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Some relations between power graphs and Cayley graphs

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Abstract Motivated by an open problem of Abawajy et al. [1] we find some relations between power graphs and Cayley graphs of finite cyclic groups. We show that the vertex deleted subgraphs of some power graphs are spanning subgraphs or equal to the complement of vertex deleted subgraphs of some unitary Cayley graphs. Also we prove that some Cayley graphs can be expressed as direct product of power graphs. Applying these relations we study the eigenvalues and energy of power graphs and the related Cayley graphs.

2000 MATHEMATICS SUBJECT CLASSIFICATION: 05C25; 05C50

1. Introduction

The study of graphical representation of semigroups or groups becomes an exciting research topic in the last few decades, leading to many fascinating results and questions. In this context the most popular class of graphs are the Cayley graphs. Cayley graphs are introduced in 1878, well studied and having many applications. The concept of power graphs is a very recent development. In this paper we have worked on an open problem of Abawajy et al. [1, Problem 10] which asked for link between power graphs and Cayley graphs.

The concept of directed power graph was first introduced and studied by Kelarev and Quinn [2–4]. The *directed power graph* of a semigroup S is a digraph with vertex set S and for $x, y \in S$ there is an arc from x to y if and only if $x \neq y$ and $y = x^m$ for some positive integer m. Following this Chakrabarty et al. [5] defined the *undirected power graph* $\mathcal{G}(G)$ of a group G as an undirected graph whose vertex set is G and two vertices u, v are adjacent if and only if $u \neq v$ and $u^m = v$ or $v^m = u$ for some positive integer m. After that the undirected power graph became the main focus of study in [6–9]. In [5] it was shown that for any finite group G, the power graph of a subgroup of G is an induced subgraph of $\mathcal{G}(G)$ and $\mathcal{G}(G)$ is complete if and only if G is a cyclic group of order 1 or p^m, for some prime p and positive integer m. In [6] Cameron has proved that for a finite cyclic group G of non-prime-power order n, the set of vertices T_n of $\mathcal{G}(G)$ which are adjacent to

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all other vertices of $\mathcal{G}(G)$, consists of the identity and the generators of G, so that $|T_n| = 1 + \phi(n)$, where $\phi(n)$ is the Euler's ϕ function. For more results on power graphs we refer the recent survey paper [1]. In this paper our main subject of study is undirected power graph and so we use the brief term '*power graph*' to refer to the *undirected power graph.*

For a finite group G and a subset S of G not containing the identity element e and satisfying $S^{-1} = \{s^{-1} : s \in S\} = S$, the *Cayley graph* of G with connection set S, $Cay(G, S)$, is an undirected graph with vertex set G and two vertices g and h are adjacent if and only if $gh^{-1} \in S$. For any positive integer n, let $\mathbb{Z}_n$ denotes the additive cyclic group of integers modulo n. If we represent the elements of $\mathbb{Z}_n$ by $\overline{0}, \overline{1}, \ldots, \overline{n-1}$, then $U_n = \{\overline{a} \in \mathbb{Z}_n : gcd(a, n) = 1\}$ is a subset of $\mathbb{Z}_n$ of order $\phi(n)$, where $\phi(n)$ is the Euler's ϕ function. The Cayley graph $Cay(\mathbb{Z}_n, U_n)$ is known as *unitary Cayley graph*, see [10]. One can observe that $Cay(\mathbb{Z}_n, \mathbb{Z}_n - \{\overline{0}\})$ is the complete graph K_n on n vertices.

For a finite simple graph G with vertex set $\{v_1, v_2, \ldots, v_n\}$, the adjacency matrix $A(G) = (a_{ij})$ is defined as an $n \times n$ matrix, where $a_{ij} = 1$ if $v_i \sim v_j$, and $a_{ij} = 0$ otherwise. The eigenvalues of $A(G)$ are also called the eigenvalues of G and denoted by $\lambda_i(G), i = 1, 2, \ldots, n$. Since $A(G)$ is a symmetric matrix, $\lambda_i(G)$'s are all real and so they can be ordered as $\lambda_1(G) \geqslant \lambda_2(G) \geqslant \cdots \geqslant \lambda_n(G)$. By Perron Frobenious theorem, see [11], $\lambda_1(G)$ is always positive and $\lambda_1(G) > |\lambda_i(G)|$ for all $i = 2, 3, \ldots, n$. For the graph G, the *energy* $E(G)$ of G, introduced by Gutman [12], is the sum of the absolute values of all its eigenvalues. The concept of energy of graphs arose in chemistry. The total π-electron energy of some conjugated carbon molecule, computed using H*ü*ckel theory, coincides with the energy of its "molecular" graph [13]. One can easily check that the eigenvalues of the complete graph K_n are $n-1$ and -1 with respective multiplicities 1 and $n-1$ and so $E(K_n) = 2(n-1)$. A graph G on n vertices is called *hyperenergetic* if $E(G) > 2(n-1)$ [13]. In [14] the authors found energy of all unitary Cayley graphs and determined conditions for which they are hyperenergetic.

Due to the applications of Cayley graphs in automata theory as explained in the monograph [15] and other versatile areas, the authors of [1] have given an open problem (Problem 10) to investigate the relations of power graphs and Cayley graphs. In this paper we find some relations between power graphs of finite cyclic groups $\mathbb{Z}_n$ and the Cayley graphs. Applying these relations we obtain the eigenvalues and energy of $\mathcal{G}(\mathbb{Z}_n)$ as well as of the related Cayley graphs and also find the relations between the energy of power graphs and Cayley graphs.

2. Relations between power graphs and Cayley graphs

It is known [10] that if $n = p$ is a prime number, then the unitary Cayley graph $Cay(\mathbb{Z}_n, U_n)$ is the complete graph K_n and if $n = p^\alpha$ is a prime-power then it is a complete p-partite graph. So we have the observations below.

(i) For any prime $p, \mathcal{G}(\mathbb{Z}_p) = K_p = Cay(\mathbb{Z}_p, U_p)$.
(ii) If $n = p^\alpha$ for some prime p and a positive integer $\alpha > 1$ then $Cay(\mathbb{Z}_n, U_n)$ is a regular spanning subgraph of $\mathcal{G}(\mathbb{Z}_n)$.

Notations: Let T_n be a subset of $\mathbb{Z}_n$ consists of the identity and generators i.e. $T_n = U_n \cup \{\overline{0}\}$. We denote the vertex deleted subgraph $\mathcal{G}(\mathbb{Z}_n) - T_n$ of the graph $\mathcal{G}(\mathbb{Z}_n)$ by $\mathcal{G}^*(\mathbb{Z}_n)$ and similarly $Cay^*(\mathbb{Z}_n, U_n) = Cay(\mathbb{Z}_n, U_n) - T_n$. Again for any graph G let $\overline{G}$ be the complement of G.

Theorem 2.1 gives a relation between $\mathcal{G}^*(\mathbb{Z}_n)$ and $Cay^*(\mathbb{Z}_n, U_n)$ for some values of n. From the definition of power graph it is clear that the vertices of T_n are adjacent to all other vertices in $\mathcal{G}(\mathbb{Z}_n)$. So roughly speaking, this theorem gives an expression of $\mathcal{G}(\mathbb{Z}_n)$ in terms of $Cay(\mathbb{Z}_n, U_n)$ for a class of values of n. Now since $Cay(\mathbb{Z}_n, U_n)$ is highly symmetric and also widely studied in the literature, this theorem may help us to investigate the structure and various properties of $\mathcal{G}(\mathbb{Z}_n)$. For instance in the next section we apply this theorem to investigate the eigenvalues and energy of $\mathcal{G}(\mathbb{Z}_{pq})$ which may not be so easy to get otherwise.

Theorem 2.1. *If $n = p^\alpha q^\beta$, where p, q are distinct primes and α, β are positive integers, then $\mathcal{G}^*(\mathbb{Z}_n)$ is a spanning subgraph of $\overline{Cay^*(\mathbb{Z}_n, U_n)}$. These two graphs are equal if and only if $\alpha = 1 = \beta$.*

Proof. Both the graphs have the same vertex set $\mathbb{Z}_n - T_n$, where $T_n = U_n \cup \{\overline{0}\}$. Let $E_p = \{a\overline{p} \in \mathbb{Z}_n : q \nmid a\}, E_q = \{b\overline{q} \in \mathbb{Z}_n : p \nmid b\}$ and $E_{pq} = \{t\overline{pq} \in \mathbb{Z}_n\} - \{\overline{0}\}$. Then E_p, E_q, and E_{pq} are pairwise disjoint sets whose union is $\mathbb{Z}_n - T_n$.

First we look into the adjacency among the vertices in the graph $Cay^*(\mathbb{Z}_n, U_n)$. If possible suppose that for some integers c and $d, c\overline{p} \sim d\overline{p}$. Then there exists $\overline{s} \in U_n$ such that for some integer r,

$$c\overline{p} + \overline{s} = d\overline{p} \Rightarrow s = (d - c + rp^{\alpha-1}q^\beta)p$$

which is a contradiction because $p \nmid s$. So for any integers c and $d, c\overline{p} \nsim d\overline{p}$. Similarly it can be shown that for any integers c and $d, c\overline{q} \nsim d\overline{q}$. Thus no vertex of E_p, E_q is adjacent to a vertex in the same set and each vertex of E_{pq} is an isolated vertex in the graph $Cay^*(\mathbb{Z}_n, U_n)$. Now for any $\overline{s} \in U_n, \gcd(s, n) = 1$ and so there exist integers u and v such that $su + nv = 1$. Consider any vertex $a\overline{p}$ from E_p and $b\overline{q}$ from E_q. Then

$$ap - bq = (ap - bq)(su + nv) \equiv (ap - bq)su \quad (mod\ n).$$

Since $p \nmid apu - bqu$ as well as $q \nmid apu - bqu, (ap - bq)$ $u\overline{s} \in U_n$ and so $a\overline{p} \sim b\overline{q}$. Hence $Cay^*(\mathbb{Z}_n, U_n)$ is a complete bipartite graph with bipartition $E_p \cup E_q$ along with the isolated vertices E_{pq}. So in the graph $\overline{Cay^*(\mathbb{Z}_n, U_n)}$, none of the vertices of E_p is adjacent to any vertex of E_q and these are the only non-adjacency of vertices in $\overline{Cay^*(\mathbb{Z}_n, U_n)}$.

Next we check for the non-adjacency of vertices in the graph $\mathcal{G}^*(\mathbb{Z}_n)$. If possible suppose that $a\overline{p}$ of E_p is adjacent to $b\overline{q}$ of E_q in the graph $\mathcal{G}^*(\mathbb{Z}_n)$. Then for some positive integers m_1 and m_1',

$$a\overline{p} = m_1 b\overline{q} \quad \text{or} \quad b\overline{q} = m_1' a\overline{p}.$$

First consider $a\overline{p} = m_1 b\overline{q}$ which implies $ap = m_1 bq + m_2 p^\alpha q^\beta$ for some $m_2 \in \mathbb{Z}$. But then $q \mid ap$ which is a contradiction as q is a prime and $q \nmid p$ as well as $q \nmid a$. Hence $a\overline{p} \neq m_1 b\overline{q}$. Similarly it can be proved that $b\overline{q} \neq m_1' a\overline{p}$. Thus none of the vertices of E_p is adjacent to any vertex of E_q. Hence $\mathcal{G}^*(\mathbb{Z}_n)$ is a spanning subgraph of $\overline{Cay^*(\mathbb{Z}_n, U_n)}$.

For $\alpha = 1 = \beta$ i.e. for $n = pq, E_{pq} = \phi, E_p = \{\overline{p}, 2\overline{p}, \ldots, (q-1)\overline{p}\}$ and $E_q = \{\overline{q}, 2\overline{q}, \ldots, (p-1)\overline{q}\}$. Thus $Cay^*(\mathbb{Z}_{pq}, U_{pq})$ is a complete bipartite graph with bipartition $E_p \cup E_q$ and so $\overline{Cay^*(\mathbb{Z}_{pq}, U_{pq})}$ is the disjoint union of K_{q-1} and K_{p-1}. Again $E_p \cup \{\overline{0}\}$ and $E_q \cup \{\overline{0}\}$, being closed under addition modulo pq, are subgroups of $\mathbb{Z}_{pq}$ of order q and p respectively. Therefore both $E_p \cup \{\overline{0}\}$ and $E_q \cup \{\overline{0}\}$ are cyclic groups of prime order and so $\mathcal{G}(E_p) = K_{q-1}, \mathcal{G}(E_q) = K_{p-1}$. Thus $\mathcal{G}^*(\mathbb{Z}_{pq})$ is the disjoint union of K_{q-1} and K_{p-1} and so $\mathcal{G}^*(\mathbb{Z}_{pq}) = \overline{Cay^*(\mathbb{Z}_{pq}, U_{pq})}$.

Next we take $\alpha > 1$. If possible suppose that $\overline{p}^2$ of E_p is adjacent to $\overline{pq}$ of E_{pq} in the graph $\mathcal{G}^*(\mathbb{Z}_n)$. Then for some positive integers m and m',

$$\overline{p}^2 = m\overline{pq} \quad \text{or} \quad \overline{pq} = m'\overline{p}^2.$$

First consider $\overline{p}^2 = m\overline{pq}$. Then for some integer $r, p = (m + rp^{\alpha-1}q^{\beta-1})q$ which is a contradiction. Hence $\overline{p}^2 \neq m\overline{pq}$. Similarly it can be proved that $\overline{pq} \neq m'\overline{p}^2$. Thus for $\alpha > 1$, the two graphs $\mathcal{G}^*(\mathbb{Z}_n)$ and $\overline{Cay^*(\mathbb{Z}_n, U_n)}$ can not be equal. The same thing is also valid for $\beta > 1$. Hence $\mathcal{G}^*(\mathbb{Z}_n) = \overline{Cay^*(\mathbb{Z}_n, U_n)}$ if only if $\alpha = 1 = \beta$. □

Next we give another relation between power graphs and Cayley graphs using the direct product concept of graphs as well as that of groups. Recall that the direct product $G_1 \otimes G_2$ of graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph with vertex set $V_1 \times V_2$, the cartesian product of V_1 and V_2, and (u_1, u_2) is adjacent to (v_1, v_2) in $G_1 \otimes G_2$ if and only if u_1, v_1 are adjacent in G_1 and u_2, v_2 are adjacent in G_2. Also, for the groups $\mathbb{Z}_{p_i^{\alpha_i}}, 1 \leqslant i \leqslant k$, the direct product $\prod_{i=1}^{k} \mathbb{Z}_{p_i^{\alpha_i}}$ is the group with elements $\{(\overline{u_1}, \ldots, \overline{u_k}) : \overline{u_i} \in \mathbb{Z}_{p_i^{\alpha_i}}\}$ and binary operation $(\overline{u_1}, \ldots, \overline{u_k}) + (\overline{v_1}, \ldots, \overline{v_k}) = (\overline{u_1} + \overline{v_1}, \ldots, \overline{u_k} + \overline{v_k})$, where addition in i^{th} component is modulo $p_i^{\alpha_i}$.

In Theorem 2.2 we show that with the appropriate choice of the connection set S, the Cayley graph $Cay(\mathbb{Z}_n, S)$ is either the power graph of $\mathbb{Z}_n$ or the direct product of the power graphs of suitable proper subgroups of $\mathbb{Z}_n$. Lemma 2.1 below will be useful to prove this theorem. However this Lemma is similar to Lemma 2.6 of [16] and so here we omit its proof.

Lemma 2.1. *For distinct primes p_i, positive integers α_i and subsets $S_i \subset \mathbb{Z}_{p_i^{\alpha_i}}, (i = 1, 2, \ldots, k)$, $Cay(\mathbb{Z}_{p_1^{\alpha_1}}, S_1) \otimes \cdots \otimes Cay(\mathbb{Z}_{p_k^{\alpha_k}}, S_k) = Cay\left(\prod_{i=1}^{k} \mathbb{Z}_{p_i^{\alpha_i}}, \prod_{i=1}^{k} S_i\right)$.*

Notations: For a natural number $n > 1$, we take its prime factorization as $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$, where $p_1, p_2, \ldots, p_k$ are distinct primes and $\alpha_1, \alpha_2, \ldots, \alpha_k$ are positive integers. It is known [17] that $\mathbb{Z}_n \cong \prod_{i=1}^{k} \mathbb{Z}_{p_i^{\alpha_i}}$ as groups (under addition) through the isomorphism $g : \mathbb{Z}_n \to \prod_{i=1}^{k} \mathbb{Z}_{p_i^{\alpha_i}}$ defined by $g([a]_n) = ([a]_{p_1^{\alpha_1}}, [a]_{p_2^{\alpha_2}}, \ldots, [a]_{p_k^{\alpha_k}})$. We consider the mapping $f = g^{-1}$. Then $f : \prod_{i=1}^{k} \mathbb{Z}_{p_i^{\alpha_i}} \to \mathbb{Z}_n$ is also a group isomorphism. The image of an element in $\prod_{i=1}^{k} \mathbb{Z}_{p_i^{\alpha_i}}$ under f can be computed by applying the Chinese remainder theorem.

Theorem 2.2. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the prime factorization of a natural number $n > 1, S_i = \mathbb{Z}_{p_i^{\alpha_i}} - \{\overline{0}\}, i = 1, 2, \ldots, k$, and S be the image of $\prod_{i=1}^{k} S_i$ under the above mentioned group isomorphism $f : \prod_{i=1}^{k} \mathbb{Z}_{p_i^{\alpha_i}} \to \mathbb{Z}_n$. Then $Cay(\mathbb{Z}_n, S)$ is isomorphic to $\mathcal{G}(\mathbb{Z}_{p_1^{\alpha_1}}) \otimes \mathcal{G}(\mathbb{Z}_{p_2^{\alpha_2}}) \otimes \cdots \otimes \mathcal{G}(\mathbb{Z}_{p_k^{\alpha_k}})$.*

Proof. Clearly $\mathcal{G}(\mathbb{Z}_{p_i^{\alpha_i}}) = K_{p_i^{\alpha_i}} = Cay(\mathbb{Z}_{p_i^{\alpha_i}}, S_i)$ for all $i = 1, 2, \ldots, k$. Now from Lemma 2.1, $Cay(\mathbb{Z}_{p_1^{\alpha_1}}, S_1) \otimes Cay(\mathbb{Z}_{p_2^{\alpha_2}}, S_2) \otimes \cdots \otimes Cay(\mathbb{Z}_{p_k^{\alpha_k}}, S_k) = Cay\left(\prod_{i=1}^{k} \mathbb{Z}_{p_i^{\alpha_i}}, \prod_{i=1}^{k} S_i\right)$. Next we show that $Cay\left(\prod_{i=1}^{k} \mathbb{Z}_{p_i^{\alpha_i}}, \prod_{i=1}^{k} S_i\right)$ is isomorphic to $Cay(\mathbb{Z}_n, S)$. In fact the group isomorphism $f : \prod_{i=1}^{k} \mathbb{Z}_{p_i^{\alpha_i}} \to \mathbb{Z}_n$ is also a graph isomorphism i.e. it preserves adjacency as shown below:

$$(\overline{u_1}, \ldots, \overline{u_k}) \sim (\overline{v_1}, \ldots, \overline{v_k}) \text{ in } Cay\left(\prod_{i=1}^{k} \mathbb{Z}_{p_i^{\alpha_i}}, \prod_{i=1}^{k} S_i\right),$$

$$\Longleftrightarrow (\overline{u_1}, \ldots, \overline{u_k}) - (\overline{v_1}, \ldots, \overline{v_k}) \in \prod_{i=1}^{k} S_i,$$

$$\Longleftrightarrow f((\overline{u_1}, \ldots, \overline{u_k}) - (\overline{v_1}, \ldots, \overline{v_k})) \in f\left(\prod_{i=1}^{k} S_i\right), \text{ as } f \text{ is a bijection}$$

$$\Longleftrightarrow f(\overline{u_1}, \ldots, \overline{u_k}) - f(\overline{v_1}, \ldots, \overline{v_k}) \in S, \text{ as}$$

f is a group homomorphism

$$\Longleftrightarrow f(\overline{u_1}, \ldots, \overline{u_k}) \sim f(\overline{v_1}, \ldots, \overline{v_k}) \text{ in } Cay(\mathbb{Z}_n, S).$$

Therefore f preserves adjacency and we get $Cay(\mathbb{Z}_n, S) \cong Cay(\mathbb{Z}_{p_1^{\alpha_1}}, S_1) \otimes \cdots \otimes Cay(\mathbb{Z}_{p_k^{\alpha_k}}, S_k) = \mathcal{G}(\mathbb{Z}_{p_1^{\alpha_1}}) \otimes \cdots \otimes \mathcal{G}(\mathbb{Z}_{p_k^{\alpha_k}})$. □

We illustrate this theorem by giving an example.

Example 1: Let us consider the group $\mathbb{Z}_{12} \cong \mathbb{Z}_4 \times \mathbb{Z}_3$ with $\mathbb{Z}_n = \{\overline{0}_n, \overline{1}_n, \ldots \overline{n-1}_n, \}$. Therefore $S_1 = \{\overline{1}_4, \overline{2}_4, \overline{3}_4\}, S_2 = \{\overline{1}_3, \overline{2}_3\}$ and $S = f(S_1 \times S_2)$. The graphs $\mathcal{G}(\mathbb{Z}_4), \mathcal{G}(\mathbb{Z}_3)$ and $\mathcal{G}(\mathbb{Z}_4) \otimes \mathcal{G}(\mathbb{Z}_3)$ are given in Fig. 1 and Fig. 2. The images of f and so the elements of S can be computed applying the Chinese remainder theorem. Here $S = \{\overline{1}_{12}, \overline{2}_{12}, \overline{5}_{12}, \overline{7}_{12}, \overline{10}_{12}, \overline{11}_{12}\}$ and so $Cay(\mathbb{Z}_{12}, S)$ can be drawn as given in Fig. 3. From Figs. 2 and 3 one can check that f is a graph isomorphism from the graph $\mathcal{G}(\mathbb{Z}_4) \otimes \mathcal{G}(\mathbb{Z}_3)$ to the graph $Cay(\mathbb{Z}_{12}, S)$.

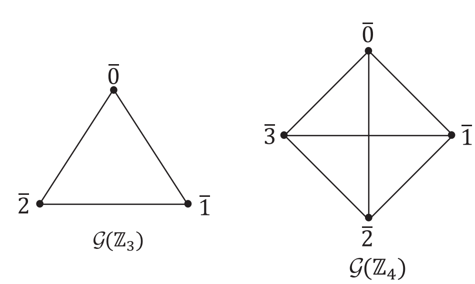

Fig. 1 $\mathcal{G}(\mathbb{Z}_3)$ and $\mathcal{G}(\mathbb{Z}_4)$.

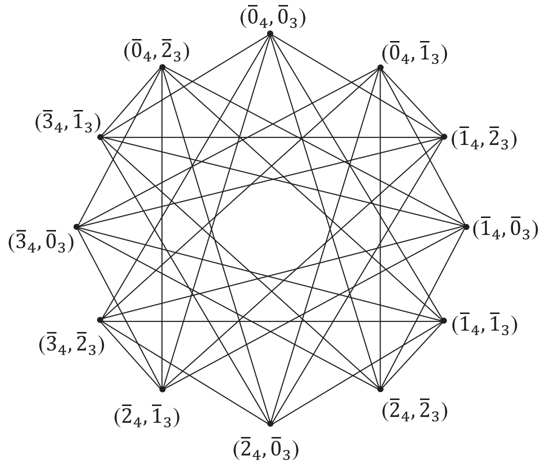

Fig. 2 $\mathcal{G}(\mathbb{Z}_4) \otimes \mathcal{G}(\mathbb{Z}_3)$.

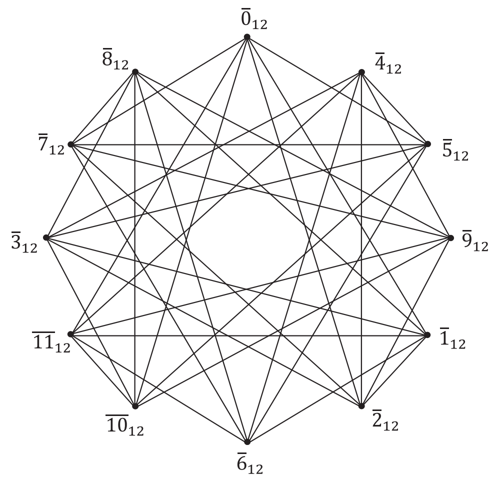

Fig. 3 $Cay(\mathbb{Z}_{12}, S)$.

3. Energy of $\mathcal{G}(Z_n)$ and $Cay(Z_n, S)$

For the rest of this paper the connection set S appears in $Cay(\mathbb{Z}_n, S)$ will be that subset of $\mathbb{Z}_n$ given in Theorem 2.2. In this section we apply the relation between power graph and Cayley graph to investigate the eigenvalues and energy of $\mathcal{G}(\mathbb{Z}_n)$ as well as that of $Cay(\mathbb{Z}_n, S)$. Since the eigenvalues and energy of unitary Cayley graph is well studied (for example see [10,14]) we compare the energies of the graphs $\mathcal{G}(\mathbb{Z}_n)$ and $Cay(\mathbb{Z}_n, S)$ with that of the unitary Cayley graph. In the following theorem we apply Theorem 2.1 to find the eigenvalues and energy of $\mathcal{G}(\mathbb{Z}_{pq})$.

Theorem 3.1. *For the power graph $\mathcal{G}(\mathbb{Z}_{pq})$, where p and q are two distinct primes, we have the following.*

(i) $\lambda_2(\mathcal{G}(\mathbb{Z}_{pq})) \leqslant \frac{p+q}{2} - 2$ *and* $\lambda_j(\mathcal{G}(\mathbb{Z}_{pq})) = -1$, $j = 3, 4, \ldots, pq-1$;
(ii) $-\frac{p+q}{2} \leqslant \lambda_{pq}(\mathcal{G}(\mathbb{Z}_{pq})) \leqslant -1$
(iii) $E(\mathcal{G}(\mathbb{Z}_{pq})) \leqslant 2pq + p + q - 6$

Proof. (i) It is well known [11] that for any graph G on n vertices $\lambda_2(G) + \lambda_n(\overline{G}) \leqslant -1$. Also if G is connected and $\overline{G}$ is the union of a complete bipartite graph and some isolated vertices then from [18] one gets that $\lambda_j(G) = -1$, for $j = 3, 4, \ldots, n-1$. Now for two distinct primes p and q, $\mathcal{G}(\mathbb{Z}_{pq})$ is a connected graph and from Theorem 2.1, $\overline{\mathcal{G}(\mathbb{Z}_{pq})}$ is the union of $Cay^*(\mathbb{Z}_{pq}, U_{pq})$ and the vertices of T_{pq} i.e. $\overline{\mathcal{G}(\mathbb{Z}_{pq})}$ is the graph $K_{q-1,p-1}$ plus $\phi(pq) + 1$ isolated vertices. Thus

$$\lambda_j(\mathcal{G}(\mathbb{Z}_{pq})) = -1, \quad j = 3, 4, \ldots, pq-1 \tag{1}$$

and $\lambda_2(\mathcal{G}(\mathbb{Z}_{pq})) - \sqrt{(p-1)(q-1)} \leqslant -1$ (since the smallest eigenvalue of $K_{q-1,p-1}$ is $-\sqrt{(p-1)(q-1)}$). This implies

$$\lambda_2(\mathcal{G}(\mathbb{Z}_{pq})) \leqslant \sqrt{(p-1)(q-1)} - 1 \leqslant \frac{p+q}{2} - 2 \quad (\text{A.M} \geqslant \text{G.M}). \tag{2}$$

(ii) Since $\lambda_{pq}(\mathcal{G}(\mathbb{Z}_{pq})) \leqslant \lambda_{pq-1}(\mathcal{G}(\mathbb{Z}_{pq}))$ the right hand side inequality follows from (i). Again it is known that (for instance, see [11]) for any graph G on n vertices, $\lambda_1(G) \leqslant n-1$ and the sum of the eigenvalues of G is zero. Using these facts and Eqs. (1), (2) we get the left hand side inequality.
(iii) From (ii) it follows that

$$|\lambda_{pq}(\mathcal{G}(\mathbb{Z}_{pq}))| \leqslant \frac{p+q}{2}. \tag{3}$$

Thus from Eqs. (1)–(3) and using $0 < \lambda_1(\mathcal{G}(\mathbb{Z}_{pq})) \leqslant pq - 1$ we get

$$E(\mathcal{G}(\mathbb{Z}_{pq})) = \sum_{i=1}^{pq} |\lambda_i(\mathcal{G}(\mathbb{Z}_{pq}))| \leqslant 2pq + p + q - 6. \quad \square \tag{4}$$

It is known [13] that for any two graphs G_1 and G_2, $E(G_1 \otimes G_2) = E(G_1)E(G_2)$. Since $E(\mathcal{G}(\mathbb{Z}_{p_i^{\alpha_i}})) = 2(p_i^{\alpha_i} - 1)$, for $1 \leqslant i \leqslant k$, applying Theorem 2.2 we get the energy of $Cay(\mathbb{Z}_n, S)$ as given below.

Theorem 3.2. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the prime factorization of any natural number $n > 1$. Then*

$$\begin{aligned} E(Cay(\mathbb{Z}_n, S)) &= E(\mathcal{G}(\mathbb{Z}_{p_1^{\alpha_1}}))E(\mathcal{G}(\mathbb{Z}_{p_2^{\alpha_2}})) \cdots E(\mathcal{G}(\mathbb{Z}_{p_k^{\alpha_k}})) \\ &= 2^k (p_1^{\alpha_1} - 1)(p_2^{\alpha_2} - 1) \cdots (p_k^{\alpha_k} - 1). \end{aligned} \tag{5}$$

In the following few results we give comparison of energies of power graph and Cayley graph.

Corollary 3.1. *For $n = pq$, where p, q are distinct odd primes, $E(\mathcal{G}(\mathbb{Z}_n)) \leqslant E(Cay(\mathbb{Z}_n, S))$.*

Proof. For $p = 3, q = 5$ we get from (4) and (5),

$$E(\mathcal{G}(\mathbb{Z}_{pq})) \leqslant 32 = E(Cay(\mathbb{Z}_{pq}, S)). \tag{6}$$

Now for both $p, q \geqslant 5$ it is easy to verify that $5(p+q-2) < 2pq$ which implies that $p + q - 6 < 2pq - 4p - 4q + 4$. Thus using (4) and (5)

$$\begin{aligned} E(\mathcal{G}(\mathbb{Z}_{pq})) &< 4pq - 4p - 4q + 4 < 4(p-1)(q-1) \\ &= E(Cay(\mathbb{Z}_{pq}, S)). \ \square \end{aligned}$$

Corollary 3.2. *For $n = 2q$, where q is an odd prime, $E(\mathcal{G}(\mathbb{Z}_n)) > E(Cay(\mathbb{Z}_n, S))$.*

Proof. It is well known [11] that for any graph G on n vertices $\lambda_1(G) \geqslant \frac{2M}{n}$, where M is the number of edges in G. By Corollary 4.3 of [5] the number of edges, M of $\mathcal{G}(\mathbb{Z}_{2q})$ is given by

$$2M = [4 - \phi(2) - 1]\phi(2) + [2q - \phi(q) - 1]\phi(q) + [4q - \phi(2q) - 1]\phi(2q) = 2 + 4q(q-1).$$

Therefore

$$\lambda_1(\mathcal{G}(\mathbb{Z}_{2q})) \geqslant \frac{2M}{2q} = 2(q-1) + \frac{1}{q}. \quad (7)$$

Now it is well known (see [13]) that for any graph $G, E(G) \geqslant 2\lambda_1(G)$. Combining this fact with Eqs. (5) and (7) we get that

$$E(\mathcal{G}(\mathbb{Z}_{2q})) \geqslant 4(q-1) + \frac{2}{q} = E(Cay(\mathbb{Z}_{2q}, S)) + \frac{2}{q}. \; \square$$

Theorem 3.3. *For any natural number $n > 1$, $E(Cay(\mathbb{Z}_n, S)) \geqslant E(Cay(\mathbb{Z}_n, U_n))$.*

Proof. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the prime factorization of a natural number $n > 1$. Since for all $i = 1, 2, \ldots, k, p_i$'s are primes and α_i's are positive integers then $p_i^{\alpha_i} - 1 \geqslant p_i^{\alpha_i} - p_i^{\alpha_i - 1}$.

$$\begin{aligned} So \quad & (p_1^{\alpha_1} - 1)(p_2^{\alpha_2} - 1) \cdots (p_k^{\alpha_k} - 1) \\ & \geqslant \phi(p_1^{\alpha_1})\phi(p_2^{\alpha_2}) \cdots \phi(p_k^{\alpha_k}) \\ & = \phi(n), \quad \text{since } \phi \text{ is multiplicative.} \end{aligned} \quad (8)$$

Now by Theorem 3.7 of [14], $E(Cay(\mathbb{Z}_n, U_n)) = 2^k \phi(n)$. So from Eqs. (5) and (8) we get

$$E(Cay(\mathbb{Z}_n, S)) \geqslant 2^k \phi(n) = E(Cay(\mathbb{Z}_n, U_n)). \quad \square$$

From Corollary 3.2 and Theorem 3.3 the following corollary is immediate.

Corollary 3.3. *For $n = 2q$, where q is an odd prime, $E(\mathcal{G}(\mathbb{Z}_n)) > E(Cay(\mathbb{Z}_n, U_n))$.*

Next theorem gives necessary and sufficient conditions of $Cay(\mathbb{Z}_n, S)$ to be hyperenergetic. Comparing this theorem with Theorem 3.10 in [14] we see that the values of n for which $Cay(\mathbb{Z}_n, U_n)$ is hyperenergetic, $Cay(\mathbb{Z}_n, S)$ is also hyperenergetic. However there are some values of n for which $Cay(\mathbb{Z}_n, S)$ is hyperenergetic but $Cay(\mathbb{Z}_n, U_n)$ is not.

Theorem 3.4. *Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ be the prime factorization of any natural number $n > 1$. Then $Cay(\mathbb{Z}_n, S)$ is hyperenergetic if and only if (i) $k \geqslant 3$ or (ii) $k = 2, n$ is odd or (iii) $k = 2, n$ is of the form $n = 2^\alpha q^\beta$ for some positive integers α, β with $\alpha \geqslant 2$ and q is an odd prime.*

Proof. We consider three cases.

Case I: For $k = 1, n = p^\alpha$, $Cay(\mathbb{Z}_n, S)$ is the complete graph K_n on n vertices and so is not hyperenergetic.

Case II: Here $k = 2, n = p^\alpha q^\beta (p < q)$. We consider three subcases:

Subcase 1: $p = 2, \alpha = 1, n = 2q^\beta, 2 < q$. Then using Eq. (5) we get

$$E(Cay(\mathbb{Z}_n, S)) = 4(q^\beta - 1) = 2(2.q^\beta - 1) - 2 = 2(n-1) - 2.$$

Thus $E(Cay(\mathbb{Z}_n, S)) < 2(n-1)$ and so $Cay(\mathbb{Z}_n, S)$ is not hyperenergetic.

Subcase 2: $p = 2, \alpha \geqslant 2, n = 2^\alpha q^\beta, 2 < q$. Then using Eq. (5) we get

$$\begin{aligned} E(Cay(\mathbb{Z}_n, S)) &= 4(2^\alpha - 1)(q^\beta - 1) \\ &= 2(n-1) + 2.2^\alpha q^\beta - 4.2^\alpha - 4.q^\beta + 6 \\ &= 2(n-1) + A(say), \end{aligned} \quad (9)$$

where

$$\begin{aligned} A &= 2.2^\alpha q^\beta - 4.2^\alpha - 4.q^\beta + 6 \\ &= 2^\alpha(q^\beta - 4) + q^\beta(2^\alpha - 4) + 6. \end{aligned} \quad (10)$$

Since $\alpha \geqslant 2, (2^\alpha - 4) \geqslant 0$. First we consider $q^\beta \neq 3$. Then as q is a prime with $q > 2, (q^\beta - 4) > 0$ and so from Eq. (10) we get that $A > 0$. Next we consider $q^\beta = 3$. Then from Eq. (10) we get that $A = 2^{\alpha+1} - 6 > 0$ as $\alpha \geqslant 2$. Therefore $A > 0$ for all positive integers α, β with $\alpha \geqslant 2$ and for all prime q with $q > 2$. Then from Eq. (9) we get $E(Cay(\mathbb{Z}_n, S)) > 2(n-1)$ and so $Cay(\mathbb{Z}_n, S)$ is hyperenergetic.

Subcase 3: $p, q \geqslant 3, n = p^\alpha q^\beta, p < q$. For this case authors in [14] have shown that $E(Cay(\mathbb{Z}_n, U_n)) > 2(n-1)$. Then by Theorem 3.3, we get that $Cay(\mathbb{Z}_n, S)$ is hyperenergetic.

Case III: Here $k \geqslant 3$. This case is similar to subcase 3 of Case II. $\square$

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