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Dot product graphs and domination number

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Abstract

Let A be a commutative ring with $1 \neq 0$ and $R = A \times A$. The unit dot product graph of R is defined to be the undirected graph $UD(R)$ with the multiplicative group of units in R , denoted by $U(R)$, as its vertex set. Two distinct vertices x and y are adjacent if and only if $x \cdot y = 0 \in A$, where $x \cdot y$ denotes the normal dot product of x and y . In 2016, Abdulla studied this graph when $A = \mathbb{Z}_n$, $n \in \mathbb{N}$, $n \geq 2$. Inspired by this idea, we study this graph when A has a finite multiplicative group of units. We define the congruence unit dot product graph of R to be the undirected graph $CUD(R)$ with the congruent classes of the relation \sim defined on R as its vertices. Also, we study the domination number of the total dot product graph of the ring $R = \mathbb{Z}_n \times \dots \times \mathbb{Z}_n$, k times and $k < \infty$, where all elements of the ring are vertices and adjacency of two distinct vertices is the same as in $UD(R)$. We find an upper bound of the domination number of this graph improving that found by Abdulla.

Keywords: Unit dot product graph, Congruence dot product graph, Congruence total dot product graph, Domination number

AMS Subject Classification: Primary 05C25; secondary 13A99, 05C69

Introduction and preliminaries

The idea of a zero-divisor graph of a commutative ring R was introduced by Beck in [1] (1988). He considered all the elements of R to be vertices and two distinct vertices x and y are adjacent if and only if $xy = 0$, where xy denotes the multiplication in R . He was mainly interested in colorings. Beck's work was continued by Anderson and Naseer in [2] (1993), where they gave a counterexample of Beck's conjecture. In 1999, Livingston and Anderson in [3] gave a modified definition of the zero-divisor graph, denoted by $\Gamma(R)$, by taking the nonzero zero-divisors of the ring as vertices and adjacency of two distinct vertices remains unchanged, i.e., two distinct nonzero zero-divisors x and y are adjacent if and only if $xy = 0$. This definition became the standard definition of the zero-divisor graph. In the same year, they continued their work on the zero-divisor graphs with Frazier and Lauve in [4]. They studied the cliques which are complete subgraphs of $\Gamma(R)$ and the relationship between graph isomorphisms and ring isomorphisms.

In 2003, Redmond in [5] introduced the ideal-based zero-divisor graph $\Gamma_I(R)$ with vertex set $\{x \in R - I \mid xy \in I \text{ for some } y \in R - I\}$, where I is an ideal of R and two

distinct vertices x and y are adjacent if and only if $xy \in I$. This graph is considered to be a generalization of zero-divisor graphs of rings. In 2002, Mulay in [6] provided the idea of the zero-divisor graph determined by equivalence classes. Later on, Spiroff and Wickham in [7] denoted this graph by $\Gamma_E(R)$ and compared it with $\Gamma(R)$. This graph was called the compressed graph by Anderson and LaGrange in [8] (2012). In the compressed graph, the relation on R is given by $r \sim s$ if and only if $\text{ann}(r) = \text{ann}(s)$, where $\text{ann}(r) = \{v \in R \mid rv = 0\}$ is the annihilator of r . This relation is an equivalence relation on R . The vertex set of the compressed graph is the set of all equivalence classes induced by \sim except the classes $[0]_{\sim}$ and $[1]_{\sim}$. The equivalence class of r is $[r]_{\sim} = \{a \in R \mid r \sim a\}$ and two distinct vertices $[r]_{\sim}$ and $[s]_{\sim}$ are adjacent if and only if $rs = 0$. There have been other ways to associate a graph to a ring R . For surveys on the topic of zero-divisor graphs, see [9, 10].

In 2015, Badawi introduced in [11] the dot product graph associated with a commutative ring R . In 2016, his student Abdulla in his master thesis [12] introduced the unit dot product graph and the equivalence dot product graph on a commutative ring with $1 \neq 0$. We are interested here primarily in these graphs.

In 2016, Anderson and Lewis introduced the congruence-based zero-divisor graph in [13], which is a generalization of the zero-divisor graphs mentioned above. The vertices of this graph are the congruence classes of the nonzero zero-divisors of R induced by a congruence relation defined on the ring R . Two distinct vertices are adjacent if and only if their product is zero. The concept of congruence relation is used in this paper.

In 2017, Chebolu and Lockridge in [14] found all cardinal numbers occurring as the cardinality of the group of all units in a commutative ring with $1 \neq 0$. This is very helpful to us as we want to graph the units of a ring R .

In the second section, we generalize a result of [12] concerning the unit dot product graph of a commutative ring R , where $R = \mathbb{Z}_n \times \mathbb{Z}_n$, replacing \mathbb{Z}_n by a commutative ring A such that $U(A)$ is finite. In the third section, a congruence relation on the unit dot product graph is defined and some of its properties are characterized. In the last section, we discuss the domination number of some graphs.

We recall some definitions which are used in this paper. Let G be an undirected graph. Two vertices v_1 and v_2 are said to be adjacent if v_1, v_2 are connected by an edge of G . A finite sequence of edges from a vertex v_1 of G to a vertex v_2 of G is called a path of G . We say that G is connected if there is a path between any two distinct vertices and it is totally disconnected if no two vertices in G are adjacent. For two vertices x and y in G , the distance between x and y , denoted by $d(x, y)$, is defined to be the length of a shortest path from x to y , where $d(x, x) = 0$ and $d(x, y) = \infty$ if there is no such path. The diameter of G is $\text{diam}(G) = \sup\{d(x, y) \mid x \text{ and } y \text{ are vertices in } G\}$. A cycle of length n , $n \geq 3$, in G is a path of the form $x_1 - x_2 - \dots - x_n - x_1$, where $x_i \neq x_j$ when $i \neq j$. The girth of G , denoted by $\text{gr}(G)$, is the length of the shortest cycle in G and $\text{gr}(G) = \infty$ if G contains no cycle. A graph G is said to be complete if any two distinct vertices are adjacent and the complete graph with n vertices is denoted by K_n . A complete bipartite graph is a graph which may be partitioned into two disjoint nonempty vertex sets A and B such that two distinct vertices are adjacent if and only if they are in distinct vertex sets. This graph is denoted by $K_{m,n}$, where $|A| = m$ and $|B| = n$.

Throughout the paper, R and A denote commutative rings with $1 \neq 0$. Its set of zero-divisors is denoted by $Z(R)$ and $Z(R)^* = Z(R) - \{0\}$. As usual, \mathbb{Z} , \mathbb{Z}_n and $GF(p^n)$

denote the integers, integers modulo n , and finite field with p^n elements, respectively, where p is a prime number and n is a positive integer. $\phi(n)$ is the Euler phi function of a given positive integer n , which counts the positive integers up to n that are relatively prime to n .

Unit dot product graph of a commutative ring

The unit dot product graph of R was introduced in [12], denoted by $UD(R)$. This graph is a subgraph of the total dot product graph, denoted by $TD(R)$, where its vertex set is all the elements of R . Some of its properties were characterized when $R = A \times A$ and $A = \mathbb{Z}_n$. In this section, we generalize the $UD(R)$, as A will be a commutative ring with $1 \neq 0$, whose multiplicative group of units is finite.

In the proof of Theorems 2 and 3, we use the order of the multiplicative group of units $U(R)$ of R . In this context, the following theorem is helpful.

Theorem 1 (Th. 8, [14]) *Let λ be a cardinal number. There exists a commutative ring R with $|U(R)| = \lambda$ if and only if λ is equal to*

1. An odd number of the form $\prod_{i=1}^t (2^{n_i} - 1)$ for some positive integers n_1, \dots, n_t
2. An even number
3. An infinite cardinal number

We are interested only in commutative rings $R = A \times A$, where A is a commutative ring with $1 \neq 0$ and $U(A)$ has a finite order. For instance, from [14], rings in the form $R_{2m} = \frac{\mathbb{Z}[x]}{(x^2, mx)}$ are examples of such a ring. Here, $U(A)$ has an even order equal to $2m$, where $m \in \mathbb{N}$. The units in these rings are in the form $1 + bx$ and $-1 + bx$, $0 \leq b \leq m - 1$. If the order of $U(A)$ is odd, then this odd number will be in the form $\prod_{i=1}^t (2^{n_i} - 1)$ for some positive integers n_1, \dots, n_t and the characteristic of the ring must be equal to 2.

The following two Theorems 2 and 3 characterize the graph of the rings $R = R_{2m} \times R_{2m}$ and $R = A \times A$, respectively.

Theorem 2 *Let $R = R_{2m} \times R_{2m}$. Then, $UD(R)$ is the union of m disjoint $K_{2m,2m}$'s.*

Proof Since $|U(R_{2m})| = 2m$, then $UD(R)$ has exactly $4m^2$ vertices. Let $v_1 = u(1, a)$ and $v_2 = v(1, b)$ in R , for some $u, v, a, b \in U(R_{2m})$. From [14], the units are in the form $1 + ax$ and $-1 + ax$, where $0 \leq a \leq m - 1$, so we have $v_1 = u(1, 1 + ax)$ and $v_2 = v(1, -1 + bx)$ in R for some $u, v \in U(R_{2m})$, $0 \leq a, b \leq m - 1$. Hence, v_1 is adjacent to v_2 if and only if $v_1 \cdot v_2 = uv(b - a)x = 0$ in R_{2m} . This is equivalent to $b = a$, since uv is a unit in R_{2m} . Thus, for each $0 \leq a \leq m - 1$, let $V_a = \{u(1, 1 + ax) \mid u \in U(R_{2m})\}$ and $W_a = \{u(1, -1 + ax) \mid u \in U(R_{2m})\}$. For different units u and u' in $U(R_{2m})$, we cannot have $u(1, 1 + ax) = u'(1, 1 + ax)$ or $u(1, -1 + ax) = u'(1, -1 + ax)$, so $|V_a| = |W_a| = 2m$. If $u(1, 1 + ax) = u'(1, -1 + ax)$, then $u = u'$ and $u(1 + ax) = u'(-1 + ax)$. So $u(1 + ax) = u(-1 + ax)$ which implies that $u = -u$, a contradiction. Thus, $V_a \cap W_a = \emptyset$. It is clear that every two distinct vertices in V_a or in W_a are not adjacent. By construction of V_a and W_a , every vertex in V_a is adjacent to every vertex in W_a . Thus, the vertices in $V_a \cup W_a$ form the graph $K_{2m,2m}$ that is a complete bipartite subgraph of $TD(R)$. By construction, $UD(R)$ is the union of m disjoint $K_{2m,2m}$'s. □

Example 1 When $m = 1$, we have $R_2 = \frac{\mathbb{Z}[x]}{(x^2, x)}$ which is isomorphic to \mathbb{Z} . The graph of $UD(R_2 \times R_2)$ will be a complete bipartite graph of 4 vertices which are $(1,1)$, $(1,-1)$, $(-1,1)$, and $(-1,-1)$. Thus, its diameter = 2 and girth = 4 (Fig. 1).

The following theorem deals with the case $R = A \times A$, where $|U(A)|$ is odd. In this case, the unit -1 in A (from Cauchy Theorem) must have order 1. Then, $\text{Char}(A) = 2$.

Theorem 3 Let $R = A \times A$. If the order of the multiplicative group $U(A)$ is odd, then $UD(R)$ is the union of $\frac{m-1}{2}$ disjoint $K_{m,m}$'s and one K_m .

Proof From [14], the order of $U(A)$ is an odd number if and only if this odd number is of the form $\prod_{i=1}^t (2^{n_i} - 1)$ for some positive integers n_1, \dots, n_t . Let m be the odd order of $U(A)$, so $UD(R)$ has exactly m^2 vertices. Let $v_1 = u(1, a)$ and $v_2 = v(1, b)$ in R , for some $u, v, a, b \in U(A)$. v_1 is adjacent to v_2 if and only if $v_1 \cdot v_2 = uv + uvab = 0$. This will occur if and only if $1 + ab = 0$. This is equivalent to $a = b^{-1}$, since uv is a unit in A and $\text{Char}(R) = 2$. Thus, for each, $a \neq 1 \in U(A)$, let $V_a = \{u(1, a) \mid u \in U(A)\}$ and $W_a = \{u(1, a^{-1}) \mid u \in U(A)\}$. For different units u and u' in $U(A)$, we cannot have $u(1, a) = u'(1, a^{-1})$, so $|V_a| = |W_a| = m$. If $u(1, a) = u'(1, a^{-1})$, then $u = u'$ and $ua = u'a^{-1}$. So, $u(a - a^{-1}) = 0$, i.e. $a = a^{-1}$, which implies that $a^2 = 1$ a contradiction since $U(A)$ has an odd order. Thus, $V_a \cap W_a = \emptyset$. It is clear that every two distinct vertices in V_a or in W_a are not adjacent. By construction of V_a and W_a , every vertex in V_a is adjacent to every vertex in W_a . Thus, the vertices in $V_a \cup W_a$ form the graph $K_{m,m}$ that is a complete bipartite subgraph of $UD(R)$. By construction, there are exactly $\frac{m-1}{2}$ disjoint complete bipartite $K_{m,m}$ subgraphs of $UD(R)$. For $a = 1$, we have m vertices in the form of $u(a, a)$. Since $\text{Char}(R) = 2$, these m vertices form the graph K_m , that is a complete subgraph of $UD(R)$. Hence, $UD(R)$ is the union of $\frac{m-1}{2}$ disjoint $K_{m,m}$'s and one K_m . \square

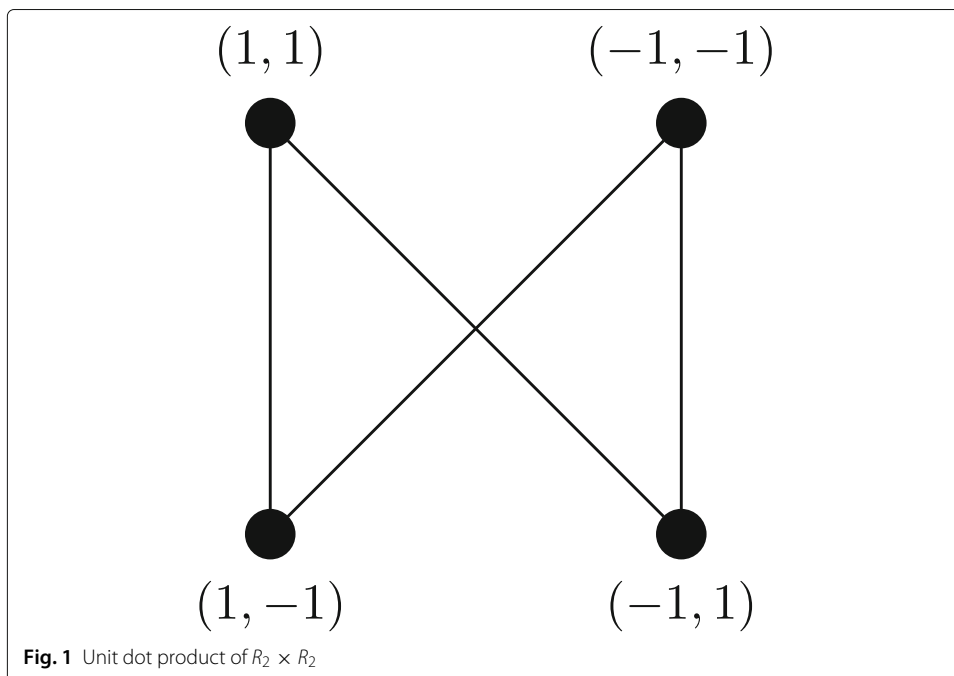


Fig. 1 Unit dot product of $R_2 \times R_2$

Congruence dot product graph of a commutative ring

In 2016, Anderson and Lewis in [13] introduced the congruence-based zero-divisor graph $\Gamma_{\sim}(R) = \Gamma(R/\sim)$, where \sim is a multiplicative congruence relation on R and showed that R/\sim is a commutative semigroup with zero. They showed that the zero-divisor graph of R , the compressed zero-divisor graph of R , and the ideal based zero-divisor graph of R are examples of the congruence-based zero-divisor graphs of R . In this paper, we are interested in the multiplicative congruence relation \sim on R , which is an equivalence relation on the multiplicative monoid R with the additional property that if $x, y, z, w \in R$ with $x \sim y$ and $z \sim w$, then $xz \sim yw$.

The equivalence unit dot product graph of $U(R)$ was introduced in [12], where $R = A \times A$ and $A = \mathbb{Z}_m$. The equivalence relation \sim on $U(R)$ was defined such that $x \sim y$, where $x, y \in U(R)$, if $x = (c, c)y$ for some $(c, c) \in U(R)$. Let $EU(R)$ be the set of all distinct equivalence classes of $U(R)$. If $X \in EU(R)$, then $\exists a \in U(A)$ such that $X = [(1, a)]_{\sim} = \{u(1, a) \mid u \in U(A)\}$. Thus, the equivalence unit dot product graph of $U(R)$ is the (undirected) graph $EUD(R)$ with vertices $EU(R)$. Two distinct vertices X and Y are adjacent if and only if $x \cdot y = 0 \in A$, where $x \cdot y$ denotes the normal dot product of x and y .

From the definition of the congruence relation, we find that the relation defined by Abdulla is not only an equivalence relation but also a congruence relation. In fact, let $x \sim y$ and $w \sim v$. So, $x = (c_1, c_1)y$ and $w = (c_2, c_2)v$ for some $(c_1, c_1), (c_2, c_2) \in U(R)$. Then, $xw = (c_1, c_1)y(c_2, c_2)v = (c_1, c_1)(c_2, c_2)yv = (c, c)yv$ and hence $xw \sim yv$. We denote this congruence unit dot product graph by $CUD(R)$, and its set of vertices is the set of all distinct congruence classes of $U(R)$, denoted by $CU(R)$.

In this section, we characterize the generalized case of the congruence unit dot product graph $CUD(R)$, as we will apply the congruence relation on the unit dot product graph we introduced in the first section.

Theorem 4 *Let $R = R_{2m} \times R_{2m}$. Then, $CUD(R)$ is the union of m disjoint $K_{1,1}$'s.*

Proof For each $a \in U(R_{2m})$, let V_a and W_a be as in the proof of Theorem 2. Then, $V_a, W_a \in CU(R)$. Indeed, for each $a \in U(R_{2m})$, there exist V_a and $W_a \in CU(R)$ each has cardinality $2m$. We conclude that each $K_{2m,2m}$ of $UD(R)$ is a $K_{1,1}$ of $CUD(R)$. From Theorem 2 the result follows. □

Example 2 *In Example 1, we graphed the unit dot product graph of $R_2 \times R_2$, and now, we graph the congruence dot product graph of the same ring. This graph will be a complete graph of 2 vertices as R_2 is isomorphic to \mathbb{Z} . So, we will have only two congruence classes $[(1, 1)]_{\sim} = \{(1, 1), (-1, -1)\}$ and $[(1, -1)]_{\sim} = \{(1, -1), (-1, 1)\}$ (Fig. 2).*

Theorem 5 *Let $R = A \times A$. If the order of $U(A)$ is odd, then $CUD(R)$ is the union of $\frac{m-1}{2}$ disjoint $K_{1,1}$'s and one K_1 .*

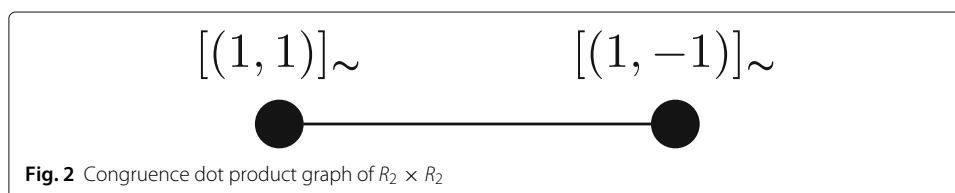


Fig. 2 Congruence dot product graph of $R_2 \times R_2$

Proof For each $a \in U(A)$, let V_a and W_a be as in the proof of Theorem 3. Then, $V_a, W_a \in CU(R)$. Indeed, for each $a \in U(R)$ and $a \neq 1$, there exist V_a and $W_a \in CU(R)$ each of cardinality m . For $a = 1$, we have one congruence class V , where $V = \{u(a, a) \mid u \in U(A)\}$. We conclude that each $K_{m,m}$ of $UD(R)$ is a $K_{1,1}$ of $CUD(R)$, and each K_m of $UD(R)$ is a K_1 of $CUD(R)$. From Theorem 3, the result follows. \square

Let $R = \mathbb{Z}_n \times \mathbb{Z}_n$. We make a little change on the congruence relation defined above by taking the vertices from the whole ring R not only from $U(R)$. Define a relation on R such that $x \sim y$, where $x, y \in R$, if $x = (c, c)y$ for some $(c, c) \in U(R)$. It is clear that \sim is an equivalence relation on R and also a congruence relation.

The congruence total dot product graph of R is defined to be the undirected graph $CTD(R)$, and its vertices are the congruent classes of all the elements of R induced by the defined congruence relation \sim . Two distinct classes $[X]_{\sim}$ and $[Y]_{\sim}$ are adjacent if and only if $x \cdot y = 0 \in \mathbb{Z}_n$, where $x \cdot y$ denotes the normal dot product of x and y . Also, the congruence zero-divisor dot product graph, denoted by $CZD(R)$, is defined to be an undirected graph whose vertices are the congruent classes of the nonzero zero-divisor elements in R and adjacency between distinct vertices remains as defined before.

Obviously, this congruence relation is well-defined. Indeed, let $x, x', y, y' \in R$ be such that $y = (y_1, y_2)$ and $y' = (y'_1, y'_2)$ and let $u, u' \in U(R)$ be such that $u = (c_1, c_1)$ and $u' = (c'_1, c'_1)$, where $y_1, y'_1, y_2, y'_2, c_1, c'_1 \in \mathbb{Z}_n$. Assume that $x \sim y$ and $x' \sim y'$. Then, $x \cdot x' = 0$ if and only if $(c_1 y_1)(c'_1 y'_1) + (c_1 y_2)(c'_1 y'_2) = 0$. This happens if and only if $y_1 y'_1 + y_2 y'_2 = 0$, since $c_1 c'_1$ is a unit in \mathbb{Z}_n .

Theorem 6 *Let $A = \mathbb{Z}_p$, where p is a prime number and $R = A \times A$. Then, $CTD(R)$ is disconnected and $CZD(R) = \Gamma_{\sim}(R)$ is a complete graph of 2 vertices.*

Proof If $CTD(R)$ was connected, then $\exists x, y \in R$ such that x is adjacent to y . $x \cdot y = 0$ if and only if $xy = 0$, leads to a contradiction with (Theorem 2.1, [11]). So, $CZD(R) = \Gamma_{\sim}(R)$ is connected. Since A is a field, then all the nonzero zero-divisors in R will be in two classes only, which are $[(a, 0)]_{\sim}$ and $[(0, b)]_{\sim}, \forall a, b \in U(A)$ and since $(a, 0) \cdot (0, b) = 0$, so it is a complete graph of two vertices. \square

If $A = \mathbb{Z}_p$ and $R = \mathbb{Z}_p \times \dots \times \mathbb{Z}_p, k$ times and $k < \infty$, then the diameter and girth of $CZD(R)$ and $CTD(R)$ are the same as the case of $TD(R)$ and $ZD(R)$, which was discussed before in [11]. This reduces the number of vertices but adjacency is the same in both cases.

Example 3 *If $A = \mathbb{Z}$, then $R = \mathbb{Z} \times \mathbb{Z}$. Here, the only units in the form (c, c) are $(1, 1)$ and $(-1, -1)$ so the classes of the zero-divisors will be in the form $[(a, 0)]_{\sim} = \{(a, 0), (-a, 0)\}$ and $[(0, a)]_{\sim} = \{(0, a), (0, -a)\}, \forall a \in U(A)$. For two distinct vertices $(a, 0) \cdot (b, 0) \neq 0$, because $ab \neq 0$. Then, there will be an edge only between classes in the form $[(a, 0)]_{\sim}$ and $[(0, b)]_{\sim}$, which means $diam(CZD(R)) = 2$ and $gr(CZD(R)) = 4$.*

Theorem 7 *Let $R = \mathbb{Z}_n \times \mathbb{Z}_n$ for $n \in \mathbb{N}$ and n is not a prime number. Then, $CTD(R)$ is a connected graph with $diam(CTD(R)) = 3$ and $gr(CTD(R)) = 3$.*

Proof The proof is similar to that of Theorem 2.3 [11], taking into consideration that the vertices we used are in distinct classes. \square

Domination number

Let G be a graph with V as its set of vertices. We recall that a subset $S \subseteq V$ is called a dominating set of G if every vertex in V is either in S or is adjacent to a vertex in S . The domination number $\gamma(G)$ of G is the minimum cardinality among the dominating sets of G . The study of the domination number started around 1960s; however, there are some domination-related problems before that date. Namely, about 100 years earlier, in 1862, De Jaenisch [15] posed the problem of finding the minimum number of queens required to cover (attack) each square of an $n \times n$ chess board. In 1892, there were three basic types of problems that chess players studied during this time reported by Rouse Ball in [16]. For more details on this topic, see [17, 18].

In this section, we find a new upper bound of the domination number of the total dot product graph of $\mathbb{Z}_n \times \dots \times \mathbb{Z}_n$, k times and $k < \infty$. It is an improvement of the upper bound of the same graph given in [12].

Theorem 8 *Let $n \geq 4$ be an integer that is not prime, $A = \mathbb{Z}_n$ and $R = A \times A$. Then, write $n = p_1^{k_1} \dots p_m^{k_m}$, where p_i 's, $1 \leq i \leq m$, are distinct prime numbers. Let $M = \{\frac{n}{p_i} \mid 1 \leq i \leq m\}$. Then*

1. *If n is even, then $D = \{(0, b) \mid b \in M\} \cup \{(d, 0) \mid d \in M\} \cup \{(\frac{n}{2}, \frac{n}{2})\}$ is a minimal dominating set of $TD(R)$, and thus, $\gamma(TD(R)) \leq 2m + 1$.*
2. *If n is odd, then $D = \{(0, b) \mid b \in M\} \cup \{(d, 0) \mid d \in M\} \cup \{(1, c) \mid c \in U(A)\}$ is a minimal dominating set of $TD(R)$, and thus, $\gamma(TD(R)) \leq 2m + \phi(n)$.*

Proof 1. Let n be even and $x = (x_1, x_2)$ a vertex in $TD(R)$. We consider two cases:

- (a) Assume that x is a unit. Since $(x_1, x_2) \cdot (\frac{n}{2}, \frac{n}{2}) = \frac{n}{2}(x_1 + x_2) = nc' = 0$ (because $x_1 + x_2$ is an even number), then x is adjacent to a vertex in D ,
- (b) Assume that x_2 is a zero-divisor of A , i.e., $p_i \mid x_2$ in A for some p_i , $1 \leq i \leq m$. Then, $v = (0, \frac{n}{p_i}) \in D$ is adjacent to x in $TD(R)$ (the same case is true if x_1 is a zero-divisor of A).

This shows that D is a dominating set of $TD(R)$. We show that it is minimal. We have to find when $(a, p_i) \cdot (\frac{n}{2}, \frac{n}{2}) = 0$, $a \in A$. It is clear that if both a and p_i are even or odd together, then $(a, p_i) \cdot (\frac{n}{2}, \frac{n}{2}) = 0$. But if a is even and p_i is odd or the opposite, we have $(a, p_i) \cdot (\frac{n}{2}, \frac{n}{2}) \neq 0$. Now, when we remove the vertex $(0, \frac{n}{p_i})$, the vertex (a, p_i) is not adjacent to any other vertex in D , where a and p_i are different. The same argument holds if we remove the vertex $(\frac{n}{p_i}, 0)$. Moreover, when we remove the vertex $(\frac{n}{2}, \frac{n}{2})$, the vertex $u = (u_1, u_2)$ which is a unit will not be adjacent to any other vertex in D . Thus, D is a minimal dominating set and therefore $\gamma(TD(R)) \leq 2m + 1$.

2. Let n be odd, $\phi(n) = r$ and $x = (x_1, x_2)$ a vertex in $TD(R)$. We consider two cases:
 - (a) Assume that x is a unit. From Theorem 3.3 parts 2 & 3 [12], $UD(R)$ is either a union of $\frac{r}{2}$ disjoint $K_{r,r}$'s or a union of $\frac{r}{2} - 2^{m-1}$ disjoint $K_{r,r}$ and 2^m disjoint K_r 's. In both graphs, every unit $(1, c)$ is adjacent to r units in the form $u(1, c)$ for all $u \in U(A)$,

- (b) Assume that x_2 is a zero-divisor of A , i.e., $p_i|x_2$ in A for some $p_i, 1 \leq i \leq m$. Then, $v = (0, \frac{n}{p_i}) \in D$ is adjacent to x in $TD(R)$ (the same case takes place if x_1 is a zero-divisor of A).

This shows that D is a dominating set of $TD(R)$. In order to show that it is minimal, let us first remove the vertex $v = (0, \frac{n}{p_i})$ from D for some $i, 1 \leq i \leq m$. We have $(a, p_i) \cdot (1, c) = a + cp_i = 0$ if and only if $a = -cp_i$. So, when we remove $(0, \frac{n}{p_i})$, we will find a vertex (a, p_i) for some $a \in A$ which is not adjacent to any other vertices in D (as an example, take $a = 1$). Thus, v cannot be removed from D . The same argument is true if we remove $(\frac{n}{p_i}, 0)$. If we remove the unit $(1, c)$, we will have distinct r units that are not adjacent to any other vertex in D . Thus, D is a minimal dominating set, and then, $\gamma(TD(R)) \leq 2m + \phi(n)$. \square

Example 4 Let $R = \mathbb{Z}_4 \times \mathbb{Z}_4$. As a result of part 1 in the previous theorem, $\gamma(TD(R)) \leq 2 \times 1 + 1 = 3$. The following figure shows that the dominating set of R is $\{(0, 2), (2, 0), (2, 2)\}$ (Fig. 3).

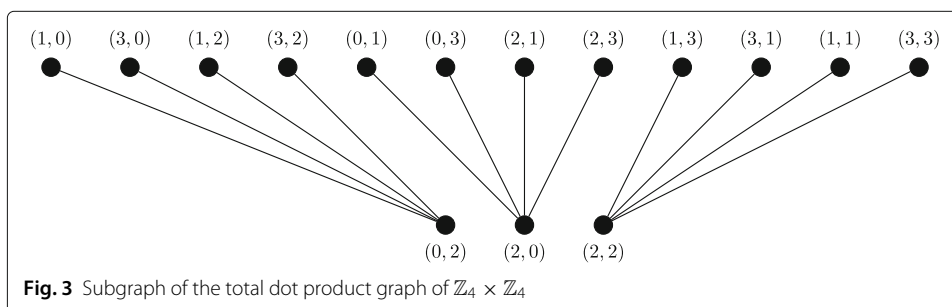
We note that the upper bound of the domination number of the congruence total dot product graph of $\mathbb{Z}_n \times \mathbb{Z}_n$ is the same as the previous result of the total dot product graph, taking into consideration that the vertices we used are in distinct classes.

Example 5 Let $R = \mathbb{Z}_4 \times \mathbb{Z}_4$ as in example 4. The vertices of $CTD(R)$ are the congruence classes $[(0, 1)]_{\sim} = \{(0, 1), (0, 3)\}$, $[(0, 2)]_{\sim} = \{(0, 2)\}$, $[(1, 0)]_{\sim} = \{(1, 0), (3, 0)\}$, $[(1, 1)]_{\sim} = \{(1, 1), (3, 3)\}$, $[(1, 2)]_{\sim} = \{(1, 2), (3, 2)\}$, $[(1, 3)]_{\sim} = \{(1, 3), (3, 1)\}$, $[(2, 0)]_{\sim} = \{(2, 0)\}$, $[(2, 1)]_{\sim} = \{(2, 1), (2, 3)\}$, $[(2, 2)]_{\sim} = \{(2, 2)\}$. The following graph shows that $\gamma(CTD(R)) = 3$ and its dominating set is $\{[(0, 2)]_{\sim}, [(2, 0)]_{\sim}, [(2, 2)]_{\sim}\}$ (Fig. 4).

The following corollary is a generalization of Theorem 8 when $R = \mathbb{Z}_n \times \dots \times \mathbb{Z}_n$, k times, $k < \infty$ and n is even.

Corollary 1 Let $n \geq 4$ be an even integer, $A = \mathbb{Z}_n$ and $R = A \times \dots \times A$, k times and $k < \infty$. Then, write $n = p_1^{k_1} \dots p_m^{k_m}$, where p_i 's, $1 \leq i \leq m$, are distinct prime numbers. Let $M = \{\frac{n}{p_i} \mid 1 \leq i \leq m\}$. Then, $D = \{(0, \dots, 0, b) \mid b \in M\} \cup \{(d, 0, \dots, 0) \mid d \in M\} \cup \{(\frac{n}{2}, 0, \dots, 0, \frac{n}{2})\}$ is a minimal dominating set of $TD(R)$, and thus, $\gamma(TD(R)) \leq 2m + 1$.

Proof Let $x = (x_1, \dots, x_k)$ be a vertex in $TD(R)$. We consider two cases:



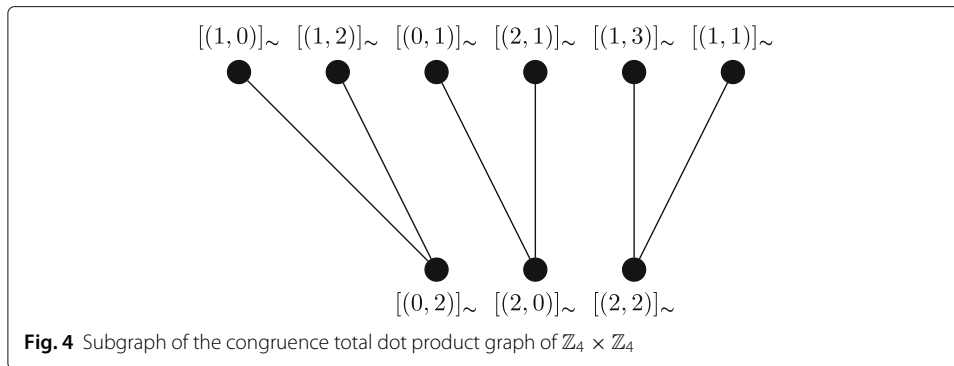


Fig. 4 Subgraph of the congruence total dot product graph of $\mathbb{Z}_4 \times \mathbb{Z}_4$

1. Assume that x is a unit, i.e., each coordinate is an odd number. Then, $(x_1, \dots, x_k) \cdot (\frac{n}{2}, 0, \dots, 0, \frac{n}{2}) = 0$,
2. Assume that x_i is a zero-divisor of A , $1 \leq i \leq n$. If $i = 1$ or n , then x is adjacent to $(0, \dots, 0, \frac{n}{p_i})$ or $(\frac{n}{p_i}, 0, \dots, 0)$ and both of them are in D . But if $i \neq 1$ or $i \neq n$ such that x_1 and x_n are units, then it is adjacent to $(\frac{n}{2}, 0, \dots, 0, \frac{n}{2}) \in D$.

This shows that D is a dominating set of $TD(R)$. We show that it is minimal. Since $(a, 0, 0, p_i) \cdot (\frac{n}{2}, 0, \dots, 0, \frac{n}{2}) = \frac{n}{2}(a + p_i)$ for some $a \in A$, which is equal to zero if and only if a and p_i are odd or even together. By removing $(0, \dots, 0, \frac{n}{p_i})$ from D , we will find a vertex $(a, 0, 0, p_i)$ in $TD(R)$ that is not adjacent to any other vertex in D . The same argument works if we remove $(\frac{n}{p_i}, 0, \dots, 0)$ from D . Also, by removing $(\frac{n}{2}, 0, \dots, 0, \frac{n}{2})$, we will find a vertex in $TD(R)$ where the first and the last coordinates are units that is not adjacent to any other vertex in D . Thus, D is a minimal dominating set, and then, $\gamma(TD(R)) \leq 2m + 1$. \square

Again here, we note that the upper bound of the domination number of the congruence total dot product graph of $\mathbb{Z}_n \times \dots \times \mathbb{Z}_n$ is the same as the previous result of the total dot product graph, taking into account that the vertices we used are in distinct classes.

Theorem 9 For the unit dot product graph of R :

1. If $R = R_{2m} \times R_{2m}$, then $D = \{(1, a) \mid a \in U(R_{2m})\}$ and $\gamma(UD(R)) = 2m$.
2. If $R = A \times A$, where A is a commutative ring with $1 \neq 0$ and $|U(A)|$ is an odd number, say m , then $D = \{(1, c) \mid c \in U(A)\}$ and $\gamma(UD(R)) = m$.

Proof 1. Let $x = (c_1, c_2)$ be a vertex in $UD(R)$ and assume that x is not in D . Let $c = c_1 c_2^{-1} \in U(R_{2m})$. Then, (c_1, c_2) is adjacent to $(1, c)$ in D . Assume that $(1, c)$ is removed from D for some c in $U(R_{2m})$. Then, $(-c, 1)$ is not adjacent to any other vertex in D . Hence, D is a minimal dominating set, and thus, $\gamma(UD(R)) = 2m$.

2. Holds by the same idea of the proof of part 1. \square

The following theorem is a direct consequence of Theorem 6 in this paper. In that theorem, we find that $CTD(R)$ is a complete graph of 2 vertices, and as a result its dominating set contains only one vertex of these 2 vertices.

Theorem 10 Let p be a prime number, $n \geq 1$, $m = p^n - 1$, $A = GF(p^n)$ and $R = A \times A$. Then $\gamma(CZD(R)) = 1$.

Conclusion

In our future work, we are looking forward to working on one of the following open questions:

1. In the first section, we studied the case when $R = R_{2m} \times R_{2m}$. We are interested in defining the unit dot product graph in the general case when $R = R_{2m} \times \dots \times R_{2m}$, n times and $n < \infty$. For n odd, the simplest case is $R = R_{2m} \times \dots \times R_{2m}$, (i.e., $m = 1$). By straight forward calculations, the unit dot product graph will be isolated vertices. But the case when $m \geq 2$ and n is odd is still an open question. For even n , the case is more complicated.
2. Define the unit dot product graph on a commutative ring $R = A \times A$ relaxing the conditions on the ring A and characterizing the case when $|U(A)|$ is infinite.
3. Determine the domination number for all of the previous cases using the results of this paper.

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