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ORIGINAL ARTICLE

# Some integral inequalities for logarithmically convex functions

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**Abstract** The main aim of the present note is to establish new Hadamard like integral inequalities involving log-convex function. We also prove some Hadamard-type inequalities, and applications to the special means are given.

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## 1. Introduction

Logarithmically convex (log-convex) functions are of interest in many areas of mathematics and science. They have been found to play an important role in the theory of special functions and mathematical statistics (see, e.g., [1–4]).

Let $I$ be an interval of real numbers. The function $f : I \to \mathbb{R}$ is said to be *convex*on $I$ if for all $x, y \in I$ and $t \in [0,1]$, one has the inequality:

$$f(tx + (1-t)y) \leqslant tf(x) + (1-t)f(y). \tag{1.1}$$

A function $f:I \to (0,\infty)$ is said to be *log-convex* or *multiplicatively convex* if $\log(f)$ is convex, or equivalently, if for all $x$, $y \in I$ and $t \in [0,1]$, one has the inequality (see [4, p. 7]):

$$f(tx + (1-t)y) \leqslant [f(x)]^t[f(y)]^{1-t}. \tag{1.2}$$

We note that if $f$ and $g$ are convex functions and $g$ is monotonic nondecreasing, then $gof$ is convex. Moreover, since $f = exp(\log f)$, it follows that a log-convex function is convex, but the converse is not true [4, p. 7]. This fact is obvious from (1.2) as by the arithmetic-geometric mean inequality, we have

$$[f(x)]^t[f(y)]^{1-t} \leqslant tf(x) + (1-t)f(y) \tag{1.3}$$

for all $x$, $y \in I$ and $t \in [0,1]$.

If the above inequality (1.2) is reversed, then $f$ is called *logarithmically concave*, or simply log-*concave*. Apparently, it would seem that log-concave (log-convex) functions would be unremarkable because they are simply related to concave (convex) functions. But they have some surprising properties. It is well known that the product of log-concave (log-convex) functions is also log-concave (log-convex). Moreover, the

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sum of log-convex functions is also log-convex, and a convergent sequence of log-convex (log-concave) functions has a log-convex (log-concave) limit function provided that the limit is positive. However, the sum of log-concave functions is not necessarily log-concave. Due to their interesting properties, the log-convex (log-concave) functions frequently appear in many problems of classical analysis and probability theory.

The next inequality (see for example [4, p.137]) is well known in the literature as the Hermite–Hadamard inequality

$$f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant \frac{f(a)+f(b)}{2} \tag{1.4}$$

where $f: I \to \mathbb{R}$ is a convex function on the interval $I$ of real numbers $a, b \in I$ with $a < b$.

For some recent results related to this classic result, see the books [1–4] and the papers [5–12] where further references are given.

In [7], Dragomir and Mond proved that the following inequalities of Hermite–Hadamard type hold for log-convex functions:

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &\leqslant \exp\left[\frac{1}{b-a}\int_a^b \ln[f(x)]dx\right] \\ &\leqslant \frac{1}{b-a}\int_a^b G(f(x), f(a+b-x))dx \\ &\leqslant \frac{1}{b-a}\int_a^b f(x)dx \leqslant L(f(a), f(b)) \\ &\leqslant \frac{f(a)+f(b)}{2} \end{aligned} \tag{1.5}$$

where $G(p,q) := \sqrt{pq}$ is the geometric mean and $L(p,q) := \frac{p-q}{\ln p - \ln q}(p \neq q)$ is the logarithmic mean of the positive real numbers $p$, $q$ (for $p = q$, we put $L(p,p) = p$).

In [8], Pachpatte proved that the inequalities hold for two log-convex functions:

$$\frac{4}{b-a}\int_a^b f(x)g(x)dx \leqslant [f(a)+f(b)]L(f(a), f(b)) + [g(a) + g(b)]L(g(a), g(b)) \tag{1.6}$$

In this paper, we prove another refinement of the Hermite–Hadamard Inequality for log-convex functions. Some applications for special means are also given.

Throughout this paper, we will use the following notations and conventions. Let $I \subseteq \mathbb{R} = (-\infty, +\infty)$, and $a$, $b \in I$ with $0 < a < b$ and

$$A(a,b) = \frac{a+b}{2}, \quad G(a,b) = \sqrt{ab}, \quad H = H(a,b) = \frac{2ab}{a+b},$$

$$L(a,b) = \frac{b-a}{\ln b - \ln a}, \quad K(a,b) = \sqrt{\frac{a^2+b^2}{2}}$$

be the arithmetic mean, geometric mean, harmonic mean, logarithmic mean, and quadratic mean, respectively.

## 2. Inequalities for log-convex functions

We shall start with the following refinement of the Hermite–Hadamard inequality for log-convex functions.

**Theorem 1.** *Let $f: I \to (0, \infty)$ be a log-convex function on I and $a$, $b \in I$ with $a < b$. Then, the following inequality holds:*

$$\frac{1}{b-a}\int_a^b f(x)f(a+b-x)dx \leqslant A(f(a), f(b))L(f(a), f(b)). \tag{2.1}$$

**Proof.** Since $f$ is log-convex function on $I$, we have that

$$f(ta+(1-t)b) \leqslant [f(a)]^t[f(b)]^{1-t} \tag{2.2}$$

$$f((1-t)a+tb) \leqslant [f(a)]^{1-t}[f(b)]^t \tag{2.3}$$

for all $a$, $b \in I$ and $t \in [0,1]$. It is easy to observe that

$$\frac{1}{b-a}\int_a^b f(x)f(a+b-x)dx = \int_0^1 f(ta+(1-t)b)f((1-t)a+tb)dt. \tag{2.4}$$

Using the elementary inequality $G(p,q) \leqslant K(p,q)$ $(p, q \geqslant 0$ real) and making the change of variable, we get

$$\begin{aligned} &\frac{1}{b-a}\int_a^b f(x)f(a+b-x)dx \\ &\leqslant \frac{1}{2}\int_0^1 [\{f(ta+(1-t)b)\}^2 + \{f((1-t)a+tb)\}^2]dt \\ &\leqslant \frac{1}{2}\int_0^1 [\{[f(a)]^t[f(b)]^{1-t}\}^2 + \{[f(a)]^{1-t}[f(b)]^t\}^2]dt \\ &= \frac{1}{2}\left\{f^2(b)\int_0^1 \left(\frac{f(a)}{f(b)}\right)^{2t}dt + f^2(a)\int_0^1\left(\frac{f(b)}{f(a)}\right)^{2t}dt\right\} \\ &= \frac{1}{4}\left\{f^2(b)\int_0^2 \left(\frac{f(a)}{f(b)}\right)^{u}du + f^2(a)\int_0^2\left(\frac{f(b)}{f(a)}\right)^{u}du\right\} \\ &= \frac{1}{4}\left\{f^2(b)\left[\frac{\left(\frac{f(a)}{f(b)}\right)^u}{\log\frac{f(a)}{f(b)}}\right]_0^2 + f^2(a)\left[\frac{\left(\frac{f(b)}{f(a)}\right)^u}{\log\frac{f(b)}{f(a)}}\right]_0^2\right\} \\ &= \frac{1}{4}\left\{\frac{f^2(b)\left[\frac{f^2(a)}{f^2(b)}-1\right]}{\log f(a) - \log f(b)} + \frac{f^2(a)\left[\frac{f^2(b)}{f^2(a)}-1\right]}{\log f(b) - \log f(a)}\right\} \\ &= \frac{1}{4}\left\{\frac{f^2(a)-f^2(b)}{\log f(a) - \log f(b)} + \frac{f^2(b)-f^2(a)}{\log f(b) - \log f(a)}\right\} \\ &= \frac{1}{2}\frac{(f(a)+f(b))(f(a)-f(b))}{\log f(a) - \log f(b)} \\ &= A(f(a), f(b))L(f(a), f(b)) \end{aligned} \tag{2.5}$$

Rewriting (2.5), we get the required inequality in (2.1). The proof is complete. □

The following theorem also holds.

**Theorem 2.** *Let $f: I \to (0, \infty)$ be an increasing and a log-convex function on $I$ and $a, b \in I$ with $a < b$. Then, the following inequality holds:*

$$L(f(a), f(b))f\left(\frac{a+b}{2}\right) \leqslant \frac{1}{8(b-a)}\int_a^b f^4(x)dx + \frac{1}{8}K^2(f(a), f(b))A(f(a), f(b))L(f(a), f(b)) + 1 \tag{2.6}$$

**Proof.** Since $f$ is log-convex function on $I$, we have that

$$f(ta+(1-t)b) \leqslant [f(a)]^t[f(b)]^{1-t} \tag{2.7}$$

for all $a,\ b \in I$ and $t \in [0,1]$. Using the inequality [2, p. 9], $8xy \leqslant x^4 + y^4 + 8(x, y \in \mathbb{R})$, we have that

$$8f(ta+(1-t)b)[f(a)]^t[f(b)]^{1-t} \leqslant f^4(ta+(1-t)b) + [f(a)]^{4t}[f(b)]^{4(1-t)} + 8 \tag{2.8}$$

for all $a, b \in I$, $t \in [0,1]$. Integrating this inequality (2.8) over $t$ on $[0,1]$, we get the inequality:

$$8\int_0^1 f(ta+(1-t)b)[f(a)]^t[f(b)]^{1-t}dt \leqslant \int_0^1 f^4(ta+(1-t)b)dt + \int_0^1 [f(a)]^{4t}[f(b)]^{4(1-t)}dt + 8$$

and

$$\int_0^1 f(ta+(1-t)b)dt \int_0^1 [f(a)]^t[f(b)]^{1-t}dt \leqslant \int_0^1 f(ta+(1-t)b)[f(a)]^t[f(b)]^{1-t}dt.$$

Then, we obtain

$$8\int_0^1 f(ta+(1-t)b)dt \int_0^1 [f(a)]^t[f(b)]^{1-t}dt \leqslant \int_0^1 f^4(ta+(1-t)b)dt + \int_0^1 [f(a)]^{4t}[f(b)]^{4(1-t)}dt + 8.$$

As it easy to see that:

$$\begin{aligned}
&\textstyle\int_0^1 f(ta+(1-t)b)dt = \frac{1}{b-a}\int_a^b f(x)dx,\\
&\textstyle\int_0^1 [f(a)]^t[f(b)]^{1-t}dt = \frac{f(b)-f(a)}{\ln f(b)-\ln f(a)} = L(f(a),f(b)),\\
&\textstyle\int_0^1 [f(a)]^{4t}[f(b)]^{4(1-t)}dt = f^4(b)\int_0^1 \left(\frac{f(a)}{f(b)}\right)^{4t}dt\\
&\textstyle\quad = \frac{1}{4}f^4(b)\int_0^4 \left(\frac{f(a)}{f(b)}\right)^u du\\
&\textstyle\quad = \frac{1}{4}f^4(b)\left[\frac{\left(\frac{f(a)}{f(b)}\right)^u}{\ln\frac{f(a)}{f(b)}}\right]_0^4\\
&\textstyle\quad = \frac{1}{4}\frac{f^4(a)-f^4(b)}{\ln f(a)-\ln f(b)}\\
&\textstyle\quad = \frac{f^2(a)+f^2(b)}{2}\frac{f(a)+f(b)}{2}\frac{f(a)-f(b)}{\ln f(a)-\ln f(b)},
\end{aligned}$$

respectively; then, the following inequality is obtained

$$\frac{8L(f(a),f(b))}{b-a}\int_a^b f(x)dx \leqslant \frac{1}{(b-a)}\int_a^b f^4(x)dx + K^2(f(a),f(b))A(f(a),f(b))L(f(a),f(b)) + 8$$

and by using the left half of the Hadamard's inequality given in (1.4) on the left side of the above inequalities, then, the inequality (2.6) is proved. □

**Theorem 3.** *Let $f,\ g: I \to (0,\infty)$ be increasing and log-convex functions on $I$ and $a,b \in I$ with $a < b$. Then, the following inequality holds:*

$$f\left(\frac{a+b}{2}\right)L(g(a),g(b)) + g\left(\frac{a+b}{2}\right)L(f(a),f(b)) \leqslant \frac{1}{b-a}\int_a^b f(x)g(x)dx + L(G^2(f(a),g(a)),G^2(f(b),g(b))) \tag{2.9}$$

**Proof.** Since $f,g$ are log-convex functions, we have that

$$f(ta+(1-t)b) \leqslant [f(a)]^t[f(b)]^{1-t}$$
$$g(ta+(1-t)b) \leqslant [g(a)]^t[g(b)]^{1-t}$$

for all $a,\ b \in I$, and $t \in [0,1]$. Now, using the elementary inequality [2, p. 4], $(a-b)(c-d) \geqslant 0(a,b,c,d \in \mathbb{R}$ and $a < b, c < d)$, we get inequality:

$$f(ta+(1-t)b)[g(a)]^t[g(b)]^{1-t} + g(ta+(1-t)b)[f(a)]^t[f(b)]^{1-t} \leqslant f(ta+(1-t)b)g(ta+(1-t)b) + [f(a)g(a)]^t[f(b)g(b)]^{1-t}.$$

Integrating this inequality over $t$ on $[0,1]$, we deduce that:

$$\begin{aligned}
(A:=)&\int_0^1 f(ta+(1-t)b)[g(a)]^t[g(b)]^{1-t}dt + \int_0^1 g(ta+(1-t)b)[f(a)]^t[f(b)]^{1-t}dt\\
&\leqslant (B:=)\int_0^1 f(ta+(1-t)b)g(ta+(1-t)b)dt + \int_0^1 [f(a)g(a)]^t[f(b)g(b)]^{1-t}dt.
\end{aligned}$$

Using $A$ and $B$ expressions to analyze respectively and increasing of $f$, $g$ and the left half of the Hadamard's inequality given in (1.4) on the left side of the above inequalities, we get

$$\begin{aligned}
A &= \int_0^1 f(ta+(1-t)b)[g(a)]^t[g(b)]^{1-t}dt + \int_0^1 g(ta+(1-t)b)[f(a)]^t[f(b)]^{1-t}dt\\
&\geqslant \int_0^1 f(ta+(1-t)b)dt\int_0^1 [g(a)]^t[g(b)]^{1-t}dt + \int_0^1 g(ta+(1-t)b)dt\int_0^1 [f(a)]^t[f(b)]^{1-t}dt\\
&= \frac{1}{b-a}\int_a^b f(x)dxL(g(a),g(b)) + \frac{1}{b-a}\int_a^b g(x)dxL(f(a),f(b))\\
&\geqslant f\left(\frac{a+b}{2}\right)L(g(a),g(b)) + g\left(\frac{a+b}{2}\right)L(f(a),f(b))
\end{aligned}$$

and

$$\begin{aligned}
B &= \int_0^1 f(ta+(1-t)b)g(ta+(1-t)b)dt + \int_0^1 [f(a)g(a)]^t[f(b)g(b)]^{1-t}dt\\
&= \int_0^1 f(ta+(1-t)b)g(ta+(1-t)b)dt + f(b)g(b)\int_0^1 \left[\frac{f(a)g(a)}{f(b)g(b)}\right]^t dt\\
&= \frac{1}{b-a}\int_a^b f(x)g(x)dx + L(f(a)g(a),f(b)g(b))
\end{aligned}$$

respectively, $(A \leqslant B)$; then, the inequality (2.9) is proved. □

**Theorem 4.** *Let $f: I \to (0,\infty)$ be a increasing and log-convex function on $I$ and $a,\ b \in I$ with $a < b$. Then, the following inequality holds:*

$$\frac{1}{b-a}\int_a^b f^2(x)dx + A(f(a),f(b))L(f(a),f(b)) + \Psi(a,b)$$
$$\geqslant f\left(\frac{a+b}{2}\right)L(f(a),f(b))$$
$$+ 2A(f(a),f(b))L(f(a),f(b)) - L^2(f(a),f(b))$$
$$+ \frac{f(a)}{(b-a)^2}\int_a^b (b-x)f(x)dx + \frac{f(b)}{(b-a)^2}\int_a^b (x-a)f(x)dx \tag{2.10}$$

*where* $\Psi(a,b) = f^2(a) + f(a)f(b) + f^2(b)$.

**Proof.** Since $f$ is log-convex function on $I$, we have that

$$f(ta+(1-t)b) \leqslant [f(a)]^t[f(b)]^{1-t} \leqslant tf(a) + (1-t)f(b) \tag{2.11}$$

for all $a$, $b \in I$, and $t \in [0,1]$. Using the elementary inequality [2, p. 8], $xy + yz + zx \leqslant x^2 + y^2 + z^2 (x,y,z \in \mathbb{R})$, we observe that

$$f^2(ta+(1-t)b) + [f(a)]^{2t}[f(b)]^{2(1-t)}$$
$$+ t^2f^2(a) + 2t(1-t)f(a)f(b) + (1-t)^2f^2(b)$$
$$\geqslant f(ta+(1-t)b)[f(a)]^t[f(b)]^{1-t}$$
$$+ t[f(a)]^{1+t}[f(b)]^{1-t} + (1-t)[f(a)]^t[f(b)]^{2-t}$$
$$+ f(a)tf(ta+(1-t)b) + f(b)(1-t)f(ta+(1-t)b)$$

Integrating this inequality over $t$ on $[0,1]$, we deduce that:

$$(A=)\int_0^1 f^2(ta+(1-t)b)dt + \int_0^1 [f(a)]^{2t}[f(b)]^{2(1-t)}dt$$
$$+f^2(a)\int_0^1 t^2dt + 2f(a)f(b)\int_0^1 t(1-t)dt$$
$$+f^2(b)\int_0^1 (1-t)^2dt$$
$$\geqslant (B=)\int_0^1 f(ta+(1-t)b)[f(a)]^t[f(b)]^{1-t}dt$$
$$+\int_0^1 t[f(a)]^{1+t}[f(b)]^{1-t}dt + \int_0^1 (1-t)[f(a)]^t[f(b)]^{2-t}dt$$
$$+f(a)\int_0^1 tf(ta+(1-t)b)dt + f(b)\int_0^1 (1-t)f(ta+(1-t)b)dt \tag{2.12}$$

As explained in the proof of inequality (2.9) given above, using $A$ and $B$ expressions to analyze, respectively, and increasing of $f$, and the left half of the Hadamard's inequality given in (1.4) on the left side of the above inequalities, we get

$$\int_0^1 f^2(ta+(1-t)b)dt = \frac{1}{b-a}\int_a^b f^2(x)dx,$$

By substituting $2t = u$, it is easy to observe that

$$\int_0^1 [f(a)]^{2t}[f(b)]^{2(1-t)}dt = f^2(b)\int_0^1 \left[\frac{f(a)}{f(b)}\right]^{2t}dt$$
$$= \frac{1}{2}f^2(b)\int_0^2 \left[\frac{f(a)}{f(b)}\right]^u du = \frac{1}{2}f^2(b)\left[\frac{\left(\frac{f(a)}{f(b)}\right)^u}{\ln\frac{f(a)}{f(b)}}\right]_0^2 = \frac{1}{2}f^2(b)\left[\frac{\frac{f^2(a)}{f^2(b)}-1}{\ln\frac{f(a)}{f(b)}}\right]$$
$$= \frac{1}{2}\frac{f^2(a)-f^2(b)}{\ln f(a) - \ln f(b)} = \frac{(f(a)+f(b))(f(a)-f(b))}{2(\ln f(a) - \ln f(b))}$$
$$= A(f(a),f(b))L(f(a),f(b)),$$

and

$$f^2(a)\int_0^1 t^2dt + 2f(a)f(b)\int_0^1 t(1-t)dt + f^2(b)\int_0^1 (1-t)^2dt$$
$$= \frac{f^2(a)+f(a)f(b)+f^2(b)}{3}.$$

Then, we get

$$(A=)\int_0^1 f^2(ta+(1-t)b)dt + \int_0^1 [f(a)]^{2t}[f(b)]^{2(1-t)}dt$$
$$+f^2(a)\int_0^1 t^2dt + 2f(a)f(b)\int_0^1 t(1-t)dt + f^2(b)\int_0^1 (1-t)^2dt$$
$$= \frac{1}{b-a}\int_a^b f^2(x)dx + A(f(a),f(b))L(f(a),f(b))$$
$$+ \frac{f^2(a)+f(a)f(b)+f^2(b)}{3}.$$

For proof of the right of (2.12), by using increasing of $f$ and using the left half of the Hadamard's inequality given in (1.4), we get

$$\int_0^1 f(ta+(1-t)b)[f(a)]^t[f(b)]^{1-t}dt$$
$$\geqslant \int_0^1 f(ta+(1-t)b)dt\int_0^1 [f(a)]^t[f(b)]^{1-t}dt$$
$$= \frac{1}{b-a}\int_a^b f(x)dx\frac{f(a)-f(b)}{\ln f(a) - \ln f(b)}$$
$$\geqslant f\left(\frac{a+b}{2}\right)L(f(a),f(b)),$$

By applying a simple integration by parts formula, $\left(t = u, \left(\frac{f(a)}{f(b)}\right)^t dt = dv\right)$ it is easy to observe that:

$$\int_0^1 t[f(a)]^{1+t}[f(b)]^{1-t}dt$$
$$= f(a)f(b)\int_0^1 t\left(\frac{f(a)}{f(b)}\right)^t dt$$
$$= f(a)f(b)\left\{\left[\frac{t\left(\frac{f(a)}{f(b)}\right)^t}{\ln\frac{f(a)}{f(b)}}\right]_0^1 - \frac{1}{\ln\frac{f(a)}{f(b)}}\int_0^1 \left(\frac{f(a)}{f(b)}\right)^t dt\right\}$$
$$= f(a)f(b)\left\{\frac{\frac{f(a)}{f(b)}}{\ln\frac{f(a)}{f(b)}} - \frac{1}{\ln\frac{f(a)}{f(b)}}\left[\frac{\left(\frac{f(a)}{f(b)}\right)^t}{\ln\frac{f(a)}{f(b)}}\right]_0^1\right\}$$
$$= f(a)f(b)\left\{\frac{\frac{f(a)}{f(b)}}{\ln\frac{f(a)}{f(b)}} - \frac{\frac{f(a)}{f(b)}-1}{\left(\ln\frac{f(a)}{f(b)}\right)^2}\right\}$$
$$= \frac{f^2(a)}{\ln\frac{f(a)}{f(b)}} - \frac{f^2(a)-f(a)f(b)}{\left(\ln\frac{f(a)}{f(b)}\right)^2} \tag{2.13}$$

Similarly, $\left(1-t = u, \left(\frac{f(a)}{f(b)}\right)^t dt = dv\right)$ it is easy to observe that:

$$\int_0^1 (1-t)[f(a)]^t[f(b)]^{2-t}dt$$
$$= f^2(b)\int_0^1 (1-t)\left(\frac{f(a)}{f(b)}\right)^t dt$$

$$= f^2(b)\left\{\left[\frac{(1-t)\left(\frac{f(a)}{f(b)}\right)^t}{\ln\frac{f(a)}{f(b)}}\right]_0^1 + \frac{1}{\ln\frac{f(a)}{f(b)}}\int_0^1\left(\frac{f(a)}{f(b)}\right)^t dt\right\}$$

$$= f^2(b)\left\{\frac{-1}{\ln\frac{f(a)}{f(b)}} + \frac{1}{\ln\frac{f(a)}{f(b)}}\left[\frac{\left(\frac{f(a)}{f(b)}\right)^t}{\ln\frac{f(a)}{f(b)}}\right]_0^1\right\}$$

$$= f^2(b)\left\{\frac{-1}{\ln\frac{f(a)}{f(b)}} + \frac{\frac{f(a)}{f(b)}-1}{\left(\ln\frac{f(a)}{f(b)}\right)^2}\right\}$$

$$= \frac{-f^2(b)}{\ln\frac{f(a)}{f(b)}} + \frac{f(a)f(b)-f^2(b)}{\left(\ln\frac{f(a)}{f(b)}\right)^2} \tag{2.14}$$

So by adding these equalities (2.13) and (2.14), we find that

$$\int_0^1 t[f(a)]^{1+t}[f(b)]^{1-t}dt + \int_0^1(1-t)[f(a)]^t[f(b)]^{2-t}dt$$

$$= \frac{f^2(a)}{\ln\frac{f(a)}{f(b)}} - \frac{f^2(a)-f(a)f(b)}{\left(\ln\frac{f(a)}{f(b)}\right)^2} + \frac{-f^2(b)}{\ln\frac{f(a)}{f(b)}} + \frac{f(a)f(b)-f^2(b)}{\left(\ln\frac{f(a)}{f(b)}\right)^2}$$

$$= \frac{f^2(a)-f^2(b)}{\ln f(a)-\ln f(b)} - \left(\frac{f(a)-f(b)}{\ln f(a)-\ln f(b)}\right)^2$$

$$= \frac{2(f(a)+f(b))(f(a)-f(b))}{2(\ln f(a)-\ln f(b))} - \left(\frac{f(a)-f(b)}{\ln f(a)-\ln f(b)}\right)^2$$

$$= 2A(f(a),f(b))L(f(a),f(b)) - L^2(f(a),f(b))$$

And by substituting $ta+(1-t)b = x$, it is easy to observe that

$$f(a)\int_0^1 tf(ta+(1-t)b)dt = \frac{f(a)}{(b-a)^2}\int_a^b(b-x)f(x)dx$$

$$f(b)\int_0^1(1-t)f(ta+(1-t)b)dt = \frac{f(b)}{(b-a)^2}\int_a^b(x-a)f(x)dx$$

When above equalities and inequalities are taken into account, $(A \geqslant B)$, then the inequality (2.10) is proved. □

## 3. Applications

The function $f(x) = \frac{1}{x}, x \in (0,\infty)$ is log-convex on $(0,\infty)$. Then we have

$$\frac{1}{b-a}\int_a^b\frac{dx}{x} = L^{-1}(a,b)$$

$$f\left(\frac{a+b}{2}\right) = A^{-1}(a,b)$$

$$\frac{f(a)+f(b)}{2} = H^{-1}(a,b).$$

Now, applying the inequality (2.1) for the function $f(x) = \frac{1}{x}$, we get the inequality:

$$\frac{1}{b-a}\int_a^b f(x)f(a+b-x)dx = \frac{1}{b-a}\int_a^b\frac{dx}{x(a+b-x)}$$

$$= \frac{1}{(b-a)(a+b)}\int_a^b\left\{\frac{1}{x}+\frac{1}{a+b-x}\right\}dx$$

$$= \frac{2}{a+b}L^{-1}(a,b) = A^{-1}(a,b)L^{-1}(a,b)$$

$$\leqslant A(f(a),f(b))L(f(a),f(b)) = \frac{\frac{1}{a}+\frac{1}{b}}{2}\cdot\frac{\frac{1}{a}-\frac{1}{b}}{\ln\frac{1}{a}-\ln\frac{1}{b}}$$

$$= \frac{a+b}{2ab}\frac{b-a}{ab\ln\frac{b}{a}} = \frac{L(a,b)}{G^2(a,b)H(a,b)} = \frac{A(a,b)L(a,b)}{G^4(a,b)}. \tag{3.1}$$

Rewriting (3.1), we get the fascinating inequalities:

$$G^4(a,b) \leqslant A(a,b)L(a,b)$$

or

$$G^2(a,b)H(a,b) \leqslant A^2(a,b)L^2(a,b).$$

Similar inequalities may be stated for the log-convex functions $f(x) = x^x,\ x > 0$ or $f(x) = e^x+1, x \in \mathbb{R}$, etc. We omit the details.

## References

[1] S.S. Dragomir, C.E.M. Pearce, Selected Topics on Hermite–Hadamard Inequalities and Applications, RGMIA Monographs, Victoria University, 2000.
[2] R.B. Manfrino, R.V. Delgado, J.A.G. Ortega, Inequalities a Mathematical Olympiad Approach, Birkhäuser, 2009.
[3] D.S. Mitrinović, J.E. Pečarić, A.M. Fink, Classical and New Inequalities in Analysis, Kluwer Academic Publishers, Dordrecht, 1993.
[4] J.E. Pečarić, F. Proschan, Y.L. Tong, Convex Functions, Partial Orderings and Statistical Applications, Academic Press, 1991.
[5] S.S. Dragomir, J.E. Pečarić, J. Sandor, A note on the Jensen–Hadamard inequality, Anal. Num. Theor. Approx. 19 (1990) 29–34.
[6] U.S. Kırmacı, M.E. Özdemir, Some inequalities for mappings whose derivatives are bounded and applications to specials means of real numbers, Appl. Math. Lett. 17 (2004) 641–645.
[7] S.S. Dragomir, B. Mond, Integral inequalities of Hadamard type for log-convex functions, Demonstratio Math. 31 (2) (1998) 354–364.
[8] B.G. Pachpatte, A note on integral inequalities involving two log-convex functions, Math. Ineq. Appl. 7 (4) (2004) 511–515.
[9] S.S. Dragomir, Two functions in connection to Hadamard's inequalities, J. Math. Anal. Appl. 167 (1992) 49–56.
[10] S.S. Dragomir, Some remarks on Hadamard's inequalities for convex functions, Extracta Math. 9 (2) (1994) 88–94.
[11] S.S. Dragomir, Refinements of the Hermite–Hadamard integral inequality for log-convex functions, RGMIA Res. Rep. Collect. 3 (4) (2000) 527–533.
[12] M.E. Özdemir, A theorem on mappings with bounded derivatives with applications to quadrature rules and means, Appl. Math. Lett. 13 (2000) 19–25.