{K_{a\vee b}}$,*
(3) *$\theta_{K_a} \vee \theta_{K_b} = \theta_{K_{a\wedge b}}$.*

Proof.

(1) Let $a \leqslant b$ and $(x,y) \in \theta_{K_b}$. Then $x^{\circ\circ} \wedge b = y^{\circ\circ} \wedge b$. Hence $x^{\circ\circ} \wedge b \wedge a = y^{\circ\circ} \wedge b \wedge a$. This leads to $x^{\circ\circ} \wedge a = y^{\circ\circ} \wedge a$. Then $(x,y) \in \theta_{K_a}$ and $\theta_{K_b} \subseteq \theta_{K_a}$. Conversely, let $\theta_{K_b} \subseteq \theta_{K_a}$. Then we have $(b,1) \in \theta_{K_b} \subseteq \theta_{K_a}$. This implies that $b^{\circ\circ} \wedge a = 1^{\circ\circ} \wedge a = a$. Thus $b = b^{\circ\circ} \geqslant a$.

(2) Since $a, b \leqslant a \vee b$, then by (1), $\theta_{K_{a\vee b}} \subseteq \theta_{K_a}, \theta_{K_b}$. Hence $\theta_{K_{a\vee b}} \subseteq \theta_{K_a} \cap \theta_{K_b}$. Conversely, let $(x,y) \in \theta_{K_a} \cap \theta_{K_b}$. Then $(x,y) \in \theta_{K_a}, \theta_{K_b}$. Then we have $x^{\circ\circ} \wedge a = y^{\circ\circ} \wedge a$ and $x^{\circ\circ} \wedge b = y^{\circ\circ} \wedge b$. Now

$$\begin{aligned}x^{\circ\circ} \wedge (a \vee b) &= (x^{\circ\circ} \wedge a) \vee (x^{\circ\circ} \wedge b) = (y^{\circ\circ} \wedge a) \vee (y^{\circ\circ} \wedge b)\\ &= y^{\circ\circ} \wedge (a \vee b).\end{aligned}$$

Consequently $(x,y) \in \theta_{K_{a\vee b}}$. Then $\theta_{K_a} \cap \theta_{K_b} \subseteq \theta_{K_{a\vee b}}$.

(3) It is clear that $\theta_{K_{a\wedge b}}$ is an upper bound of both θ_{K_a} and θ_{K_b} on $Con_M(L)$. Let θ_{K_z} be an upper bound of both θ_{K_a} and θ_{K_b} on $Con_M(L)$, for some $z \in L^{\circ\circ}$. Then, we obtain $\theta_{K_a} \subseteq \theta_{K_z}$ and $\theta_{K_b} \subseteq \theta_{K_z}$. This result leads to $a \geqslant z$ and $b \geqslant z$, which implies $a \wedge b \geqslant z$. Hence by (1), we have $\theta_{K_{a\wedge b}} \subseteq \theta_{K_z}$. Thus, $\theta_{K_{a\wedge b}}$ is the supremum of both θ_{K_a} and θ_{K_b} on $Con_M(L)$. □

From above Theorem 4.2, the following Theorem is an immediate consequence.

Theorem 5.5. *$Con_M(L)$ forms a de Morgan algebra on its own.*

Proof. From the above Theorem 4.2(2) and (3) we proved that the infimum and the supremum of any two elements of $Con_M(L)$ are elements of $Con_M(L)$. Then $(Con_M(L), \cap, \vee)$ is a lattice. For every $\theta_{K_a}, \theta_{K_b}, \theta_{K_c} \in Con_M(L)$, We get the following equalities.

$$\begin{aligned}\theta_{K_a} \cap (\theta_{K_b} \vee \theta_{K_c}) &= \theta_{K_a} \cap \theta_{K_{b\wedge c}} = \theta_{K_{a\vee(b\wedge c)}} = \theta_{K_{(a\vee b)\wedge(a\vee c)}}\\ &= \theta_{K_{a\vee b}} \vee \theta_{K_{a\vee c}} = (\theta_{K_a} \cap \theta_{K_b}) \vee (\theta_{K_a} \cap \theta_{K_c})\end{aligned}$$

This shows that $Con_M(L)$ is a distributive lattice. Since $\theta_{K_1} = \theta_{d_L}$ and $\theta_{K_0} = \nabla$ are the least and the greatest elements of $Con_M(L)$ respectively, then $Con_M(L)$ is bounded. Now, we define a unary operation $^-$ on $Con_M(L)$ by $\overline{\theta}_{K_a} = \theta_{K_{a^\circ}}$. Then we get the following equalities.

$$\overline{\overline{\theta}}_{K_a} = \theta_{K_a},$$
$$\overline{\theta_{K_a} \cap \theta_{K_b}} = \overline{\theta}_{K_{a\vee b}} = \theta_{K_{(a\vee b)^\circ}} = \theta_{k_{a^\circ \wedge b^\circ}} = \theta_{K_{a^\circ}} \vee \theta_{K_{b^\circ}} = \overline{\theta}_{K_a} \vee \overline{\theta}_{K_b},$$
$$\overline{\theta_{K_a} \vee \theta_{K_b}} = \overline{\theta}_{K_{a\wedge b}} = \theta_{K_{(a\wedge b)^\circ}} = \theta_{K_{a^\circ \vee b^\circ}} = \theta_{K_{a^\circ}} \cap \theta_{K_{b^\circ}} = \overline{\theta}_{K_a} \cap \overline{\theta}_{K_b}.$$

Therefore $(Con_M(L), \vee, \cap, -, \theta_{d_L}, \nabla)$ forms a de Morgan algebra on its own. □

Finally, we conclude this paper with the following.

Theorem 5.6. *Let L be a principal MS-algebra with the smallest dense element d_L. Then for any $a \in B(L)$ we have the following*

(1) *θ_{K_a} is a congruence on L,*
(2) *$\theta_{K_a} \cap \theta_{K_{a^\circ}} = \theta_{d_L}$,*
(3) *$\theta_{K_a} \vee \theta_{K_{a^\circ}} = \nabla$,*
(4) *L/θ_{K_a} is a Boolean algebra, whenever $L \in \mathbf{S}$,*
(5) *$Con_B(L) = \{\theta_{K_a} : a \in B(L)\}$ is a Boolean subalgebra of $Con_M(L)$.*

Proof.

(1) We proved in Theorem 5.3(1) that θ_{K_a} is a lattice congruence on L. Now for every element a of a Boolean subalgebra $B(L)$ of L, we prove that θ_{K_a} preserves the operation $\circ$. Since $B(L)$ is a Boolean algebra, then $a \vee a^\circ = 1$ and $a \wedge a^\circ = 0$ for all $a \in B(L)$. Now we get the following equalities

$$\begin{aligned}(x,y) \in \theta_{K_a} &\Rightarrow x^{\circ\circ} \wedge a = y^{\circ\circ} \wedge a \Rightarrow (x^{\circ\circ} \wedge a) \vee a^\circ \\ &= (y^{\circ\circ} \wedge a) \vee a^\circ \Rightarrow (x^{\circ\circ} \vee a^\circ) \wedge (a \vee a^\circ) \\ &= (y^{\circ\circ} \vee a^\circ) \wedge (a \vee a^\circ) \Rightarrow x^{\circ\circ} \vee a^\circ = y^{\circ\circ} \vee a^\circ \text{ as } a \vee a^\circ \\ &= 1 \Rightarrow (x^{\circ\circ} \vee a^\circ)^\circ = (y^{\circ\circ} \vee a^\circ)^\circ \Rightarrow x^{\circ\circ\circ} \wedge a^{\circ\circ} = y^{\circ\circ\circ} \wedge a^{\circ\circ} \\ &\Rightarrow x^{\circ\circ\circ} \wedge a = y^{\circ\circ\circ} \wedge a \text{ as } a^{\circ\circ} = a \Rightarrow (x^\circ, y^\circ) \in \theta_{K_a}\end{aligned}$$

Therefore θ_{K_a} is a congruence on L.

(2) Using Theorem 5.4 (2) and $a \vee a^\circ = 1$, we get.

$$\theta_{K_a} \cap \theta_{K_{a^\circ}} = \theta_{K_{a\vee a^\circ}} = \theta_{K_1} = \theta_{d_L}$$

(3) By Theorem 5.4 (3) and $a \wedge a^\circ = 0$, we obtain the following.

$$\theta_{K_a} \vee \theta_{K_{a^\circ}} = \theta_{K_{a\wedge a^\circ}} = \theta_{K_0} = \nabla$$

(4) By Theorem 5.3(5), for every $a \in L$, we proved that L/θ_{K_a} is a de Morgan algebra. Let $L \in \mathbf{S}$ and $a \in B(L)$. Then for every $x \in L$ we have $x \wedge x^\circ = 0$ and $x \vee x^\circ \in [d_L) \subseteq K_a$. Consequently

$$[x]\theta_{K_a} \vee [x^\circ]\theta_{K_a} = [x \vee x^\circ]\theta_{K_a} = K_a = [1]\theta_{K_a}$$

and

$$[x]\theta_{K_a} \wedge [x^\circ]\theta_{K_a} = [x \wedge x^\circ]\theta_{K_a} = [0]\theta_{K_a}$$

Then every element of L/θ_{K_a} has a complement. Then L/θ_{K_a} is a Boolean algebra.

(5) For every $a, b \in B(L)$ we have $\theta_{K_a} \vee \theta_{K_b} = \theta_{K_{a\wedge b}} \in Con_B(L)$ and $\theta_{K_a} \wedge \theta_{K_b} = \theta_{K_{a\vee b}} \in Con_B(L)$ as $a \vee b, a \wedge b \in B(L)$. Then $(Con(L), \vee, \wedge)$ is a sublattice of $Con_M(L)$. Since $0, 1 \in B(L)$, then $\theta_{K_0}, \theta_{K_1} \in Con_B(L)$. Hence $Con_B(L)$ is a bounded distributive lattice. Now we can define the unary operation $^-$ on $Con_B(L)$ by $\overline{\theta}_{K_a} = \theta_{K_{a^\circ}}$. Then we get $\overline{\theta}_{K_a} \cap \theta_{K_a} = \theta_{K_{a^\circ}} \cap \theta_{K_a} = \theta_{K_1} = \theta_{d_L}$ and $\overline{\theta}_{K_a} \vee \theta_{K_a} = \theta_{K_{a^\circ}} \vee \theta_{K_a} = \theta_{K_0} = \nabla$. Then every element of $Con_B(L)$ has a complement. Therefore $Con_B(L)$ is a Boolean subalgebra of $Con_M(L)$. □

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