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Some integrals involving *k* gamma and *k* digamma function

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Abstract

In this paper, some new integrals involving *k* gamma function and *k* digamma function have been established. An integral is established involving *k* gamma function, and its special values are discussed. Similarly, some new integrals have been established for *k* digamma function, and different elementary function is associated with it for different values of *k*. A nice representation of the Euler-Mascheroni constant and π in the form of *k* digamma function for different values of *k* is also obtained.

Keywords: *k* gamma function, *k* digamma function

Mathematics subject classification: 33B15, 41A58, 33C20

k gamma function

The *k* gamma function is a generalization of the classical gamma function introduced by Diaz and Pariguan [1], denoted and defined as

$$\Gamma_k(z) = \lim_{n\to\infty} \frac{n!k^n (nk)^{\frac{z}{k}-1}}{(z)_{n,k}}, \quad k > 0, z\in\mathbb{C}\backslash k\mathbb{Z}^{-}. \tag{1.1}$$

The symbol for $(z)_{n,\ k}$ is called Pochhammer's *k* symbol [2] and is defined as

$$(z)_{n,k} = z(z+k)(z+2k)\cdots(z+(n-1)k). \tag{1.2}$$

Due to (1.2), we see that (1.1) has simple poles at $0, -k, -2k, -3k, \cdots$. The residue of *k* gamma function at these simple poles is $\frac{1}{(-1)^n k^n n!}$, see [3]. The integral form of *k* gamma function is denoted and defined as [4]

$$\Gamma_k(z) = \int_0^\infty e^{-\frac{t^k}{k}} t^{z-1} dt. \tag{1.3}$$

The improper integral is convergent for $\mathrm{Re}(z) > 0$. The *k* gamma function reduces to the classical gamma function, i.e., $\Gamma_k \to \Gamma \ as \ k \to 1$. A simple change of variable $t^k = ky$ reveals the relationship between *k* gamma function and classical gamma function

$$\Gamma_k(z) = k^{\frac{z}{k}-1}\Gamma\left(\frac{z}{k}\right).$$

The following properties of the k gamma function have been discussed in [5, 6]

$$\Gamma_k(z+k) = z\Gamma_k(z), \tag{1.4}$$

$$\Gamma_k(z)\Gamma_k(k-z) = \frac{\pi}{\sin\left(\frac{\pi z}{k}\right)}. \tag{1.5}$$

In the "An integral involving the k gamma function" section, we will establish an integral involving k gamma function and its special cases will also be discussed. In the "Stirling formula for the k gamma function" section, the Stirling formula will be derived for the k gamma function. In the "Some integrals representing digamma function" section, we will provide few integrals involving the k digamma function. Few special cases of the k digamma function will also be presented. In the "Euler-Mascheroni constant and k digamma function" section, we will find the relationship between the Euler-Mascheroni constant in the form k digamma function for different values of k.

An integral involving the *k* gamma function

In this section, we will derive an interesting integral involving k gamma function.

Theorem 2.1 Consider a complex number p of the form $p = a + ib$. Then

$$\frac{\Gamma_k(z)}{n|p|^{\frac{z}{k}}k^{\frac{z}{k}-1}}\cos\left(\frac{z\theta}{k}\right) = \int_0^\infty e^{-au^n}(\cos bu^n)u^{\frac{nz}{k}-1}du, \tag{2.1a}$$

$$\frac{\Gamma_k(z)}{n|p|^{\frac{z}{k}}k^{\frac{z}{k}-1}}\sin\left(\frac{z\theta}{k}\right) = \int_0^\infty e^{-au^n}(\sin bu^n)u^{\frac{nz}{k}-1}du. \tag{2.1b}$$

Proof Making the substitution $t = (kpu^n)^{\frac{1}{k}} \Rightarrow dt = pnu^{n-1}(kpu^n)^{\frac{1}{k}-1}du$ into (1.3), we get

$$\Gamma_k(z) = \int_0^\infty e^{-pu^n}(kpu^n)^{\frac{z-1}{k}}pnu^{n-1}(kpu^n)^{\frac{1}{k}-1}du = np^{\frac{z}{k}}k^{\frac{z}{k}-1}\int_0^\infty e^{-pu^n}u^{\frac{nz}{k}-1}du,$$

so we get

$$\frac{\Gamma_k(z)}{np^{\frac{z}{k}}k^{\frac{z}{k}-1}} = \int_0^\infty e^{-pu^n}u^{\frac{nz}{k}-1}du. \tag{2.2}$$

Similarly, for the conjugate of p, we can write

$$\frac{\Gamma_k(z)}{n\overline{p}^{\frac{z}{k}}k^{\frac{z}{k}-1}} = \int_0^\infty e^{-\overline{p}u^n}u^{\frac{nz}{k}-1}du. \tag{2.3}$$

Adding and simplifying (2.2) and (2.3), we get

$$\frac{\Gamma_k(z)}{nk^{\frac{z}{k}-1}}\left(\frac{1}{|p|^{\frac{z}{k}}e^{\frac{iz\theta}{k}}}+\frac{1}{|p|^{\frac{z}{k}}e^{-\frac{iz\theta}{k}}}\right)=\int_0^\infty\left(e^{-(a+ib)u^n}+e^{-(a-ib)u^n}\right)u^{\frac{nz}{k}-1}du. \tag{2.4}$$

where θ and $|p|$ are the principal argument and modulus of p, respectively, so that (2.4) reduces to

$$\frac{\Gamma_k(z)}{n|p|^{\frac{z}{k}}k^{\frac{z}{k}-1}}\left(\frac{1}{e^{\frac{iz\theta}{k}}}+\frac{1}{e^{-\frac{iz\theta}{k}}}\right)=\int_0^\infty e^{-au^n}\left(e^{-ibu^n}+e^{ibu^n}\right)u^{\frac{nz}{k}-1}du.$$

By Euler's identity, we can write

$$\frac{\Gamma_k(z)}{n|p|^{\frac{z}{k}}k^{\frac{z}{k}-1}}\left(2\cos\left(\frac{z\theta}{k}\right)\right)=\int_0^\infty e^{-au^n}(2\cos(bu^n))u^{\frac{nz}{k}-1}du.$$

This yields the final integral (2.1a). Similarly, subtracting (2.2) and (2.3) and continuing in the same fashion, we get (2.1b).

Corollary 2.2 Take $a=0, b=1, n=1 \Rightarrow |p|=1, \theta=\frac{\pi}{2}$ in (2.1b), and using the relation (1.5) together with $k=1$, we see that

$$\lim_{z\to 0}\left(\frac{\sin(z\pi/2)}{z\pi/2}\right)\lim_{z\to 0}\left(\frac{\pi z}{\sin(\pi z)}\right)\frac{\pi}{\Gamma(1)}\left(\frac{z\pi}{2\pi z}\right)=\int_0^\infty \sin(u)u^{-1}du.$$

This reduces to a well-known integral

$$\int_0^\infty \frac{\sin u}{u}du=\frac{\pi}{2}.$$

Corollary 2.3 Take $z=\frac{1}{2}, a=0, b=1, n=2 \Rightarrow p=1, \theta=\frac{\pi}{2}$ into (2.1b)

$$\int_0^\infty \sin(u^2)u^{\frac{1}{k}-1}du=\frac{\Gamma_k\left(\frac{1}{2}\right)}{2k^{\frac{1}{2k}-1}}\sin\left(\frac{\pi}{4k}\right).$$

As $k\to 1$, the integral reduces to

$$\int_0^\infty \sin(u^2)du=\frac{\sqrt{2\pi}}{4}.$$

Corollary 2.4 Take $z=\frac{1}{2}, k=1, b=0, n=2$ into (2.1a); it turns out the Gaussian integral

$$\int_0^\infty e^{-au^2}du=\frac{1}{2}\sqrt{\frac{\pi}{a}}, a>0. \tag{2.5}$$

In general, for $m>0$ and $z=\frac{1}{m}, k=1, a=1, b=0$ into (2.1a), we get

$$\int_0^\infty e^{-u^m}du = \frac{1}{m}\Gamma\left(\frac{1}{m}\right).$$

The integral (2.5) can also be written as

$$\int_{-\infty}^\infty e^{-au^2}du = \sqrt{\frac{\pi}{a}}, a > 0. \tag{2.6}$$

Stirling formula for the *k* gamma function

The Stirling formula is an approximation of the factorial for large n. It associates an appropriate function to the growth of n! which is given as

$$\Gamma(n+1) = n! \approx n^n e^{-n}\sqrt{2\pi n}, n\in\mathbb{N}. \tag{3.1}$$

In fact, it is quite accurate even for small n; for example, the Stirling formula gives 99% accuracy when compared with the value of 10! A formula similar to the Stirling formula can be obtained for the k gamma function as follows:

Theorem 3.1 For $k > 0$, $\mathrm{Re}(z) > 0$,

$$\Gamma_k(z+1) = \left(\frac{z}{e}\right)^{z/k}\sqrt{\frac{2\pi}{kz^{1-2/k}}}. \tag{3.2}$$

Proof Consider

$$\Gamma_k(z+1) = \int_0^\infty e^{-\frac{t^k}{k}}t^z dt = \int_0^\infty e^{-\frac{t^k}{k}+z\ \ln t}dt. \tag{3.3}$$

Now if we let $f(t) = -\frac{t^k}{k} + z\ \ln t$ and notice that its critical value is $f'(t) = 0 \Rightarrow t = z^{1/k} = a$which gives maximum value $f''(a) = -kz^{1-2/k} < 0$ for $k > 0$. Now if we expand the function $f(t)$ by Taylor series around its critical point, we get

$$f(t) = f(a) + (t-a)f'(a) + \frac{(t-a)^2}{2!}f''(a) + O\left((t-a)^3\right).$$

Since a is the critical point of the function, the second term of the series vanishes, and the rest simplifies to

$$f(t) = -\frac{z}{k} + \frac{z}{k}\ \ln(z) - \frac{k}{2}z^{1-2/k}\left(t - z^{1/k}\right)^2 + O\left(\left(t - z^{1/k}\right)^3\right).$$

Substituting it into (3.3) and ignoring the higher order terms, we get

$$\Gamma_k(z+1) = z^{z/k}e^{-\frac{z}{k}}\int_0^\infty e^{-\frac{k}{2}z^{1-2/k}\left(t-z^{1/k}\right)^2}dt.$$

Substituting

$$t - z^{1/k} = y$$

$$\Gamma_k(z+1) = z^{z/k} e^{-\frac{z}{k}} \int_{-z1/k}^{\infty} e^{-\frac{k}{2}z^{1-2/k}\left(t-z^{1/k}\right)^2} dt. \tag{3.4}$$

Since the integrand of the integral in (3.4) is a Gaussian curve whose peak, the maximum value, lies at $t = z^{1/k}$ so at $t < 0$ the integral is negligible. Therefore, we can extend the lower limit to $-\infty$

$$\Gamma_k(z+1) \approx z^{z/k} e^{-\frac{z}{k}} \int_{-\infty}^{\infty} e^{-\frac{k}{2}z^{1-2/k}\left(t-z^{1/k}\right)^2} dt.$$

Using (2.6), we can write as

$$\Gamma_k(z+1) = z^{z/k} e^{-\frac{z}{k}} \sqrt{\frac{\pi}{\frac{k}{2}z^{1-2/k}}}. \tag{3.5}$$

This simplifies to (3.2) as claimed. Notice that for $k \to 1$, (3.5) reduces to (3.1).

Some integrals representing digamma function

The logarithmic derivative of the k gamma function for Re(z), $k > 0$ is known as k digamma function, denoted and defined as [7]

$$\psi_k(z) = \frac{\partial}{\partial z} \log\Gamma_k(z) = \frac{\Gamma_k'(z)}{\Gamma_k(z)}. \tag{4.1}$$

Taking the logarithmic derivative of the relation (1.4), we see that

$$\frac{\partial}{\partial z} \log\Gamma_k(z+k) = \frac{\partial}{\partial z} \log z + \frac{\partial}{\partial z} \log\Gamma_k(z).$$

Using (4.1), we can write

$$\psi_k(z+k) = \psi_k(z) + \frac{1}{z}. \tag{4.2}$$

The relation (4.2) is sometimes called the functional equation of k digamma function.

Remark 4.1 Notice that for $k \to 1$, $\psi_k(z) \to \psi(z)$.

A series representation of k digamma function is derived in [3] by taking the logarithmic derivative of the inverse of k—analogue Weierstrass form of the k gamma function

$$\Gamma_k(z) = z^{-1} k^{\frac{z}{k}} e^{-\frac{z}{k}\gamma} \prod_{n=1}^{\infty} \left(\frac{nk}{z+nk}\right) e^{\frac{z}{nk}}.$$

And is given by

$$\psi_k(z) = -\frac{1}{z} + \frac{1}{k} \log k - \frac{\gamma}{k} + \sum_{n=1}^{\infty} \left(\frac{1}{nk} - \frac{1}{z+nk}\right). \tag{4.3}$$

where γ is the Euler-Mascheroni constant given by the following series form

$$\gamma=\sum_{n=1}^{\infty}\frac{1}{n}-\log(n)=0.5772156649.$$

We can rearrange the series (4.3) to write

$$\psi_k(z)=\frac{1}{k}\log k-\frac{\gamma}{k}+\sum_{n=0}^{\infty}\left(\frac{1}{(n+1)k}-\frac{1}{nk+z}\right). \tag{4.4}$$

Next, we derive some integrals involving k digamma function.

Theorem 4.2 Let $s>0$ be a real number then for $k>0$

$$\int_0^1\frac{x^{s-1}}{1+x^k}dx=\frac{1}{2}\left[\psi_k\left(\frac{s+k}{2}\right)-\psi_k\left(\frac{s}{2}\right)\right]=\psi_k(s)-\psi_k\left(\frac{s}{2}\right)-\frac{1}{k}\log(2), \tag{4.5}$$

$$\int_0^1\frac{x^{s-1}\left(1-x^{(n+1)k}\right)}{1-x^{2k}}dx=\frac{1}{2}\left(\psi_k\left(\frac{s+(n+1)k}{2}\right)-\psi_k\left(\frac{s}{2}\right)\right), \tag{4.6a}$$

$$\int_0^1\frac{1}{x(1+x^k)}\sum_{n=0}^{\infty}x^{\frac{s}{2^n}}dx=\psi_k(s)-\psi_k\left(\frac{s}{2^{n+1}}\right)-\frac{n+1}{k}\log(2), \tag{4.6b}$$

$$\int_0^{\infty}\tanh(kx)e^{-sx}dx=\frac{1}{2}\left[\psi_k\left(\frac{s+2k}{4}\right)-\psi_k\left(\frac{s}{4}\right)-\frac{2}{s}\right]. \tag{4.7}$$

Proof Using the Taylor series of $\frac{1}{1+x^k}$, the LHS of (4.5) becomes

$$\int_0^1\frac{x^{s-1}}{1+x^k}dx=\int_0^1\sum_{n=1}^{\infty}(-1)^{n-1}x^{kn-k+s-1}dx.$$

Interchanging integral and summation

$$=\sum_{n=1}^{\infty}(-1)^{n-1}\int_0^1 x^{kn-k+s-1}dx=\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{1}{kn-k+s}\right).$$

Rearranging the sum into even and odd terms, we can write

$$=\sum_{n=1}^{\infty}\left(\frac{1}{2nk-2k+s}-\frac{1}{2nk-k+s}\right).$$

Adding and subtracting $\frac{1}{2nk}$ under the summation and factoring out $\frac{1}{2}$, we get

$$=\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{nk}-\frac{1}{nk+\frac{s-k}{2}}\right)-\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{nk}-\frac{1}{nk+\frac{s-2k}{2}}\right).$$

Changing the index of the sum, we get

$$=\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{(n+1)k}-\frac{1}{nk+\frac{s+k}{2}}\right)-\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{(n+1)k}-\frac{1}{nk+\frac{s}{2}}\right).$$

Adding and subtracting $(\log k)/k-\gamma/k$ and using (4.4), we get the first equality of (4.5). To prove the second equality of (4.5), we take Legendre duplication k analogous formula $r=2$ in corollary 3.14 in [8]

$$\Gamma_k(2z)=2^{\frac{2z}{k}-\frac{1}{2}}k^{\frac{1}{2}}(2\pi)^{-\frac{1}{2}}\Gamma_k(z)\Gamma_k\left(z+\frac{k}{2}\right). \tag{4.8}$$

Taking the logarithm of (4.8)

$$\log\Gamma_k(2z)=\left(\frac{2z}{k}-\frac{1}{2}\right)\log 2+\frac{1}{2}\log k-\frac{1}{2}\log(2\pi)+\log\Gamma_k(z)+\log\Gamma_k\left(z+\frac{k}{2}\right). \tag{4.9}$$

Taking derivatives of (4.9) with respect to z and using the definition (4.1), we get

$$\psi_k(2z)=\frac{1}{2}\left\{\psi_k(z)+\psi_k\left(z+\frac{k}{2}\right)\right\}+\frac{1}{k}\log(2). \tag{4.10}$$

Replacing z by $s/2$ in (4.10)

$$\psi_k(s)=\frac{1}{2}\left\{\psi_k\left(\frac{s}{2}\right)+\psi_k\left(\frac{s+k}{2}\right)\right\}+\frac{1}{k}\log(2).$$

Rearranging we get the second equality of (4.5).

$$\frac{1}{2}\left\{\psi_k\left(\frac{s+k}{2}\right)-\psi_k\left(\frac{s}{2}\right)\right\}=\psi_k(s)-\psi_k\left(\frac{s}{2}\right)-\frac{1}{k}\log(2).$$

To derive (4.6a), we replace s by $s+k$, $s+2k$, $s+3k$, $\cdots s+nk$ in (4.5); we get

$$\begin{aligned}\int_0^1\frac{x^{s+k-1}}{1+x^k}dx&=\frac{1}{2}\left(\psi_k\left(\frac{s+2k}{2}\right)-\psi_k\left(\frac{s+k}{2}\right)\right)\\ \int_0^1\frac{x^{s+2k-1}}{1+x^k}dx&=\frac{1}{2}\left(\psi_k\left(\frac{s+3k}{2}\right)-\psi_k\left(\frac{s+2k}{2}\right)\right)\\ &\vdots\\ \int_0^1\frac{x^{s+nk-1}}{1+x^k}dx&=\frac{1}{2}\left(\psi_k\left(\frac{s+(n+1)k}{2}\right)-\psi_k\left(\frac{s+nk}{2}\right)\right).\end{aligned} \tag{4.11}$$

Now observing that the RHS of (4.11) is a telescoping sum, so adding all the $(n+1)$ terms in (4.11), we get

$$\int_0^1\frac{x^{s-1}\left(1+x^k+x^{2k}+\cdots x^{nk}\right)}{1+x^k}dx=\frac{1}{2}\left(\psi_k\left(\frac{s+(n+1)k}{2}\right)-\psi_k\left(\frac{s}{2}\right)\right). \tag{4.12}$$

The series under the integral on LHS of (4.12) is a finite geometric series with common ratio x^k, so summing it, we get (4.6a). In a similar fashion, we can derive (4.6b) by successively replacing s by $s/2, s/2^2, s/2^3, \cdots s/2^n$ in the second equality of (4.5).

Remark 4.3 The ratio test shows that the infinite series $\sum_{n=0}^{\infty} x^{\frac{s}{2^n}}$ is convergent for all real value of x as long as $s > 0$.

For the relation (4.7), first, we write the integrand as an exponential function

$$\int_0^\infty \tanh(kx)e^{-sx}dx = \int_0^\infty \frac{1-e^{-2kx}}{1+e^{-2kx}}e^{-sx}dx. \tag{4.13}$$

By making the substitution $kx = -\log\sqrt{t^k} \Rightarrow dx = -\frac{1}{2t}dt$ in (4.13), we can write

$$\int_0^\infty \tanh(kx)e^{-sx}dx = \frac{1}{2}\int_0^1 \frac{1-t^k}{1+t^k}t^{\frac{s}{2}-1}dt = \frac{1}{2}\int_0^1 \frac{t^{\frac{s}{2}-1}}{1+t^k}dt - \frac{1}{2}\int_0^1 \frac{t^{\frac{s}{2}+k-1}}{1+t^k}dt.$$

By using the first equality of (4.5), we can write

$$\int_0^\infty \tanh(kx)e^{-sx}dx = \frac{1}{2}\left[\frac{1}{2}\left(\psi_k\left(\frac{s+2k}{4}\right) - \psi_k\left(\frac{s}{4}\right)\right) - \frac{1}{2}\left(\psi_k\left(\frac{s+4k}{4}\right) - \psi_k\left(\frac{s+2k}{4}\right)\right)\right].$$

After a bit simplification

$$\int_0^\infty \tanh(kx)e^{-sx}dx = \frac{1}{2}\left[\psi_k\left(\frac{s+2k}{4}\right) - \frac{1}{2}\psi_k\left(\frac{s}{4}\right) - \frac{1}{2}\psi_k\left(k+\frac{s}{4}\right)\right]. \tag{4.14}$$

Using the relation (4.2), Eq. (4.14) reduces to the required result (4.7)

Remark 4.4 Equation (4.7) can be also be written in the form of Laplace transform of $\tanh(kx)$, that is

$$L(\tanh(kx)) = \frac{1}{2}\left[\psi_k\left(\frac{s+2k}{4}\right) - \psi_k\left(\frac{s}{4}\right) - \frac{2}{s}\right].$$

Special values: Replacing $k = 1 = s$ in the first equality of (4.5), we can write

$$\log(2) = \frac{1}{2}\left[\psi(1) - \psi\left(\frac{1}{2}\right)\right].$$

Replacing $k = 2$, $s = 1$ and for $k = 4$, $s = 2$, we get from (4.5)

$$\frac{\pi}{2} = \psi_2\left(\frac{3}{2}\right) - \psi_2\left(\frac{1}{2}\right) = 2(\psi_4(3) - \psi_4(1)). \tag{4.15}$$

Replacing $k = 1/2$, $s = 1$, we get from (4.5)

$$\int_0^1 \frac{1}{1+\sqrt{x}}dx = 2 - 2\log(2) = \frac{1}{2}\left[\psi_{\frac{1}{2}}\left(\frac{3}{4}\right) - \psi_{\frac{1}{2}}\left(\frac{1}{2}\right)\right].$$

Euler-Mascheroni constant and *k* digamma function

In this section, we represent the Euler-Mascheroni constant in the form k digamma function for different values of k.

Proposition 5.1 Substituting $k = 4$ and $z = 1$ in (4.3), we see that

$$\psi_4(1) = -1 + \frac{1}{4}\log 4 - \frac{\gamma}{4} + \sum_{n=1}^{\infty}\left(\frac{1}{4n} - \frac{1}{4n+1}\right). \tag{5.1}$$

The series in (5.1) is not hard to sum. First, observe that this series can be written as

$$\sum_{n=1}^{\infty}\left(\frac{1}{4n} - \frac{1}{4n+1}\right) = \sum_{n=1}^{\infty}\int_0^1 \left(x^{4n-1} - x^{4n}\right)dx = \int_0^1 \sum_{n=1}^{\infty}\left(x^{4n-1} - x^{4n}\right)dx$$
$$= \int_0^1 \left(x^3\sum_{n=1}^{\infty}\left(x^4\right)^{n-1} - x^4\sum_{n=1}^{\infty}\left(x^4\right)^{n-1}\right)dx$$

Now employing the Taylor series

$$= \int_0^1 \left(\frac{x^3}{1-x^4} - \frac{x^4}{1-x^4}\right)dx = \int_0^1 \frac{x^3}{(1+x)(1+x^2)}dx$$
$$= \int_0^1 \left(1 - \frac{1}{2(1+x)} - \frac{x+1}{2(1+x^2)}\right)dx.$$

Evaluating the integral, we get the sum of the series

$$\sum_{n=1}^{\infty}\left(\frac{1}{4n} - \frac{1}{4n+1}\right) = 1 - \frac{\pi}{4} - \frac{3}{4}\log(2). \tag{5.2}$$

Substituting (5.2) into (5.1), we get a nice representation of the Euler-Mascheroni constant in the form of 4—digamma function

$$\gamma = -4\psi_4(1) - \frac{\pi}{2} - \log(2). \tag{5.3}$$

Or more elegantly

$$\pi + 2\gamma = -8\psi_4(1) - 2\log(2).$$

From (4.15) and (5.3), we get some more such representations

$$\begin{aligned}\gamma &= -4\psi_4(3) + \frac{\pi}{2} - \log(2),\\ \gamma &= -4\psi_4(3) + \psi_2\left(\frac{3}{2}\right) - \psi_2\left(\frac{1}{2}\right) - \log(2),\\ \gamma &= -4\psi_4(3) + \psi_2\left(\frac{3}{2}\right) - \psi_2\left(\frac{1}{2}\right) + \frac{1}{4}\psi_{\frac{1}{2}}\left(\frac{3}{4}\right) - \frac{1}{4}\psi_{\frac{1}{2}}\left(\frac{1}{2}\right) - 1.\end{aligned} \tag{5.4}$$

Adding (5.3) and (5.4), we get such representation for π

$$\pi = 4[\psi_4(3) - \psi_4(1)].$$

Conclusion

In this paper, we established and investigated few new definite integrals involving k gamma function and k digamma function. Known results of the classical gamma function and classical digamma function were obtained as special cases of k gamma function and k digamma function. We also established nice representations of π and of the Euler-Mascheroni constant which were generated, and many more can still be obtained.

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