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Fixed point theorems for self and non-self F-contractions in metric spaces endowed with a graph

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Abstract

The main results obtained in this paper are fixed point theorems for self and non-self G F-contractions on metric spaces endowed with a graph. Our new results are generalization of recently fixed point theorems for self mappings on metric spaces and also fixed point theorems for non-self mappings in Banach spaces by using the concept of new type of contractive mappings namely F-contractions.

Keywords: Fixed point, F-contraction, Non-self G F-contraction, Directed graph, G F-contraction, Non-self mapping, Rothe's boundary condition

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Introduction

The well-known Banach contraction theorem [1] has plenty of extensions in the literature (see, for example, [2, 3]). That theorem states that every self-mapping f defined on complete metric space (S, d) satisfying

$$d\left(fr, fs\right) \leq \alpha d\left(r, s\right) \ \forall \ r, s \in S, \tag{1}$$

where $\alpha \in (0, 1)$ has a unique fixed point, i.e., there exists a unique $r^* \in S$ such that $fr^* = r^*$.

The extension of Banach contraction theorem for non-self multi-valued mappings was first studied by Assad and Kirk [4] in 1972. After this initiation, lot of fixed point theorems for non-self mappings have been proved by various authors, see, for example, [5–7] and [8].

Firstly, the study of fixed point theorem for single-valued monotone mappings in a metric space endowed with a partial ordering has been investigated by Ran and Reurings [9] and presented its applications to matrix equations. After this, many results in this direction were studied by different authors; see [10–12] and [13]. These theorems are actually hybrids of two fundamental theorems of fixed point theory: the Kanaster-Tarski theorem [14] and the Banach Contraction Principle. Jachymski [15] established the fixed point theorems by using graphs which is the generalization of concept of partial ordering in

metric spaces. Jachymski [15, Theorem 3.2] generalized the Banach contraction theorem for self-mappings on complete metric spaces endowed with the graph, where as Berinde [16, Theorem 3.1] for non-self mappings to Banach spaces endowed with a graph by using the inwardness condition defined in [17]. There are few other fixed point theorems for non-self mappings to Banach spaces endowed with a graph , see, for example, [18] and [19].

Recently, Wardowski [20] introduced a new type of contraction by using a particular function $F : \mathbb{R}^+ \to \mathbb{R}$ called F-contraction and gave examples to show the validity of such extensions in complete metric spaces. The author proved a new fixed point theorem by using this concept of F-contraction.

This paper has been organized in the following manner: In the "Preliminaries" section, we will give the brief introduction of a new type of contraction called F-contraction. In the last section, we present a few preliminary notations and our main aim is to study the fixed point theorems for self-mappings as well as non-self mappings using F-contractions for metric spaces endowed with a graph. These theorems are the generalization of fixed point theorems discussed by Berinde [16] on Banach spaces endowed with a graph and Wardowski [20] on complete metric spaces.

Preliminaries

In this section, we present some definitions, examples and results from [20], which will be used in this article. Throughout this paper, consider $\mathbb{R}$ be the set of all real numbers, $\mathbb{R}^+$ be the set of all positive real numbers and $\mathbb{N}$ be the set of all positive integers.

Definition 1 *Let the mapping $F : \mathbb{R}^+ \to \mathbb{R}$ satisfies the following conditions:*

(f1)   *F is strictly increasing;*
(f2)   *for each sequence $\{r_n\} \subset \mathbb{R}^+$ $\lim_{n \to +\infty} r_n = 0$ iff $\lim_{n \to +\infty} F(r_n) = -\infty$;*
(f3)   *there exists $k \in (0, 1)$ provided that $\lim_{\lambda \to 0^+} \lambda^k F(\lambda) = 0$.*

The collection of all such mappings is denoted by Ω.

Definition 2 *Let (S, d) be a metric space. A mapping $\Upsilon : S \to S$ is said to be F-contraction if there exist $F \in \Omega$ and $\tau > 0$ provided that*

$$d(\Upsilon s, \Upsilon r) > 0 \implies \tau + F(d(\Upsilon s, \Upsilon r)) \leq F(d(s, r)), \tag{2}$$

for all $r, s \in S$.

Example 1 *Let $F \in \Omega$ be defined by $F(\alpha) = \ln \alpha$. For any $k \in (0, 1)$, it is clear that every mapping $\Upsilon : S \to S$ satisfying (2) is an F-contraction such that*

$$d(\Upsilon s, \Upsilon r) \leq e^{-\tau} d(r, s) \; \forall r, s \in S, \; \Upsilon r \neq \Upsilon s.$$

Example 2 *Consider $F \in \Omega$ be defined by $F(\alpha) = \dfrac{-1}{\sqrt{\alpha}}, \alpha > 0$. In this case, for any $k \in (1/2, 1)$, every F-contraction Υ satisfies*

$$d(\Upsilon r, \Upsilon s) \leq \frac{1}{\left(1 + \tau \sqrt{d(r, s)}\right)^2} d(r, s) \; \forall \, r, s \in S, \; \Upsilon r \neq \Upsilon s.$$

Wardowski stated the *F*-contraction theorem for self mappings in complete metric spaces as follows.

Theorem 1 *Let a mapping* $T : S \rightarrow S$ *be an F-contraction and* (S, d) *be a complete metric space. Then, T has a unique fixed point* $s^* \in S$ *and for every* $s \in S$ *the sequence* $(T^n s)_{n \in \mathbb{N}}$ *converges to* s^*.

Remark 1 *From (f1) and (2), we can conclude that every F-contraction* Υ *is a contractive mappping, i.e.,*

$$d(\Upsilon r, \Upsilon s) \leq d(r, s) \text{ for all } r, s \in S \text{ and } \Upsilon r \neq \Upsilon s. \tag{3}$$

Thus, every F-contraction is continuous mapping.

Fixed point theorems in metric spaces endowed with a graph

By using the concept of *F*-contractions, we establish fixed point theorems for self as well as non-self mappings in complete metric spaces endowed with a graph.

Some graph theory terminologies will be presented here. Let (S, d) be metric space and $\triangle$ denote the diagonal of Cartesian product $S \times S$. Let $G = (V(G), E(G))$ be a directed graph such that $E(G)$, the set of its edges consists of all loops, that is, $\triangle \subset E(G)$ and $V(G)$, the vertex set coincides with S. Let G has no parallel edges (arcs). For more details of these terminologies and notations see [21] and [22].

G^{-1} is the converse graph of G, i.e., the edge set of G^{-1} is obtained by reversing the edges of G, defined as:

$$E(G^{-1}) = \{(r, s) \in S \times S : (s, r) \in E(G)\}.$$

If s, r are vertices in the graph G, then a path from s to r of length t is a sequence $\{s_i\}_{i=1}^{t}$ of $t + 1$ vertices of G such that $s_0 = s$, $s_t = r$ and $(s_{i-1}, s_i) \in E(G)$, $i = 1, 2 \cdots t$.

A graph G is called connected if there exist at least a path between two arbitrary vertices. If $\tilde{G} = \left(S, E\left(\tilde{G}\right)\right)$ is the symmetric graph obtained by placing together the vertices of both G and G^{-1}, that is,

$$E\left(\tilde{G}\right) = E(G) \cup E\left(G^{-1}\right),$$

then G is said to be weakly connected whenever $\tilde{G}$ is connected.

If $G = (V(G), E(G))$ is a graph and $V(G) \supset H$, then the graph $(H, E(G))$ with

$$E(H) = E(G) \cap (H \times H)$$

is said to be the subgraph of G determined by H, denoted by G_H.

Self *F*-contraction case

A mapping $\Upsilon : S \rightarrow S$ is said to be defined on a metric space endowed with a graph G if it satisfies

$$\forall r, s \in S, \ (r, s) \in E(G) \text{ implies } (\Upsilon r, \Upsilon s) \in E(G). \tag{4}$$

A mapping $\Upsilon : S \rightarrow S$ defined on metric space endowed with a graph G, is said to be a G *F*-contraction, if there is a constant $\tau > 0$ such that $\forall r, s \in S$ with $(r, s) \in E(G)$, we have

$$[d(\Upsilon r, \Upsilon s) > 0 \implies \tau + F(d(\Upsilon r, \Upsilon s)) \leq F(d(r, s))]. \tag{5}$$

If $\Upsilon r = r$, then the element $r \in S$ is said to be the fixed point of mapping Υ.

Theorem 2 *Suppose (S, d, G) be a complete metric space endowed with a weakly connected and directed graph G such that the following property (T) holds, that is, for any sequence $\{r_n\}_{n=1}^{\infty} \subset S$ with $r_n \rightarrow r$ as $n \rightarrow \infty$ and $(r_n, r_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, there exists a subsequence $\{r_{s_n}\}_{n=1}^{\infty}$ satisfying*

$$\left(r_{s_n}, r\right) \in E(G), \ \forall n \in \mathbb{N}. \tag{6}$$

Let $\Upsilon : S \rightarrow S$ be a G F-contraction. If the set

$$S_{\Upsilon} = \{r \in S : (r, \Upsilon r) \in E(G)\} \tag{7}$$

is nonempty, then the mapping Υ has a unique fixed point in S.

Proof Let $r_0 \in S_{\Upsilon}$. It follows from (7) that $(r_0, \Upsilon r_0) \in E(G)$ and by using (4), we obtain

$$\left(\Upsilon^n r_0, \Upsilon^{n+1} r_0\right) \in E(G), \ \forall n \in \mathbb{N}. \tag{8}$$

Denote $r_n := \Upsilon^n r_0$ for all $n \in \mathbb{N}$. Then, by the fact that Υ is a $G F$-contraction and in view of (4), we get

$$F(d(r_n, r_{n+1})) \leq F(d(r_{n-1}, r_n)) - \tau, \tag{9}$$

for all $n \in \mathbb{N}$. Denote $\alpha_n = d(r_n, r_{n+1})$, $n = 0, 1, \ldots$

Let $r_{n+1} \neq r_n$, for every $n \in \mathbb{N} \cup \{0\}$. Then, $\alpha_n > 0$ for all $n \in \mathbb{N} \cup \{0\}$ and by using (2), we get

$$F(\alpha_n) \leq F(\alpha_{n-1}) - \tau \leq F(\alpha_{n-2}) - 2\tau \leq \cdots \leq F(\alpha_0) - n\tau. \tag{10}$$

Hence, $\lim_{n \to \infty} F(\alpha_n) = -\infty$. By the property (f2), we obtain that $\alpha_n \rightarrow 0$ as $n \rightarrow \infty$. From (f3), there exists $k \in (0, 1)$ such that $\lim_{n \to \infty} \alpha_n^k F(\alpha_n) = 0$. By (10), the following holds for all $n \in \mathbb{N}$

$$\alpha_n^k F(\alpha_n) - \alpha_n^k F(\alpha_0) \leq \alpha_n^k (F(\alpha_0) - n\tau) - \alpha_n^k F(\alpha_0) = -\alpha_n^k n\tau. \tag{11}$$

Letting $n \rightarrow \infty$ in (11), we deduce $\lim_{n \to \infty} n\alpha_n^k = 0$. From (11), we observe that there exists $n' \in \mathbb{N}$ such that $n\alpha_n^k \leq 1$ for all $n \geq n'$. Consequently, we have

$$\alpha_n \leq \frac{1}{n^{1/k}} \text{ for all } n \geq n'. \tag{12}$$

Choose $m, n \in \mathbb{N}$ such that $m \geq n \geq n'$ and from (12), we have

$$d(r_m, r_n) \leq \alpha_{m-1} + \cdots + \alpha_n < \sum_{j=n}^{\infty} \alpha_n \leq \sum_{j=n}^{\infty} \frac{1}{j^{1/k}}.$$

The convergence of the series $\sum_{j=n}^{\infty} \frac{1}{j^{1/k}}$ implies that $\{r_n\}$ is a Cauchy sequence, hence convergent in (S, d, G). The limit of this sequence is denoted as:

$$\lim_{n \to \infty} r_n = r^*. \tag{13}$$

By using property (T) of (S, d, G), there exists a subsequence $\{r_{s_n}\}$ satisfying

$$\left(r_{s_n}, r^*\right) \in E(G), \ \forall n \in \mathbb{N}.$$

Hence, by inequality (5) and in view of (4), we get

$$F\left(d\left(\Upsilon r_{s_n}, \Upsilon r^*\right)\right) \leq Fd\left(r_{s_n}, r^*\right) - \tau < F\left(d\left(r_{s_n}, r^*\right)\right), \tag{14}$$

which implies

$$d\left(\Upsilon r_{s_n}, \Upsilon r^*\right) \le d\left(r_{s_n}, r^*\right). \tag{15}$$

Therefore, by triangle inequality, we have

$$
\begin{aligned}
d\left(r^*, \Upsilon r^*\right) &\le d\left(r^*, r_{s_n+1}\right) + d\left(r_{s_n+1}, \Upsilon r^*\right) \\
&= d\left(r^*, r_{s_n+1}\right) + d\left(\Upsilon r_{s_n}, \Upsilon r^*\right).
\end{aligned}
\tag{16}
$$

By using (15), inequality (16) yields

$$d\left(r^*, \Upsilon r^*\right) \le d\left(r^*, r_{s_n+1}\right) + d\left(r_{s_n}, r^*\right), \tag{17}$$

for all $n \ge 1$. In Eq. (17), assuming $n \to \infty$ and using (13), we have $d\left(r^*, \Upsilon r^*\right) = 0$, which implies $r^* = \Upsilon r^*$, i.e., r^* a fixed point of mapping Υ.

Note that the uniqueness of r^* follows by the $G F$-contraction condition (5).

Remark *If we use the mapping $F \in \Omega$ defined by the formula $F(\alpha) = \ln \alpha$ in Theorem 2, then for all $k \in (0, 1)$, we obtain the extension of* [16, Theorem 2.1]. $\qquad\square$

Example *Let (S, d) be the complete metric space and G be the complete graph on the set S, that is, $E(G) = S \times S$. Let the mapping $F \in \Omega$ be defined as: $F(\alpha) = \ln \alpha$, then the G F-contraction (5) is actually a F-contraction (2) which reduces to Banach contraction, i.e.,*

$$d\left(\Upsilon r, \Upsilon s\right) \le e^{-\tau} d\left(\Upsilon r, \Upsilon s\right), \text{for all } r, s \in S, \ \Upsilon r \ne \Upsilon s,$$

for any $k \in (0, 1)$ and $\tau > 0$. $\qquad\square$

Non-self *F*-contraction case

Let S be a Banach space, A be a nonempty, closed subset of S and $\Upsilon : A \to S$ be a non-self mapping. We choose $r \in A$ such that $\Upsilon r \notin A$, then there is an element $s \in \partial A$ such that

$$s = (1 - \mu) r + \mu \Upsilon r \text{ where } \mu \in (0, 1),$$

which represents the fact that

$$d(r, \Upsilon r) = d(r, s) + d(s, \Upsilon r), \ s \in \partial A \tag{18}$$

where $d(r, s) = \|r - s\|$.

Caristi [17] used a condition related to (18), called inward condition, to get the generalization of Banach contraction theorem for non-self mappings. The inward condition is more general because it does not need s in (18) to belong to ∂A.

A non-self mapping $\Upsilon : A \to S$ is said to be defined on the Banach space S endowed with a graph G, if it satisfies the property that

$$
\begin{aligned}
&\text{for all } r, s \in A \ (r, s) \in E(G) \\
&\text{with } \Upsilon r, \Upsilon s \in A, \text{ implies } (\Upsilon r, \Upsilon s) \in E(G) \cap (A \times A),
\end{aligned}
\tag{19}
$$

for the subgraph of G induced by A.

Theorem 3 *Suppose (S, d, G) be a Banach space endowed with a weakly connected and directed graph G provided that following property (T) holds, that is, for any sequence $\{r_n\} \subset S$ along with $r_n \to r$ as $n \to \infty$ and*

$$(r_n, r_{n+1}) \in E(G), \ \forall n \in \mathbb{N},$$

there exists a subsequence $\{r_{s_n}\}$ satisfying

$$\left(r_{s_n}, r\right) \in E\left(G\right), \forall\, n \in \mathbb{N}. \tag{20}$$

Let A be a nonempty, closed subset of S and $\Upsilon : A \to S$ be a G_A F-contraction, that is, there exists a constant $\tau > 0$ such that

$$\tau + F\left(d\left(\Upsilon r, \Upsilon s\right)\right) \leq F\left(d\left(r, s\right)\right) \text{ for all } (r, s) \in E\left(G_A\right), \tag{21}$$

where G_A is the subgraph of G determined by A. If the set

$$A_\Upsilon := \{r \in \partial A : (r, \Upsilon r) \in E\left(G\right)\}$$

is nonempty and Υ satisfies Rothe's boundary condition

$$\Upsilon\left(\partial A\right) \subset A, \tag{22}$$

then the mapping Υ has a unique fixed point.

Proof If $\Upsilon(A) \subset A$, then Υ is a self-map of the closed set A and the conclusion follows by Theorem 2. Now, we consider the case that $\Upsilon(A) \not\subset A$. Let $r_0 \in A_\Upsilon$. It follows that $(r_0, \Upsilon r_0) \in E\left(G\right)$ and in view of equation (4), we have

$$\left(\Upsilon^n r_0, \Upsilon^{n+1} r_0\right) \in E\left(G\right), \text{ for all } n \in \mathbb{N}. \tag{23}$$

Let we denote $r_n := \Upsilon^n r_0$, for all $n \in \mathbb{N}$. By virtue of (22) $\Upsilon r_0 \in A$.

Consider $r_1 \equiv s_1 = \Upsilon r_0$. Let $\Upsilon r_1 \in A$, set $r_2 \equiv s_2 = \Upsilon r_1$. If $\Upsilon r_1 \notin A$, then we can select an element $r_2 \in \partial A$ on the segment $[r_1, \Upsilon r_1]$, that is,

$$r_2 = (1 - \mu)\, r_1 + \mu \Upsilon r_1, \text{ where } \mu \in (0, 1).$$

By following the same method, we obtain two sequences $\{r_n\}$ and $\{s_n\}$ whose terms satisfy one of the succeeding properties:

(i) $r_n \equiv s_n = \Upsilon r_{n-1}$, if $\Upsilon r_{n-1} \in A$;

(ii) $r_n = (1 - \mu)\, r_{n-1} + \mu \Upsilon r_{n-1} \in \partial A, \mu \in (0, 1), \Upsilon r_{n-1} \notin A$.

For the simplicity of arguments in the proof, let us denote

$$U = \{r_a \in \{r_n\} : r_a = s_a = \Upsilon r_{a-1}\}$$

and

$$Z = \{r_a \in \{r_n\} : r_a \neq \Upsilon r_{a-1}\}.$$

Note that $\{r_n\} \subset A$ for all $n \in \mathbb{N}$. Moreover, if $r_a \in Z$, then both r_{a-1} and r_{a+1} belong to set U. The sequence $\{r_n\}$ can have consecutive terms r_a and r_{a+1} in set U, but this assertion is not true for the set Z. First of all we have to prove that

$$r_a \neq \Upsilon r_{a-1} \text{ implies } r_{a-1} = \Upsilon r_{a-2}.$$

Suppose contrary that $r_{a-1} \neq \Upsilon r_{a-2}$ then $r_{a-1} \in \partial A$. Since $\Upsilon\left(\partial A\right) \subset A$ then $\Upsilon r_{a-1} \in A$. Hence, $r_a = \Upsilon r_{a-1}$ which is a contradiction.

Here, we have three different cases to show that $\{r_n\}$ is Cauchy sequence which are following: □

Case 1. $r_n, r_{n+1} \in U$.

Since both elements belong to set U, therefore, we have $r_n = s_n = \Upsilon r_{n-1}$ and $r_{n+1} = s_{n+1} = \Upsilon r_n$. Hence,

$$d\left(r_{n+1}, r_n\right) = d\left(s_{n+1}, s_n\right) = d\left(\Upsilon s_n, \Upsilon s_{n-1}\right),$$

where $(s_n, s_{n-1}) \in E(G)$ by virtue of (23). Therefore, we have

$$d\left(\Upsilon s_n, \Upsilon s_{n-1}\right) = d\left(\Upsilon r_n, \Upsilon r_{n-1}\right) > 0.$$

Consequently, we get the following inequality

$$\tau + F\left(d\left(\Upsilon s_n, \Upsilon s_{n-1}\right)\right) \le F\left(d\left(s_n, s_{n-1}\right)\right), \tag{24}$$

by using (21).

Case 2. $r_n \in U, r_{n+1} \in Z$.

In this case, we have $r_n = s_n = \Upsilon r_{n-1}$, but $r_{n+1} \neq s_{n+1} = \Upsilon r_n$; therefore, we have

$$d\left(r_n, \Upsilon r_n\right) = d\left(r_n, r_{n+1}\right) + d\left(r_{n+1}, \Upsilon r_n\right).$$

The above equality implies $d\left(r_{n+1}, \Upsilon r_n\right) \neq 0$ and hence

$$d\left(r_n, r_{n+1}\right) = d\left(r_n, \Upsilon r_n\right) - d\left(r_{n+1}, \Upsilon r_n\right) < d\left(r_n, \Upsilon r_n\right) = d\left(\Upsilon r_{n-1}, \Upsilon r_n\right), \tag{25}$$

since $r_n \in U$. By using (25), we obtain

$$d\left(r_n, r_{n+1}\right) < d\left(\Upsilon r_{n-1}, \Upsilon r_n\right) = d\left(\Upsilon s_{n-1}, \Upsilon s_n\right) > 0.$$

We can obtain again inequality (24) by using the similar arguments to that in case 1.

Case 3. $r_n \in Z, r_{n+1} \in U$.

In this case, we have $r_{n+1} = \Upsilon r_n$, and $r_n \neq s_n = \Upsilon r_{n-1}$. Since $r_n \in Z$, so we have

$$d\left(r_{n-1}, \Upsilon r_{n-1}\right) = d\left(r_{n-1}, r_n\right) + d\left(r_n, \Upsilon r_{n-1}\right). \tag{26}$$

Hence, by triangle inequality

$$\begin{aligned} d\left(r_n, r_{n+1}\right) &\le d\left(r_n, \Upsilon r_{n-1}\right) + d(\Upsilon r_{n-1}, r_{n+1}) \\ &= d\left(r_n, \Upsilon r_{n-1}\right) + d(\Upsilon r_{n-1}, \Upsilon r_n) \\ &= d\left(r_n, \Upsilon r_{n-1}\right) + d(\Upsilon s_{n-1}, \Upsilon s_n). \end{aligned} \tag{27}$$

By virtue of (23) $(s_{n-1}, s_n) \in E(G)$, and the following inequality is obtained by the contraction condition (21)

$$F\left(d\left(\Upsilon s_{n-1}, \Upsilon s_n\right)\right) \le F\left(d\left(s_{n-1}, s_n\right)\right) - \tau < F\left(d\left(s_{n-1}, s_n\right)\right), \tag{28}$$

which implies

$$d(\Upsilon s_{n-1}, \Upsilon s_n) \le d\left(s_{n-1}, s_n\right) = d\left(r_{n-1}, r_n\right). \tag{29}$$

Thus, by using (26) and (29) in inequality (27), we have

$$\begin{aligned} d\left(r_n, r_{n+1}\right) &\le d\left(r_n, \Upsilon r_{n-1}\right) + d(\Upsilon s_{n-1}, \Upsilon s_n) \\ &< d\left(r_n, \Upsilon r_{n-1}\right) + d(r_{n-1}, r_n) \\ &= d\left(r_{n-1}, \Upsilon r_{n-1}\right). \end{aligned}$$

By using (23), $(r_{n-2}, r_{n-1}) = (s_{n-2}, s_{n-1}) \in E(G)$ and by virtue of contraction condition (21), we get

$$d\left(r_n, r_{n+1}\right) < d\left(r_{n-1}, \Upsilon r_{n-1}\right) = d\left(\Upsilon r_{n-2}, \Upsilon r_{n-1}\right) \le d\left(r_{n-2}, r_{n-1}\right). \tag{30}$$

Now, we summarize all the above mentioned three cases. By virtue of (24) and (30), it follows that the sequence $\{d\,(r_n, r_{n+1})\}$ satisfies the inequality

$$\tau + F\,(\max\{d\,(r_{n-2}, r_{n-1})\,, d\,(r_{n-1}, r_n)\}) \leq F\,(d\,(r_n, r_{n+1}))\,, \tag{31}$$

for all $n \geq 2$. Denote $\alpha_n = d\,(r_n, r_{n+1})$ for $n = 2, 3, \cdots$.

We obtain the following inequality by simple induction for $n \geq 2$, and using (31)

$$F\,(\alpha_n) \leq F\,(\max\{\alpha_0, \alpha_1\}) - \left[\frac{n}{2}\right]\tau, \tag{32}$$

where $\left[\frac{n}{2}\right]$ denotes the greatest integer not exceeding $\frac{n}{2}$.

Hence, $\lim_{n\to\infty} F\,(\alpha_n) = -\infty$. By the property (f2), we obtain that $\alpha_n \to 0$ as $n \to \infty$. From (f3), there exists $k \in (0, 1)$ such that $\lim_{n\to\infty} \alpha_n^k F\,(\alpha_n) = 0$. Denote $\gamma = \max\{\alpha_0, \alpha_1\}$. By (32), the following holds for all $n \geq 2$:

$$\alpha_n^k F\,(\alpha_n) - \alpha_n^k F\,(\gamma) \leq \alpha_n^k \left(F\,(\gamma) - \left[\frac{n}{2}\right]\tau\right) - \alpha_n^k F\,(\gamma) = -\alpha_n^k \left[\frac{n}{2}\right]\tau. \tag{33}$$

Assuming $n \to \infty$ in (33), we deduce $\lim_{n\to\infty} \left[\frac{n}{2}\right]\alpha_n^k = 0$. From (33), we observe that there exists $n' \in \mathbb{N}$ such that $\left[\frac{n}{2}\right]\alpha_n^k < n\alpha_n^k \leq 1$ for all $n \geq n'$. Consequently, we have

$$\alpha_n \leq \frac{1}{n^{1/k}} \text{ for all } n \geq n'. \tag{34}$$

Choose $m, n \in \mathbb{N}$ such that $m \geq n \geq n'$ and from (34), we have

$$d\,(r_m, r_n) \leq \alpha_{m-1} + \cdots + \alpha_n < \sum_{j=n}^{\infty} \alpha_n \leq \sum_{j=n}^{\infty} \frac{1}{j^{1/k}}.$$

The convergence of the series $\sum_{j=n}^{\infty} \frac{1}{j^{1/k}}$ implies that $\{r_n\}$ is a Cauchy sequence, hence convergent in (S, d, G). Since $\{r_n\} \subset A$ and A is closed, $\{r_n\}$ converges to some point $r' \in A$, i.e., $\lim_{n\to\infty} r_n = r'$.

By property (T), there exists a subsequence $\{r_{s_n}\}$ satisfying

$$\left(r_{s_n}, r'\right) \in E\,(G)\,, \text{ for all } n \in \mathbb{N}.$$

Hence, by the F-contraction condition (21), we get

$$d\left(\Upsilon r_{s_n}, \Upsilon r'\right) \leq d\left(r_{s_n}, r'\right). \tag{35}$$

Therefore, by triangle inequality, we have

$$\begin{aligned} d\left(r', \Upsilon r'\right) &\leq d\left(r', r_{s_n+1}\right) + d\left(r_{s_n+1}, \Upsilon r'\right) \\ &= d\left(r', r_{s_n+1}\right) + d\left(\Upsilon r_{s_n}, \Upsilon r'\right). \end{aligned}$$

By using (35), the above inequality yields

$$d\left(r', \Upsilon r'\right) \leq d\left(r', r_{s_n+1}\right) + d\left(r_{s_n}, r'\right), \tag{36}$$

for all $n \geq 1$. Taking limit $n \to \infty$ and using (36), we obtain $d\left(r', \Upsilon r'\right) = 0$ and get $r' = \Upsilon r'$, which shows that r' is a fixed point of Υ.

The uniqueness of r^* immediately follows by the G_A F-contraction condition (21).

Remark 3 *If we use the mapping $F \in \Omega$ defined by the formula $F\,(\alpha) = \ln\alpha$ in Theorem 3, then for all $k \in (0, 1)$, we obtain the extension of [16, Theorem 3.1].*

Example 4 *Let $S = \mathbb{R}$ be a Banach space with the usual norm and $A = (-\infty, 0]$ is a closed subset of S. Let the mapping $\Upsilon : A \to S$ be defined as:*

$$\Upsilon r = \begin{cases} 0 & \text{if } r \in [-1, 0] \\ 0.5 & \text{if } r \in (-\infty, -1) . \end{cases}$$

Let the mapping $F \in \Omega$ be given by the formula $F(\alpha) = \dfrac{-1}{\sqrt{\alpha}}$, and the edge set of graph G and the subgraph G_A determined by A is defined as:

$$E(G) = \{(r, s) \in S \times S : r \leq s\}$$

and

$$E(G_A) = \{(r, s) \in A \times A : r \leq s\},$$

respectively. It is easy to check that (19) holds, that is,

> *for all $r, s \in A$ $(r, s) \in E(G)$*
>
> *with $\Upsilon r, \Upsilon s \in A$, implies $(\Upsilon r, \Upsilon s) \in E(G) \cap (A \times A)$.*

In view of (19), for $t, u \in (-\infty, -1)$ and $r, s \in [-1, 0]$, the edges $(t, u), (t, r)$ has to be removed and for the rest of edges we have

$$(\Upsilon r, \Upsilon s) = (0, 0) \in E(G_A).$$

Moreover, G is a weakly connected and for any $k \in (0.5, 1)$, Υ is a non-self G_A F-contraction on A with $\tau = \dfrac{1}{\sqrt{d(r,s)}}$, since

$$d(\Upsilon r, \Upsilon s) = \frac{1}{2} < \frac{1}{4} \times d(r, s) \text{ for } r \in (-\infty, -1) \text{ and } s \in [-1, 0].$$

(for the rest of edges of $E(G_A)$, the F-contraction condition (21) is obvious, since the quantity in its left hand side is always zero). Property (T) holds with constant sequences $\{r_n = r\}$ satisfying the property $(r_n, r_{n+1}) \in E(G_A)$, for all $n \in \mathbb{N}$. Rothe's boundary condition is also satisfied, as $\partial A = \{0\}$ and so $\Upsilon(\partial A) \subset A$. Finally, since we have $A_\Upsilon = \{0\} \neq \emptyset$, all assumptions in Theorem 3 are satisfied and $r' = 0$ is the fixed point of Υ.

Conclusion

In this paper, we have presented the fixed point theorems for self and non-self *G*, *F*-contractions on metric spaces endowed with a graph. These theorems immediately imply the extension of recently fixed point theorems for self-mappings on metric spaces and fixed point theorems for non-self mappings in Banach spaces.

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