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ORIGINAL ARTICLE

Fekete-Szegö inequalities for p-valent starlike and convex functions of complex order

M.K. Aouf [a], R.M. EL-Ashwah [b], H.M. Zayed [c,*]

[a] *Department of Mathematics, Faculty of Science, Mansoura University, Mansoura 35516, Egypt*
[b] *Department of Mathematics, Faculty of Science, Damietta University, New Damietta 34517, Egypt*
[c] *Department of Mathematics, Faculty of Science, Menofia University, Shebin Elkom 32511, Egypt*

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Abstract In this paper, we obtain Fekete-Szegö inequalities for certain class of analytic p-valent functions $f(z)$ for which $1+\frac{1}{b}\left[\frac{1}{p}\frac{z(f*g)'(z)+\lambda z^2(f*g)''(z)}{(1-\lambda)(f*g)(z)+\lambda z(f*g)'(z)}-1\right] \prec \varphi(z) (b \in \mathbb{C}^* = \mathbb{C}\setminus\{0\})$. Sharp bounds for the Fekete-Szegö functional $|a_{p+2}-\mu a_{p+1}^2|$ are obtained.

2000 MATHEMATICAL SUBJECT CLASSIFICATION: 30C45; 30A20; 34A40

1. Introduction

Let $\mathcal{A}(p)$ denote the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k \ (p \in \mathbb{N} = \{1,2,\ldots\}), \tag{1.1}$$

which are analytic and p-valent in the open unit disk $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Let $g(z) \in \mathcal{A}(p)$, be given by

$$g(z) = z^p + \sum_{k=p+1}^{\infty} g_k z^k. \tag{1.2}$$

The Hadamard product (or convolution) of $f(z)$ and $g(z)$ is given by

$$(f*g)(z) = z^p + \sum_{k=p+1}^{\infty} a_k g_k z^k = (g*f)(z). \tag{1.3}$$

A function $f(z) \in \mathcal{A}(p)$ is said to be p-valent starlike of order α, denoted by $S_p^*(\alpha)$, if and only if

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \ (0 \leqslant \alpha < p; z \in U). \tag{1.4}$$

A function $f(z) \in \mathcal{A}(p)$ is said to be p-valent convex of order α, denoted by $C_p(\alpha)$, if and only if

$$\mathrm{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \alpha \ (0 \leqslant \alpha < p; z \in U). \tag{1.5}$$

The classes $S_p^*(\alpha)$ and $C_p(\alpha)$ were defined by Owa [1]. From (1.4) and (1.5), it follows that

$$f(z) \in C_p(\alpha) \Longleftrightarrow \frac{zf'(z)}{p} \in S_p^*(\alpha). \tag{1.6}$$

* Corresponding author. Tel.: +20 1066757776.
E-mail addresses: mkaouf127@yahoo.com (M.K. Aouf), r_elashwah@yahoo.com (R.M. EL-Ashwah), hanaazayed42@yahoo.com (H.M. Zayed).
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For two functions f and g, analytic in U, we say that the function $f(z)$ is subordinate to $g(z)$ in U and write $f(z) \prec g(z)$, if there exists a Schwarz function $w(z)$, analytic in U with $w(0) = 0$ and $|w(z)| < 1$ such that $f(z) = g(w(z))(z \in U)$. Furthermore, if the function g is univalent in U, then we have the following equivalence (see [2]):

$$f(z) \prec g(z) \Longleftrightarrow f(0) = g(0) \text{ and } f(U) \subset g(U).$$

Let $\varphi(z)$ be an analytic function with positive real part on U satisfies $\varphi(0) = 1$ and $\varphi'(0) > 0$ which maps U onto a region starlike with respect to 1 and symmetric with respect to the real axis. Let $S^*_{b,p}(\varphi)$ be the class of functions $f(z) \in \mathcal{A}(p)$ for which

$$1 + \frac{1}{b}\left(\frac{1}{p}\frac{zf'(z)}{f(z)} - 1\right) \prec \varphi(z) \ (z \in U), \tag{1.7}$$

and $C_{b,p}(\varphi)$ be the class of functions $f(z) \in \mathcal{A}(p)$ for which

$$1 - \frac{1}{b} + \frac{1}{bp}\left(1 + \frac{zf''(z)}{f'(z)}\right) \prec \varphi(z) \ (z \in U). \tag{1.8}$$

The classes $S^*_{b,p}(\varphi)$ and $C_{b,p}(\varphi)$ were introduced and studied by Ali et al. [3]. We note that $S^*_{1,1}(\varphi) = S^*(\varphi)$ and $C_{1,1}(\varphi) = C(\varphi)$, the classes $S^*(\varphi)$ and $C(\varphi)$ were introduced and studied by Ma and Minda [4]. The classes $S^*(\alpha)$ and $C(\alpha)$ are the special cases of $S^*(\varphi)$ and $C(\varphi)$, respectively, when $\varphi(z) = \frac{1+(1-2\alpha)z}{1-z}$ $(0 \leqslant \alpha < 1)$.

For $b \in \mathbb{C}^* = \mathbb{C} \setminus \{0\}$, $0 \leqslant \lambda \leqslant 1$ and $p \in \mathbb{N}$, we let $S_{\lambda,b,p}(g,\varphi)$ be the subclass of $\mathcal{A}(p)$ consisting of functions $f(z)$ of the form (1.1), the functions $g(z)$ of the form (1.2) with $g_k > 0$ and satisfying the analytic criterion:

$$1 + \frac{1}{b}\left[\frac{1}{p}\frac{z(f*g)'(z) + \lambda z^2(f*g)''(z)}{(1-\lambda)(f*g)(z) + \lambda z(f*g)'(z)} - 1\right] \prec \varphi(z). \tag{1.9}$$

We note that for suitable choices of $g(z)$, λ, b, p and $\varphi(z)$, we obtain the following subclasses:

(1) $S_{0,b,p}(g,\varphi) = S^*_{b,p,g}(\varphi)$ (see Ali et al. [3]);
(2) $S_{0,b,p}\left(\frac{z^p}{1-z}, \varphi(z)\right) = S^*_{b,p}(\varphi)$ and $S_{1,b,p}\left(\frac{z^p}{1-z}, \varphi(z)\right) = C_{b,p}(\varphi)$ (see Ali et al. [3]);
(3) $S_{0,1,p}(g,\varphi) = S^*_{p,g}(\varphi)$ and $S_{0,1,p}\left(\frac{z^p}{1-z}, \varphi(z)\right) = S^*_p(\varphi)$ (see Ali et al. [3]);
(4) $S_{0,b,1}\left(\frac{z}{1-z}, \varphi(z)\right) = S^*_b(\varphi)$ and $S_{1,b,1}\left(\frac{z}{1-z}, \varphi(z)\right) = C_b(\varphi)$ (see Ravichandran et al. [5]);
(5) $S_{0,1,1}\left(\frac{z}{1-z}, \varphi(z)\right) = S^*(\varphi)$ (see Ma and Minda [4] and Shanmugam and Sivasubramanian [6, with $\alpha = 0$]);
(6) $S_{1,1,1}\left(\frac{z}{1-z}, \varphi(z)\right) = C(\varphi)$ (see Ma and Minda [4] and Shanmugam and Sivasubramanian [6, with $\alpha = 1$]);
(7) $S_{0,\left(1-\frac{\gamma}{p}\right)e^{-i\alpha}\cos\alpha,p}\left(\frac{z^p}{1-z}, \frac{1-z}{1+z}\right) = S^\alpha(p,\gamma)(|\alpha| < \frac{\pi}{2};\ 0 \leqslant \gamma < p)$ (see Patil and Thakare [7]).

Also, we note that:

(1)
$$S_{\lambda,\left(1-\frac{\gamma}{p}\right)e^{-i\alpha}\cos\alpha,p}\left(\frac{z^p}{1-z}, \varphi\right) = S^\alpha_{\lambda,p,\gamma}(\varphi)$$
$$= \left\{f(z) \in \mathcal{A}(p) : \frac{e^{i\alpha}\left[\frac{zf'(z)+\lambda z^2 f''(z)}{(1-\lambda)f(z)+\lambda zf'(z)}\right] - \gamma\cos\alpha - ip\sin\alpha}{(p-\gamma)\cos\alpha}\right.$$
$$\left.\prec \varphi(z)(|\alpha| < \frac{\pi}{2};\ 0 \leqslant \gamma < p; 0 \leqslant \lambda \leqslant 1)\right\};$$

(2)
$$S_{0,\left(1-\frac{\gamma}{p}\right)e^{-i\alpha}\cos\alpha,p}\left(\frac{z^p}{1-z}, \varphi\right) = S^\alpha_{p,\gamma}(\varphi)$$
$$= \left\{f(z) \in \mathcal{A}(p) : \frac{e^{i\alpha}\left(\frac{zf'(z)}{f(z)}\right) - \gamma\cos\alpha - ip\sin\alpha}{(p-\gamma)\cos\alpha}\right.$$
$$\left.\prec \varphi(z)\left(|\alpha| < \frac{\pi}{2}; 0 \leqslant \gamma < p\right)\right\};$$

(3)
$$S_{1,\left(1-\frac{\gamma}{p}\right)e^{-i\alpha}\cos\alpha,p}\left(\frac{z^p}{1-z}, \varphi\right) = C^\alpha_{p,\gamma}(\varphi)$$
$$= \left\{f(z) \in \mathcal{A}(p) : \frac{e^{i\alpha}\left(1 + \frac{zf''(z)}{f'(z)}\right) - \gamma\cos\alpha - ip\sin\alpha}{(p-\gamma)\cos\alpha}\right.$$
$$\left.\prec \varphi(z)\ \left(|\alpha| < \frac{\pi}{2}; 0 \leqslant \gamma < p\right)\right\};$$

(4)
$$S_{\lambda,b,p}\left(z^p + \sum_{k=p+1}^{\infty} \Gamma_{k,p}(\alpha_1)z^k, \varphi\right) = S_{\lambda,p,b}(\alpha_1;\varphi)$$
$$= \left\{f \in \mathcal{A}(p) : 1 + \frac{1}{b}\left[\frac{1}{p}\frac{z(H_{p,\ell,m}(\alpha_1)f(z))' + \lambda z^2(H_{p,\ell,m}(\alpha_1)f(z))''}{(1-\lambda)(H_{p,\ell,m}(\alpha_1)f(z)) + \lambda z(H_{p,\ell,m}(\alpha_1)f(z))'} - 1\right]\right.$$
$$\left.\prec [\varphi(z)\ (\ell \leqslant m+1; \ell, m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})\right\}$$

where the operator

$$H_{p,\ell,m}(\alpha_1)(z) = z^p + \sum_{k=p+1}^{\infty} \Gamma_{k,p}(\alpha_1)z^k,$$

$$\Gamma_{k,p}(\alpha_1) = \frac{(\alpha_1)_{k-p}\cdots(\alpha_\ell)_{k-p}}{(\beta_1)_{k-p}\cdots(\beta_m)_{k-p}}\frac{1}{(k-p)!}, \tag{1.10}$$

$\alpha_1, \ldots, \alpha_\ell$ and $\beta_1, \ldots, \beta_m$ are real parameters, $\beta_j \neq 0, -1, -2, \ldots; j = 1, \ldots, m$, was introduced and studied by Dziok and Srivastava [8];

(5)
$$S_{\lambda,b,p}\left(z^p + \sum_{k=p+1}^{\infty}\left[\frac{p+\ell+\gamma(k-p)}{p+\ell}\right]^m z^k, \varphi(z)\right) = S_{\lambda,p,b}(m,\gamma,\ell;\varphi)$$
$$= \left\{f \in \mathcal{A}(p) : 1 + \frac{1}{b}\left[\frac{1}{p}\frac{z\left(I^m_p(\gamma,\ell)f(z)\right)' + \lambda z^2\left(I^m_p(\gamma,\ell)f(z)\right)''}{(1-\lambda)\left(I^m_p(\gamma,\ell)f(z)\right) + \lambda z\left(I^m_p(\gamma,\ell)f(z)\right)'} - 1\right]\right.$$
$$\left.\prec \varphi(z)(\gamma \geqslant 0; \ell \geqslant 0; m \in \mathbb{Z}; p \in \mathbb{N})\right\};$$

where the operator

$$I^m_p(\gamma,\ell)(z) = z^p + \sum_{k=p+1}^{\infty}\left[\frac{p+\ell+\gamma(k-p)}{p+\ell}\right]^m z^k, \tag{1.11}$$

was introduced and studied by Prajapat [9], (see also, Catas [10] and El-Ashwah and Aouf [11] with $m \in \mathbb{N}_0$).

In this paper, we obtain the Fekete-Szegö inequalities for functions in the class $S_{\lambda,b,p}(g,\varphi)$.

2. Fekete-Szegö problem

Let Ω be the class of functions of the form

$$w(z) = w_1 z + w_2 z^2 + w_3 z^3 + \dots, \tag{2.1}$$

in the open unit disk U satisfying $|w(z)| < 1$.

To prove our results, we need the following lemmas.

Lemma 1 [12]. *If $w \in \Omega$, then for any complex number t,*

$$\left|w_2 - t w_1^2\right| \leqslant \max\{1; |t|\}.$$

The result is sharp for the functions given by

$w(z) = z$ or $w(z) = z^2$.

Lemma 2. [3,4] *If $w \in \Omega$, then*

$$\left|w_2 - t w_1^2\right| \leqslant \begin{cases} -t & \text{if } t \leqslant -1, \\ 1 & \text{if } -1 \leqslant t \leqslant 1, \\ t & \text{if } t \geqslant 1. \end{cases}$$

When $t < -1$ or $t > 1$, the equality holds if and only if $w(z) = z$ or one of its rotations. If $-1 < t < 1$, then the equality holds if and only if $w(z) = z^2$ or one of its rotations. If $t = -1$, the equality holds if and only if

$$w(z) = \frac{z(z+\eta)}{1+\eta z} \quad (0 \leqslant \eta \leqslant 1),$$

or one of its rotations. If $t = 1$, the equality holds if and only if

$$w(z) = -\frac{z(z+\eta)}{1+\eta z} \quad (0 \leqslant \eta \leqslant 1).$$

or one of its rotations. Also the above upper bound is sharp and it can be improved as follows when $-1 < t < 1$:

$$\left|w_2 - t w_1^2\right| + (t+1)|w_1|^2 \leqslant 1 \quad (-1 < t \leqslant 0),$$

and

$$\left|w_2 - t w_1^2\right| + (1-t)|w_1|^2 \leqslant 1 \quad (0 < t < 1).$$

Lemma 3 13. *If $w \in \Omega$, then for any real numbers q_1 and q_2, the following sharp estimates holds:*

$$\left|w_3 + q_1 w_1 w_2 + q_2 w_1^3\right| \leqslant H(q_1, q_2), \tag{2.2}$$

where

$$H(q_1,q_2) = \begin{cases} 1 & \text{for } (q_1,q_2) \in D_1 \cup D_2, \\ |q_2| & \text{for } (q_1,q_2) \in \bigcup_{k=3}^{7} D_k, \\ \frac{2}{3}(|q_1|+1)\left(\frac{|q_1|+1}{3(|q_1|+1+q_2)}\right)^{\frac{1}{2}} & \text{for } (q_1,q_2) \in D_8 \cup D_9, \\ \frac{q_2}{3}\left(\frac{q_1^2-4}{q_1^2-4q_2}\right)\left(\frac{q_1^2-4}{3(q_2-1)}\right)^{\frac{1}{2}} & \text{for } (q_1,q_2) \in D_{10} \cup D_{11} - \{\pm 2, 1\}, \\ \frac{2}{3}(|q_1|-1)\left(\frac{|q_1|-1}{3(|q_1|-1-q_2)}\right)^{\frac{1}{2}} & \text{for } (q_1,q_2) \in D_{12}. \end{cases}$$

The extremal functions, up to rotations, are of the form

$$w(z) = z^3, \quad w(z) = z, \quad w(z) = w_0(z)$$
$$= \frac{(z[(1-\lambda)\varepsilon_2 + \lambda\varepsilon_1] - \varepsilon_1\varepsilon_2 z)}{1 - [(1-\lambda)\varepsilon_1 + \lambda\varepsilon_2]z}, \quad w(z) = w_1(z) = \frac{z(t_1 - z)}{1 - t_1 z},$$

$$w(z) = w_2(z) = \frac{z(t_2 + z)}{1 + t_2 z}, \quad |\varepsilon_1| = |\varepsilon_2| = 1,$$

$$\varepsilon_1 = t_0 - e^{\frac{-i\theta_0}{2}}(a \mp b), \quad \varepsilon_2 = -e^{\frac{-i\theta_0}{2}}(ia \pm b),$$

$$a = t_0 \cos\frac{\theta_0}{2}, \quad b = \sqrt{1 - t_0^2 \sin^2\frac{\theta_0}{2}}, \quad \lambda = \frac{b \pm a}{2b},$$

$$t_0 = \left[\frac{2q_2(q_1^2+2) - 3q_1^2}{3(q_2-1)(q_1^2-4q_2)}\right]^{\frac{1}{2}}, \quad t_1 = \left(\frac{|q_1|+1}{3(|q_1|+1+q_2)}\right)^{\frac{1}{2}},$$

$$t_2 = \left(\frac{|q_1|-1}{3(|q_1|-1-q_2)}\right)^{\frac{1}{2}},$$

$$\cos\frac{\theta_0}{2} = \frac{q_1}{2}\left[\frac{q_2(q_1^2+8) - 2(q_1^2+2)}{2q_2(q_1^2+2) - 3q_1^2}\right].$$

The sets $D_k, k = 1, 2, \dots, 12$, are defined as follows:

$$D_1 = \left\{(q_1,q_2) : |q_1| \leqslant \frac{1}{2},\ |q_2| \leqslant 1\right\},$$
$$D_2 = \left\{(q_1,q_2) : \frac{1}{2} \leqslant |q_1| \leqslant 2,\ \frac{4}{27}(|q_1|+1)^3 - (|q_1|+1) \leqslant q_2 \leqslant 1\right\},$$
$$D_3 = \left\{(q_1,q_2) : |q_1| \leqslant \frac{1}{2},\ q_2 \leqslant -1\right\},$$
$$D_4 = \left\{(q_1,q_2) : |q_1| \geqslant \frac{1}{2},\ q_2 \leqslant -\frac{2}{3}(|q_1|+1)\right\},$$
$$D_5 = \{(q_1,q_2) : |q_1| \leqslant 2, q_2 \geqslant 1\},$$
$$D_6 = \left\{(q_1,q_2) : 2 \leqslant |q_1| \leqslant 4,\ q_2 \geqslant \frac{1}{12}(q_1^2+8)\right\},$$
$$D_7 = \left\{(q_1,q_2) : |q_1| \geqslant 4,\ q_2 \geqslant \frac{2}{3}(|q_1|-1)\right\},$$
$$D_8 = \left\{(q_1,q_2) : \frac{1}{2} \leqslant |q_1| \leqslant 2, -\frac{2}{3}(|q_1|+1) \leqslant q_2 \leqslant \frac{4}{27}(|q_1|+1)^3 - (|q_1|+1)\right\},$$
$$D_9 = \left\{(q_1,q_2) : |q_1| \geqslant 2, -\frac{2}{3}(|q_1|+1) \leqslant q_2 \leqslant \frac{2|q_1|(|q_1|+1)}{q_1^2+2|q_1|+4}\right\},$$
$$D_{10} = \left\{(q_1,q_2) : 2 \leqslant |q_1| \leqslant 4,\ \frac{2|q_1|(|q_1|+1)}{q_1^2+2|q_1|+4} \leqslant q_2 \leqslant \frac{1}{12}(q_1^2+8)\right\},$$
$$D_{11} = \left\{(q_1,q_2) : |q_1| \geqslant 4,\ \frac{2|q_1|(|q_1|+1)}{q_1^2+2|q_1|+4} \leqslant q_2 \leqslant \frac{2|q_1|(|q_1|-1)}{q_1^2-2|q_1|+4}\right\},$$
$$D_{12} = \left\{(q_1,q_2) : |q_1| \geqslant 4,\ \frac{2|q_1|(|q_1|-1)}{q_1^2-2|q_1|+4} \leqslant q_2 \leqslant \frac{2}{3}(|q_1|-1)\right\}.$$

Unless otherwise mentioned, we assume throughout this paper that $b \in \mathbb{C}^*$, $0 \leqslant \lambda \leqslant 1$, $k \geqslant p+1$, $p \in \mathbb{N}$ and the function $g(z)$ is given by (1.2) with $g_k > 0$.

Theorem 1. *Let $\varphi(z) = 1 + B_1 z + B_2 z^2 + \dots$, $B_1 > 0$. If $f(z)$ given by (1.1) belongs to the class $S_{\lambda,b,p}(g,\varphi)$ and μ is a complex number, then*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leqslant \frac{pB_1|b|}{2g_{p+2}}\left(\frac{1+\lambda(p-1)}{1+\lambda(p+1)}\right) \cdot$$
$$\cdot \max\left\{1, \left|\frac{B_2}{B_1} + pB_1 b\left[1 - 2\mu\frac{g_{p+2}}{g_{p+1}^2}\left(1 - \frac{\lambda^2}{(1+\lambda p)^2}\right)\right]\right|\right\}, \tag{2.3}$$

and

$$|a_{p+3}| \leqslant \frac{pB_1|b|}{3g_{p+3}}\left(\frac{1+\lambda(p-1)}{1+\lambda(p+2)}\right)H(q_1,q_2), \tag{2.4}$$

where

$$q_1 = \frac{4B_2 + 3pbB_1^2}{2B_1}, \tag{2.5}$$

$$q_2 = \frac{2B_3 + 3pbB_1B_2 + p^2b^2B_1^3}{2B_1}. \tag{2.6}$$

The result is sharp.

Proof. If $f(z) \in S_{\lambda,b,p}(g,\varphi)$, then there is a Schwarz function

$$w(z) = w_1z + w_2z^2 + w_3z^3 + \ldots \in \Omega,$$

such that

$$1 + \frac{1}{b}\left[\frac{1}{p}\frac{z(f*g)'(z) + \lambda z^2(f*g)''(z)}{(1-\lambda)(f*g)(z) + \lambda z(f*g)'(z)} - 1\right] = \varphi(w(z)). \tag{2.7}$$

Since

$$\begin{aligned}\frac{1}{p}&\frac{z(f*g)'(z) + \lambda z^2(f*g)''(z)}{(1-\lambda)(f*g)(z) + \lambda z(f*g)'(z)} \\ &= 1 + \frac{(1+\lambda p)}{p[1+\lambda(p-1)]}a_{p+1}g_{p+1}z \\ &\quad + \left(\frac{2[1+\lambda(p+1)]}{p[1+\lambda(p-1)]}a_{p+2}g_{p+2} - \frac{(1+\lambda p)^2}{p[1+\lambda(p-1)]^2}a_{p+1}^2g_{p+1}^2\right)z^2 \\ &\quad + \left(\frac{3[1+\lambda(p+2)]}{p[1+\lambda(p-1)]}a_{p+3}g_{p+3} + \frac{(1+\lambda p)^3}{p[1+\lambda(p-1)]^3}a_{p+1}^3g_{p+1}^3\right. \\ &\quad \left. - \frac{3[1+\lambda p][1+\lambda(p+1)]}{p[1+\lambda(p-1)]^2}a_{p+1}a_{p+2}g_{p+1}g_{p+2}\right)z^3 + \ldots, \end{aligned} \tag{2.8}$$

and

$$\begin{aligned}\varphi(w(z)) = 1 &+ w_1B_1z + \left(w_1^2B_2 + w_2B_1\right)z^2 \\ &+ \left(w_3B_1 + w_1^3B_3 + 2w_1w_2B_2\right)z^3 + \ldots,\end{aligned} \tag{2.9}$$

then

$$a_{p+1} = \frac{pbB_1w_1}{g_{p+1}}\left(\frac{1+\lambda(p-1)}{1+\lambda p}\right),$$

$$a_{p+2} = \frac{pB_1b}{2g_{p+2}}\left(\frac{1+\lambda(p-1)}{1+\lambda(p+1)}\right)\left[w_2 + w_1^2\frac{B_2}{B_1} + pw_1^2B_1b\right],$$

and

$$\begin{aligned}a_{p+3} = \frac{pB_1b}{3g_{p+3}}\left(\frac{1+\lambda(p-1)}{1+\lambda(p+2)}\right)&\left\{w_3 + \left(\frac{4B_2 + 3pbB_1^2}{2B_1}\right)w_1w_2\right. \\ &\left. + \left(\frac{2B_3 + 3pbB_1B_2 + p^2b^2B_1^3}{2B_1}\right)w_1^3\right\}.\end{aligned}$$

Therefore, we have

$$a_{p+2} - \mu a_{p+1}^2 = \frac{pB_1b}{2g_{p+2}}\left(\frac{1+\lambda(p-1)}{1+\lambda(p+1)}\right)\left\{w_2 - \upsilon w_1^2\right\}, \tag{2.10}$$

where

$$\upsilon = \left[2\mu\frac{g_{p+2}}{g_{p+1}^2}\left(1 - \frac{\lambda^2}{(1+\lambda p)^2}\right)pB_1b - \frac{B_2}{B_1} - pB_1b\right].$$

The result (2.3) follows by an application of Lemma 1 and the result (2.4) follows by an application of Lemma 3. The result is sharp for the functions

$$1 + \frac{1}{b}\left[\frac{1}{p}\frac{z(f*g)'(z) + \lambda z^2(f*g)''(z)}{(1-\lambda)(f*g)(z) + \lambda z(f*g)'(z)} - 1\right] = \varphi(z^2), \tag{2.11}$$

and

$$1 + \frac{1}{b}\left[\frac{1}{p}\frac{z(f*g)'(z) + \lambda z^2(f*g)''(z)}{(1-\lambda)(f*g)(z) + \lambda z(f*g)'(z)} - 1\right] = \varphi(z). \tag{2.12}$$

This completes the proof of Theorem 1. □

Putting $\lambda = 0$ in Theorem 1, we obtain the following corollary which improves the result obtained by Ali et al. [3, Theorem 2].

Corollary 1. *Let $\varphi(z) = 1 + B_1z + B_2z^2 + \ldots$, $B_1 > 0$. If $f(z)$ given by (1.1) belongs to the class $S^*_{b,p,g}(\varphi)$ and μ is a complex number, then*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leqslant \frac{pB_1|b|}{2g_{p+2}}\max\left\{1, \left|\frac{B_2}{B_1} + pB_1b\left[1 - 2\mu\frac{g_{p+2}}{g_{p+1}^2}\right]\right|\right\}.$$

The result is sharp.

Putting $p = 1$, $\lambda = 0$ and $g(z) = \frac{z}{1-z}$ in Theorem 1, we obtain the following corollary which improves the result obtained by Ravichandran et al. [5, Theorem 4.1].

Corollary 2. *Let $\varphi(z) = 1 + B_1z + B_2z^2 + \ldots$, $B_1 > 0$. If $f(z)$ given by (1.1) belongs to the class $S^*_b(\varphi)$ and μ is a complex number, then*

$$\left|a_3 - \mu a_2^2\right| \leqslant \frac{B_1|b|}{2}\max\left\{1, \left|\frac{B_2}{B_1} + B_1b[1 - 2\mu]\right|\right\}.$$

The result is sharp.

Putting $g(z) = \frac{z^p}{1-z}$ and $b = \left(1 - \frac{\gamma}{p}\right)e^{-i\alpha}\cos\alpha\left(|\alpha| < \frac{\pi}{2}; 0 \leqslant \gamma < p\right)$ in Theorem 1, we obtain the following corollary.

Corollary 3. *Let $\varphi(z) = 1 + B_1z + B_2z^2 + \ldots$, $B_1 > 0$. If $f(z)$ given by (1.1) belongs to the class $S^{\alpha}_{\lambda,p,\gamma}(\varphi)$ and μ is a complex number, then*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leqslant \frac{B_1(p-\gamma)\cos\alpha}{2}\left(\frac{1+\lambda(p-1)}{1+\lambda(p+1)}\right).$$

$$.\max\left\{1, \left|\frac{B_2}{B_1} + B_1\left[1 - 2\mu\left(1 - \frac{\lambda^2}{(1+\lambda p)^2}\right)\right](p-\gamma)e^{-i\alpha}\cos\alpha\right|\right\}.$$

The result is sharp.

Remark 1. Putting $\lambda = 0, p = 1$ and $\varphi(z) = \frac{1+z}{1-z}$ in Corollary 3, we obtain the result obtained by Keogh and Merkes [12, Theorem 1].

Putting $g(z) = \frac{z^p}{1-z}$ and $\lambda = 1$ in Theorem 1, we obtain the following corollary.

Corollary 4. *Let $\varphi(z) = 1 + B_1z + B_2z^2 + \ldots$, $B_1 > 0$. If $f(z)$ given by (1.1) belongs to the class $C_{b,p}(\varphi)$ and μ is a complex number, then*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| \leqslant \frac{p^2B_1|b|}{2(p+2)}\max\left\{1, \left|\frac{B_2}{B_1} + pB_1b\left[1 - 2\mu\frac{p(p+2)}{(p+1)^2}\right]\right|\right\}.$$

The result is sharp.

By using Lemma 2, we can obtain the following theorem.

Theorem 2. *Let* $\varphi(z)=1+B_1z+B_2z^2+\ldots,(B_i>0,i\in\mathbb{N},b>0)$. *If* $f(z)$ *given by* (1.1) *belongs to the class* $S_{\lambda,b,p}(g,\varphi)$ *and* μ *is a real number, then*

$$\left|a_{p+2}-\mu a_{p+1}^2\right|\leqslant\begin{cases}\frac{pb}{2g_{p+2}}\left(\frac{1+\lambda(p-1)}{1+\lambda(p+1)}\right)\left\{B_2+pB_1^2b\left[1-2\mu\frac{g_{p+2}}{g_{p+1}^2}\left(1-\frac{\lambda^2}{(1+\lambda p)^2}\right)\right]\right\} & \text{if } \mu\leqslant\sigma_1,\\ \frac{pB_1b}{2g_{p+2}}\left(\frac{1+\lambda(p-1)}{1+\lambda(p+1)}\right) & \text{if } \sigma_1\leqslant\mu\leqslant\sigma_2,\\ \frac{pb}{2g_{p+2}}\left(\frac{1+\lambda(p-1)}{1+\lambda(p+1)}\right)\left\{-B_2+pB_1^2b\left[2\mu\frac{g_{p+2}}{g_{p+1}^2}\left(1-\frac{\lambda^2}{(1+\lambda p)^2}\right)-1\right]\right\} & \text{if } \mu\geqslant\sigma_2,\end{cases}\tag{2.13}$$

where

$$\sigma_1=\frac{\{(B_2-B_1)+pB_1^2b\}g_{p+1}^2}{2pB_1^2bg_{p+2}}\frac{(1+\lambda p)^2}{[(1+\lambda p)^2-\lambda^2]},$$

and

$$\sigma_2=\frac{\{(B_2+B_1)+pB_1^2b\}g_{p+1}^2}{2pB_1^2bg_{p+2}}\frac{(1+\lambda p)^2}{[(1+\lambda p)^2-\lambda^2]}.$$

The result is sharp.

Proof. First, let $\mu\leqslant\sigma_1$, then

$$\begin{aligned}\left|a_{p+2}-\mu a_{p+1}^2\right|&\leqslant\frac{pB_1b}{2g_{p+2}}\left(\frac{1+\lambda(p-1)}{1+\lambda(p+1)}\right)\\&\quad\times\left\{\frac{B_2}{B_1}+pB_1b\left[1-2\mu\frac{g_{p+2}}{g_{p+1}^2}\left(1-\frac{\lambda^2}{(1+\lambda p)^2}\right)\right]\right\}\\&\leqslant\frac{pb}{2g_{p+2}}\left(\frac{1+\lambda(p-1)}{1+\lambda(p+1)}\right)\\&\quad\times\left\{B_2+pB_1^2b\left[1-2\mu\frac{g_{p+2}}{g_{p+1}^2}\left(1-\frac{\lambda^2}{(1+\lambda p)^2}\right)\right]\right\}.\end{aligned}$$

Let, now $\sigma_1\leqslant\mu\leqslant\sigma_2$. Then, using the above calculations, we obtain

$$\left|a_{p+2}-\mu a_{p+1}^2\right|\leqslant\frac{pB_1b}{2g_{p+2}}\left(\frac{1+\lambda(p-1)}{1+\lambda(p+1)}\right).$$

Finally, if $\mu\geqslant\sigma_2$, then

$$\begin{aligned}\left|a_{p+2}-\mu a_{p+1}^2\right|&\leqslant\frac{pB_1b}{2g_{p+2}}\left(\frac{1+\lambda(p-1)}{1+\lambda(p+1)}\right)\\&\quad\times\left\{-\frac{B_2}{B_1}+pB_1b\left[2\mu\frac{g_{p+2}}{g_{p+1}^2}\left(1-\frac{\lambda^2}{(1+\lambda p)^2}\right)-1\right]\right\}\\&\leqslant\frac{pb}{2g_{p+2}}\left(\frac{1+\lambda(p-1)}{1+\lambda(p+1)}\right)\\&\quad\times\left\{-B_2+pB_1^2b\left[2\mu\frac{g_{p+2}}{g_{p+1}^2}\left(1-\frac{\lambda^2}{(1+\lambda p)^2}\right)-1\right]\right\}.\end{aligned}$$

To show that the bounds are sharp, we define the functions $K_{\varphi n}(n\geqslant 2)$ by

$$1+\frac{1}{b}\left[\frac{1}{p}\frac{z(K_{\varphi n}*g)'(z)+\lambda z^2(K_{\varphi n}*g)''(z)}{(1-\lambda)(K_{\varphi n}*g)(z)+\lambda z(K_{\varphi n}*g)'(z)}-1\right]=\varphi(z^{n-1}),\quad K_{\varphi n}(0)=0=K'_{\varphi n}(0)-1,$$

and the functions F_β and $G_\beta(0\leqslant\beta\leqslant 1)$ by

$$1+\frac{1}{b}\left[\frac{1}{p}\frac{z(F_\beta*g)'(z)+\lambda z^2(F_\beta*g)''(z)}{(1-\lambda)(F_\beta*g)(z)+\lambda z(F_\beta*g)'(z)}-1\right]=\varphi\left(\frac{z(z+\beta)}{1+\beta z}\right),\quad F_\beta(0)=0=F'_\beta(0)-1,$$

and

$$1+\frac{1}{b}\left[\frac{1}{p}\frac{z(G_\beta*g)'(z)+\lambda z^2(G_\beta*g)''(z)}{(1-\lambda)(G_\beta*g)(z)+\lambda z(G_\beta*g)'(z)}-1\right]=\varphi\left(-\frac{z(z+\beta)}{1+\beta z}\right),\quad G_\beta(0)=0=G'_\beta(0)-1.$$

Cleary the functions $K_{\varphi n}$, F_β and $G_\beta\in S_{\lambda,b,p}(g,\varphi)$. Also we write $K_\varphi=K_{\varphi 2}$.

If $\mu<\sigma_1$ or $\mu>\sigma_2$, then the equality holds if and only if f is K_φ or one of its rotations. When $\sigma_1<\mu<\sigma_2$, then the equality holds if f is $K_{\varphi 3}$ or one of its rotations. If $\mu=\sigma_1$, then the equality holds if and only if f is F_β or one of its rotations. If $\mu=\sigma_2$, then the equality holds if and only if f is G_β or one of its rotations. This completes the proof of Theorem 2. □

Putting $\lambda=0$ in Theorem 2, we obtain the following corollary.

Corollary 5. *Let* $\varphi(z)=1+B_1z+B_2z^2+\ldots,(B_i>0,i\in\mathbb{N},b>0)$. *If* $f(z)$ *given by* (1.1) *belongs to the class* $S^*_{b,p,g}(\varphi)$ *and* μ *is a real number, then*

$$\left|a_{p+2}-\mu a_{p+1}^2\right|\leqslant\begin{cases}\frac{pb}{2g_{p+2}}\left\{B_2+pB_1^2b\left[1-2\mu\frac{g_{p+2}}{g_{p+1}^2}\right]\right\} & \text{if } \mu\leqslant\sigma_1,\\ \frac{pB_1b}{2g_{p+2}} & \text{if } \sigma_1\leqslant\mu\leqslant\sigma_2,\\ \frac{pb}{2g_{p+2}}\left\{-B_2+pB_1^2b\left[2\mu\frac{g_{p+2}}{g_{p+1}^2}-1\right]\right\} & \text{if } \mu\geqslant\sigma_2,\end{cases}$$

where

$$\sigma_1=\frac{\{(B_2-B_1)+pB_1^2b\}g_{p+1}^2}{2pB_1^2bg_{p+2}},$$

and

$$\sigma_2=\frac{\{(B_2+B_1)+pB_1^2b\}g_{p+1}^2}{2pB_1^2bg_{p+2}}.$$

The result is sharp.

Remark 2. Putting $b=p=1$ and $g(z)=\frac{z}{1-z}$ in Corollary 5, we obtain the result obtained by Murugusundaramoorthy et al. [14, Corollary 2.2] and the result obtained by Shanmugam and Sivasubramanian [6, Theorem 2.1 with $\alpha=0$].

Putting $g(z)=\frac{z^p}{1-z}$ and $\lambda=1$ in Theorem 2, we obtain the following corollary.

Corollary 6. *Let* $\varphi(z)=1+B_1z+B_2z^2+\ldots,(B_i>0,\ i\in\mathbb{N},b>0)$. *If* $f(z)$ *given by* (1.1) *belongs to the class* $C_{b,p}(\varphi)$ *and* μ *is a real number, then*

$$\left|a_{p+2}-\mu a_{p+1}^2\right|\leqslant\begin{cases}\frac{p^2b}{2(p+2)}\left\{B_2+pB_1^2b\left[1-2\mu\frac{p(p+2)}{(p+1)^2}\right]\right\} & \text{if } \mu\leqslant\sigma_1,\\ \frac{p^2B_1b}{2(p+2)} & \text{if } \sigma_1\leqslant\mu\leqslant\sigma_2,\\ \frac{p^2b}{2(p+2)}\left\{-B_2+pB_1^2b\left[2\mu\frac{p(p+2)}{(p+1)^2}-1\right]\right\} & \text{if } \mu\geqslant\sigma_2,\end{cases}$$

where

$$\sigma_1 = \frac{\{(B_2 - B_1) + pB_1^2 b\}(p+1)^2}{2p^2 B_1^2 b(p+2)},$$

and

$$\sigma_2 = \frac{\{(B_2 + B_1) + pB_1^2 b\}(p+1)^2}{2p^2 B_1^2 b(p+2)}.$$

The result is sharp.

Putting $b = p = \lambda = 1$ and $g(z) = \frac{z}{1-z}$ in Theorem 2, we improve the result obtained by Murugusundaramoorthy et al. [14, Corollary 2.3] and also improve the result obtained by Shanmugam and Sivasubramanian [6, Theorem 2.1 with $\alpha = 1$].

Corollary 7. *Let* $\varphi(z) = 1 + B_1 z + B_2 z^2 + \ldots, (B_i > 0, i \in \mathbb{N}, b > 0)$. *If* $f(z)$ *given by* (1.1) *belongs to the class* $C(\varphi)$ *and* μ *is a real number, then*

$$|a_3 - \mu a_2^2| \leqslant \begin{cases} \frac{1}{6}\{B_2 + B_1^2 - \frac{3}{2}\mu B_1^2\} & \text{if } \mu \leqslant \sigma_1, \\ \frac{B_1}{6} & \text{if } \sigma_1 \leqslant \mu \leqslant \sigma_2, \\ \frac{1}{6}\{-B_2 - B_1^2 + \frac{3}{2}\mu B_1^2\} & \text{if } \mu \geqslant \sigma_2, \end{cases}$$

where

$$\sigma_1 = \frac{2(B_2 - B_1) + 2B_1^2}{3B_1^2},$$

and

$$\sigma_2 = \frac{2(B_2 + B_1) + 2B_1^2}{3B_1^2}.$$

The result is sharp.

Using arguments similar to those in the proof of Theorem 2, we obtain the following theorem.

Theorem 3. *Let* $\varphi(z) = 1 + B_1 z + B_2 z^2 + \ldots, (B_i > 0, i \in \mathbb{N}, b > 0)$ *and*

$$\sigma_3 = \frac{\{B_2 + pB_1^2 b\}g_{p+1}^2}{2pB_1^2 b g_{p+2}} \frac{(1+\lambda p)^2}{[(1+\lambda p)^2 - \lambda^2]}. \tag{2.14}$$

If $f(z)$ *given by* (1.1) *belongs to the class* $S_{\lambda,b,p}(g,\varphi)$ *and* μ *is a real number, then we have*

(i) *If* $\sigma_1 \leqslant \mu \leqslant \sigma_3$, *then*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| + \frac{(1+\lambda p)^2 g_{p+1}^2}{2pB_1^2 b[(1+\lambda p)^2 - \lambda^2]g_{p+2}} \left\{(B_1 - B_2) + pB_1^2 b\left[2\mu\frac{g_{p+2}}{g_{p+1}^2}\frac{\left[(1+\lambda p)^2 - \lambda^2\right]}{(1+\lambda p)^2} - 1\right]\right\}|a_{p+1}|^2 \leqslant \frac{pB_1 b}{2g_{p+2}}\left(\frac{1+\lambda(p-1)}{1+\lambda(p+1)}\right). \tag{2.15}$$

(ii) *If* $\sigma_3 \leqslant \mu \leqslant \sigma_2$, *then*

$$\left|a_{p+2} - \mu a_{p+1}^2\right| + \frac{(1+\lambda p)^2 g_{p+1}^2}{2pB_1^2 b[(1+\lambda p)^2 - \lambda^2]g_{p+2}} \times \left\{(B_1 + B_2) + pB_1^2 b\left[1 - 2\mu\frac{g_{p+2}}{g_{p+1}^2}\frac{\left[(1+\lambda p)^2 - \lambda^2\right]}{(1+\lambda p)^2}\right]\right\}|a_{p+1}|^2 \leqslant \frac{pB_1 b}{2g_{p+2}}\left(\frac{1+\lambda(p-1)}{1+\lambda(p+1)}\right), \tag{2.16}$$

where σ_1 *and* σ_2 *are given in Theorem* 2.

Remark 3. For different choices of $g(z)$, λ, b, p and $\varphi(z)$ in Theorems 1–3, we will obtain new results for different classes mentioned in the introduction.

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