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An asymptotic model for solving mixed integral equation in some domains

Mohamed Abdella Abdou[1] and Hamed Kamal Awad[2*]

*Correspondence: hamedk66@sci.dmu.edu.eg
[2] Department of Mathematics, Faculty of Science, Damanhour University, Damanhour, Egypt
Full list of author information is available at the end of the article

Abstract

In this paper, we discuss the solution of mixed integral equation with generalized potential function in position and the kernel of Volterra integral term in time. The solution will be discussed in the space $L_2(\Omega) \times C[0, T], 0 \leq t \leq T < 1$, where Ω is the domain of position and t is the time. The mixed integral equation is established from the axisymmetric problems in the theory of elasticity. Many special cases when kernel takes the potential function, Carleman function, the elliptic function and logarithmic function will be established.

Keywords: The mixed integral equation, Generalized potential function, Hypergeometric kernel, Carleman kernel, Logarithmic kernel

Mathematics Subject Classification: 45B05, 45G10

Introduction

The integral equations have been investigated from different mathematical areas of sciences and technology [1–9]. For this, many authors have been established different analytic and numeric methods. For analytical methods, see [10–15]. In addition, for numerical method one can see [16–23]. In [3], the authors considered a mixed integral equation under certain conditions, and they obtained the solution in a series form. In [23], the integral equation with potential kernel is studied. Moreover, in [24] the authors discussed the behavior of solution of mixed integral equation with generalized function.

Consider the integral equation:

$$\int_{\Omega} K(|x-y|)\Phi(y;\ t)\mathrm{d}y + \int_{0}^{t} F(t,\tau)\Phi(x;\ \tau)\mathrm{d}\tau = f(x;\ t) = \pi\vartheta[\gamma(t) + \beta(t)x - f_1(x)];$$
$$\left\{x \in R^3 \quad \vartheta = G/2(1-\nu)\right\}, \tag{1.1}$$

under the conditions

$$\int_{\Omega} \Phi(x;\ t)\mathrm{d}x = P(t), \qquad \int_{\Omega} x\Phi(x;\ t)\mathrm{d}x = M(t); \qquad t \in [0, T]. \tag{1.2}$$

The formula (1.1) is investigated in different forms, depending on the presence of time and the domain of integration of position; see [3, 4, 15, 23, 24]. In (1.1), the function $f_1(x)$ is known as a free term. The pressure and the moment functions $P(t),\ M(t)$ cause rigid displacements $\gamma(t)$ and $\beta(t)$ respectively, where $\gamma(t)$ and $\beta(t) \in C[0, T]$. The kernel of time $F(t, \tau)$ is positive continuous, while the kernel of position $K(|x-y|)$ has a singularity. Here, Ω is the domain of contact area of position. Finally, $\Phi\,(x;\ t)$ is the unknown function, will be determined.

To guarantee the existence of solution of (1.1), we assume

(1) $\left\|f\,(x;t)\,\right\|_X = H$, is a constant. In addition, the positive functions $\gamma\,(t)$ and $\beta\,(t)$ are continuous and belong to the class $C\,[0, T]$ such that $|\delta(t)| \le \delta$ and $|\beta(t)| \le \beta$, while $f_1(x) \in L_2(\Omega)$.
(2) The kernel of position satisfies $\iint\limits_{\Omega} \left|K(x-y)\right|^2 \mathrm{d}x\, \mathrm{d}y \le M^2$, where M is a constant.
(3) The function $F(t, \tau)$ with its partial derivatives are continuous in the class $C\,[0, T]$, and for all values of $t, \tau \in [0, T]$ we have $|F(t,\tau)| \le L_1 < L$ and $\left|\frac{\partial F(t,\tau)}{\partial t}\right| \le L_2 < L$, where L_1, L_2 and L are constants.
(4) The function $\Phi\,(x;t)$ satisfies Lipschitz condition for the first argument and Hölder condition for the second argument.

In the remainder part of this paper, the solution in a series form is constructed in the space $L_2\,(\Omega) \times C\,[0, T]$. Moreover, the solution is constructed in a linear combination form of eigenvalues and eigenfunctions, where the solution of Volterra is represented in eigenvalues' form, while the eigenfunctions represent the solution of Fredholm integral term in some different domains. Many applications in one, two and three dimensional are considered and derived from the problems. The results of this work can be used directly, to discuss the solution of the integral equations of the second kind, in different domains. Here, the Fredholm integral equation with logarithmic kernel is derived. Moreover, the integro-differential equation of Cauchy type kernel is established. This equation has appeared in both combined infrared gaseous radiation and molecular condition, and elastic contact studies.

Method of solutions

Consider the solution of (1.1) in the general series form

$$\Phi(x;t) = \Phi_0(x;t) + \Phi_1(x;t) \tag{2.1}$$

where $\Phi_0(x,\ t)$ is the complementary solution and $\Phi_1(x,\ t)$ is the particularly solution. Therefore, using (2.1) the formula (1.1) takes the form

$$\begin{aligned}&\int\limits_{\Omega} K(|x-y|)[\Phi_j(y;t) - \Phi_j\,(y;0)]\mathrm{d}y + \int\limits_0^t F(t,\tau)\Phi_j(x,y,\tau)\mathrm{d}\tau \\ &\quad = \pi\vartheta\delta_j[\gamma(0) + \beta(0)x - \gamma(t) - \beta(t)x], \quad j = 0, 1; \quad 0 \le t \le T < 1.\end{aligned} \tag{2.2}$$

Theorem 1 [24]. *If the position kernel $K(|x-y|)$ satisfies Fredholm condition, the integral operator*

$$K\Phi = \int\limits_{\Omega} K(|x-y|)\ \Phi(x,\,t)\mathrm{d}x \tag{2.3}$$

is compact and self-adjoint in $L_2(\Omega)\times C[0,T]$.

Theorem 2 [24]. *If the integral operator (2.3) is compact and self-adjoint, it can be written in the form*

$$\int\limits_{\Omega} K(|x-y|)\Phi(y,\,t)\mathrm{d}y = \alpha_k\ \Phi_k(x);\quad k=0,1,2,3,\ldots \tag{2.4}$$

where α_n and Φ_n are the eigenvalues and the eigenfunctions of the integral operator, respectively.

Volterra integral equation

In this section, we discuss the solution of Volterra integral equations, with continuous kernel in time. This solution represents the eigenvalues of the integral equation of (1.1). For this aim, the solution of (1.1) can be written in a closed Fourier series form in $L_2(\Omega)\times C[0,T]$ as follows:

$$\Phi_j(x;t) = \sum_{k=1}^{\infty}\left[A_{2k}^{(j)}(t)\Phi_{2k}(x) + A_{2k-1}^{(j)}(t)\Phi_{2k-1}(x)\right],\quad j=0,1, \tag{2.5}$$

where $\Phi_{2k}(x)$, $\Phi_{2k-1}(x)$ are the even and odd functions, respectively.
Using (2.4) and (2.5) in (2.2), we have the following results

$$\begin{aligned}
&A_k^{(0)}(t) + \alpha_k\int\limits_0^t A_k^{(0)}(\tau)F(t,\tau)\mathrm{d}\tau = A_k^{(0)}(0)\quad (j=0,\ k\ge 1)\\
&A_{2k}^{(0)}(t) + \alpha_{2k}\int\limits_0^t A_{2k}^{(0)}(\tau)F(t,\tau)\mathrm{d}\tau = A_{2k}^{(0)}(0)\\
&A_{2k-1}^{(0)}(t) + \alpha_{2k-1}\int\limits_0^t A_{2k-1}^{(0)}(\tau)F(t,\tau)\mathrm{d}\tau = A_{2k-1}^{(0)}(0)
\end{aligned} \tag{2.6}$$

the second and third equations of (2.6) give the same results for even and odd functions, so it is suffice to consider the first equation, where $A_k^{(0)}(0)$ is constant will be determined. In addition, taking $j=1$ in (2.5) and following the same previous way, we get

$$A_{2k}^{(1)}(t) + \alpha_{2k}\int\limits_0^t A_{2k}^{(1)}(\tau)F(t,\tau)\mathrm{d}\tau = \pi\vartheta\alpha_{2k}b_{2k}[\gamma(t)-\gamma(0)]$$

$$A_{2k-1}^{(1)}(t) + \alpha_{2k-1} \int_0^t A_{2k-1}^{(1)}(\tau)\, F(t,\tau) \mathrm{d}\tau = \pi \vartheta a_{2k-1} b_{2k-1} [\beta(t) - \beta(0)], \tag{2.7}$$

where

$\sum_{k=1}^{\infty} b_{2k} \Phi_{2k} = 1, \quad \sum_{k=1}^{\infty} b_{2k-1} \Phi_{2k-1} = x,$

$(k \geq 1, \quad 0 \leq t \leq T < 1, \quad A_{2k}^{(1)}(0) = A_{2k-1}^{(1)}(0) = 0).$

Equations (2.6)–(2.7) represent a system of Volterra integral equations which have the same kernel $F(t,\tau)$, and different free terms.

Therefore, the solution of these equations is unique in the class $C[0,T]$ depending on the free terms $A_k^{(0)}(0)$, $\gamma(t)$ and $\beta(t)$.

Put $t = 0$ in (1.1) and (2.2), and $j = 0$ in (2.1), we obtain the constant value $A_k^{(0)}(0)$ in the form

$$A_k^{(0)}(0) = \pi \ \vartheta \ \gamma(0)\, \alpha_k.$$

Moreover, put j=1, the same previous technique leads us to

$A_{2k}^{(1)}(0) = A_{2k-1}^{(1)}(0) = 0.$

Hence, the values of the constants $A_k^{(0)}(0)$, $A_{2k}^{(1)}(0)$ and $A_{2k-1}^{(1)}(0)$ are completely determined.

Convergence of the series

In view of (2.6) and (2.7), the general solution of (1.1) can be adapted in the form

$$\Phi(x;\ t) = \sum_{k=1}^{\infty} \left[A_k^{(0)}(t) + A_k^{(1)}(t)\right] \Phi_k(x), \tag{2.8}$$

where $A_k^{(0)}(t)$ and $A_k^{(1)}(t)$ satisfy the inequality

$$\sum_{k=1}^{\infty} \left[A_k^{(0)}(t) + A_k^{(1)}(t)\right]^2 < \varepsilon, \quad (\varepsilon \ll 1) \tag{2.9}$$

and $\Phi_k(x)$ is the eigenfunctions of integral operator (2.3).

Theorem 3 *For the uniformly convergent inequality* (2.9) *the series* (2.8) *holds and the solution* (1.1) *is unique in* $L_2\,(\Omega) \times C\,[0,T]$.

Proof After differentiating (2.6) with respect to t, we get a nonhomogeneous linear differential equation of the first order, which has a general solution in the form of complementary and particularly solution. From this solution, we can obtain the following relation

$$\left|A_i^{(0)}(t)\right| \leq \pi \ \vartheta \ |\alpha_i| \ \gamma(0) \ \mathrm{e}^{\left(\left|\alpha_i^{-1}\right| F\right) T}, \quad |F(t)| < F, \quad 0 < \vartheta < 1, \quad \gamma(0) \ll 1.$$

Similarly, the solution of (2.7) gives the following relation

$$\left|A_i^{(1)}(t)\right| \le \pi\,\vartheta\,|\alpha_i|\,\delta_i\,\mathrm{e}^{\left(\left|\alpha_i^{-1}\right|F\right)T}, \quad \delta_i = \begin{cases} |\gamma(t)| = \gamma & \text{for odd values} \\ |\beta(t)| = \beta & \text{for even values} \end{cases}$$

Finally, using the above solutions we get the following

$$\left\{\sum_{i=1}^{\infty}\left[A_i^{(0)}(t) + A_i^{(1)}(t)\right]^2\right\}^{\frac{1}{2}} \le \pi\,\vartheta\,B\left\{\sum_{i=1}^{\infty}\left(|\alpha_i|\mathrm{e}^{\left|\alpha_i^{-1}\right|FT}\right)^2\right\}^{\frac{1}{2}}, \quad B < 1;\ 0 < \vartheta < 1. \tag{2.10}$$

General solution of the integral equation (1.1)

In general, we consider the Volterra integral equation

$$A_k(t) + \alpha_k \int_0^t F(t,\tau)A_k(\tau)\,\mathrm{d}\tau = \pi\,\vartheta\,\alpha_k b_k w(t), \quad 0 < \vartheta < 1.$$

The general solution of the above equation is given by

$$\begin{aligned} A_k(t) = \exp\left\{\alpha_k \int_0^t F(u,u)\,\mathrm{d}u\right\}&\left(\int_0^t \exp(-\alpha_k \int_0^u F(v,v)\,\mathrm{d}v)\right. \\ &\times\left.\left[\pi\,\vartheta\,a_k b_k\,\frac{\mathrm{d}w(u)}{\mathrm{d}u} - \alpha_k \int_0^u A_k(\tau)\frac{\partial F(u,\tau)}{\partial u}\mathrm{d}\tau\right]\mathrm{d}u + C\right); \quad k = 1,2,\ldots, \end{aligned} \tag{2.11}$$

where C is a constant.

In (2.11), for even values of k we consider $w(t) = \gamma(t)$, while for odd values let $w(t) = \beta(t)$.

From the above, we can write the general solution of (1.1) in the form:

$$\Phi(x,y;t) = \sum_{k=1}^{\infty} \alpha_k\,A_k(t)\;\Phi_k(x,y). \tag{2.12}$$

In this case, the pressure and moments conditions can be adapted in the following form:

$$\begin{aligned} P(t) &= P_\mathrm{o} + \sum_{k=2,4} P_k\,\gamma_k\,\alpha_k\,A_k(t), \\ M(t) &= M_\mathrm{o} + \sum_{k=1,3} M_k\,\beta_k\,\alpha_k\,A_k(t), \end{aligned}$$

where P_o, M_0, P_k and M_k are constants.

The formula (2.12) is the solution of the integral Eq. (1.1) in the linear combination form of eigenvalues and eigenfunctions $\alpha_k\ \Phi_k(x,y)$ for the integral equation of position. While, $A_k(t)$ represents the solution of Volterra integral equation in time (2.11). The eigenfunctions $\Phi_k(x,y)$ will be obtained after knowing the kernel form of (1.1), under the pressure and moments conditions (2.13).

Some applications

Axisymmetric contact problem with generalized kernel

We consider an axisymmetric contact problem with generalized potential kernel, impressing stamp of angular form on a plane into a half space. Here, the modules of elasticity is changing according to the power law $\sigma_i = K_0\, \varepsilon_i^h$ $(0 \le h < 1)$; σ_i, ε_i are the stress and the strain rate intensities, respectively, and K_0, h are physical constants. Moreover, we consider the equation of stamp is $f_1(x, y)$, which is impressed into the lower surface by a variable force in time $P(t)$, that has eccentricity of application $e(t)$, and let $M(t)$ be a moment. The pressure and the moment cause rigid displacements $\gamma(t)$ and $\beta(t)$, respectively, where $\gamma(t)$ and $\beta(t) \in C\ [0,\ T]$. Neglecting the frictional forces between the two surfaces in the domain of contact area Ω. The given function $L(\xi, \eta)$ is continuous and has two physical meaning as shown in Eq. (3.5).

The unknown function $\Phi(x, y; t)$ represents the difference between the two normal stresses of the two surfaces.

Such problem leads to the following mixed integral equation (see [24, 25]).

$$\iint_{\Omega} L(\xi,\eta)[(x-\xi)^2+(y-\eta)^2]^{-h}\Phi(\xi,\eta,t)\mathrm{d}\xi\,\mathrm{d}\eta+\int_0^t F(t,\tau)\Phi(x,y,\tau)\mathrm{d}\tau=f(x,y;t)$$
$$=\pi\,\vartheta[\gamma(t)+\beta(t)x-f_1(x,y)];\ \left\{\Omega=\sqrt{x^2+y^2}\le a;\quad \vartheta=G(1-\nu)^{-1};\quad 0\le h<1\right\} \tag{3.1}$$

under the dynamic conditions

$$\iint_{\Omega}\Phi(\xi,\ \eta,\ t)\,\mathrm{d}\xi\,\mathrm{d}\eta=P(t)\ldots,\qquad \iint_{\Omega}\xi\eta\Phi(\xi,\ \eta,\ t)\mathrm{d}\xi\,\mathrm{d}\eta=M(t);\qquad t\in[0,T]. \tag{3.2}$$

To obtain the spectral relationships of the integral equation, we let, in (3.1), $t=0$. Hence, we have

$$\iint_{\Omega} L(\xi,\eta)\ [(x-\xi)^2+(y-\eta)^2]^{-h}\ \Phi\,(\xi,\ \eta,\ 0)\mathrm{d}\xi\,\mathrm{d}\eta=\pi\ \vartheta\ f(x,y) \tag{3.3}$$

The formula (3.3) represents an integral equation, in position with generalized potential kernel, to represent this integral equation in the form of Fredholm integral equation with Weber-Sonien integral formula, we follow.

Firstly, using the polar coordinates, $(x,\xi)=(r,\rho)\cos(\theta,\psi), (y,\eta)=(r,\rho)\sin(\theta,\psi)$. Then, using the separation variables method, $\Phi(x,y;0) = \Phi(r,\theta;0) = \Phi_n(r;0)\ \cos k\theta$.

Secondly, we consider the following two relations [25, 26],

$$1\text{–}{}_2F_1\left(\alpha,\alpha+\frac{1}{2}-\beta;\beta+\frac{1}{2};z^2\right)=(1+z)^{-2\alpha}{}_2F_1\left(\alpha,\beta;2\beta;\frac{4z}{(1+z)^2}\right),\quad (|z|<1,\ \mathrm{Re}\,\alpha>0)$$

$$2\text{–}\int_0^\infty J_\mu(bx)J_\nu(ax)\mathrm{d}x=a^\nu b^{-\nu-1}\frac{\Gamma(\frac{\mu+\nu+1}{2})}{\Gamma(1+\nu)\Gamma(\frac{\mu-\nu+1}{2})}\cdot F\left(\frac{\mu+\nu+1}{2},\ \frac{\nu-\mu+1}{2},\ \nu+1,\ a^2/b^2\right)$$

$\Gamma(x)$ is the gamma function, $(\alpha)_m \frac{\Gamma(m+\alpha)}{\Gamma(\alpha)}$ is called Pachmmer symbol and ${}_2F_1(a, b, c; z)$ is the Gauss hypergeometric function. In addition, $J_m(a\,x)$ is the Bessel function of order m and $F(\alpha, \beta; \gamma; z)$ is the hypergeometric function.

Hence, following the same technique of [23, 24], one arrives to the following integral equation

$$\int_0^1 K_n^h(r,\rho)\phi(\rho)\mathrm{d}\rho = \pi\,\vartheta\, f(r), \quad K_n^h(r,\rho) = \sqrt{r\rho}\int_0^\infty L(\rho)u^{2h-1}J_n(\rho\, u)J_n(r\, u)\,\mathrm{d}u. \tag{3.4}$$

The formula (3.4) represents Fredholm integral equation of the first kind with a kernel takes form of generalized Weber-Sonien integral formula.

To obtain the spectral relations of (3.4), we follow:

Consider the known function $L(\rho)$ of (3.4) satisfies

$$\begin{aligned} &L(\rho) = A\rho + O(\rho^3); \quad (\rho \to 0,\ A = \text{Const.} > 0),\\ &L(\rho) = 1 + \sum_{i=1}^{\ell-1}\frac{B_i}{\rho^i} + O\left(\frac{1}{\rho^\ell}\right) \quad (\rho \to \infty, \quad B_i \text{ - constant}) \end{aligned} \tag{3.5}$$

Moreover, assume in (3.4), $f(r) = P_k^{(n,\mu-\frac{1}{2})}(1-2r^2)$; $h = \mu + \frac{1}{2}$, where, $P_k^{(n,\mu-\frac{1}{2})}(1-2r^2)$ are the Jacobi polynomials.

After using Krein's method, see [7], we have the following spectral relationships form.

$$\begin{aligned} &\int_0^1 \frac{K_n(u,v)\rho^{k+1}}{(1-v^2)^{\frac{1}{2}-\mu}}\, P_k^{(n,\mu-\frac{1}{2})}(1-2v^2)\mathrm{d}v = \alpha_k^n u^k\, P_k^{(n,\mu-\frac{1}{2})}(1-2u^2),\\ &\left(u = a\,r,\ v = a\,\rho;\ 0 \le \mu < \frac{1}{2}\right), \end{aligned} \tag{3.6}$$

where the eigenvalues α_k^n are given by

$$\alpha_k^n = \pi\vartheta\, \frac{\Gamma\left(\mu + k + n + \frac{1}{2}\right)\Gamma(k+\mu+\frac{1}{2})\Gamma(\frac{1}{2}-\mu)}{n!\,(k+n)!\,\Gamma(\frac{1}{2}+\mu)}; \quad (n = 0, 1, 2, 3, \ldots). \tag{3.7}$$

The formula (3.6) represents the solution of (3.4) as a linear combination of eigenvalues and eigenfunctions form.

Finally, from the solution of (1.1), we deduce that the semi-symmetric Hertz contact problem of frictionless impression of a rigid surface (G, ν) in the domain $\Omega = \left\{(x,y,z) \in \Omega : \sqrt{x^2+y^2} \le a, z = 0\right\}$, takes the form

$$\Phi(u;t) = \sum_{k=1}^{\infty}\alpha_k^n A_k(t)u^k\, P_k^{(n,\mu-\frac{1}{2})}\,(1-2u^2) \tag{3.8}$$

where $A_k(t)$ is the solution of (3.1), α_k is given by (3.7) and $P_k^{(k,-w^-)}\,(1-2r^2)$ are the Jacobi polynomials.

Ring stamp

The solution of the problem of an elastic half-space, after neglecting the frictional forces and external forces, inclined ring stamp problem, $a \le r \le b$, takes the form

$$K_n(u,v) = \frac{\sqrt{\pi}\,\Gamma(n+1/2)e^{2k}}{2^{2n-1}\,n!\,(u+v)} F\left(n+\frac{1}{2},\ n+\frac{1}{2},\ 2n+1,\ e^2\right),\ \left(e = E\left(\frac{2\sqrt{uv}}{u+v}\right)\right) \tag{3.9}$$

$\Gamma(x)$ is the Gamma function, and $E\left(\frac{2\sqrt{r\rho}}{r+\rho}\right)$ represents the complete elliptic integral form.

Using the hypergeometric series 9.111, pp.1054 of [27],

$$F(\alpha,\delta,\gamma,z) = \frac{1}{\beta(\delta,\ \gamma-\delta)} \int_0^t t^{\delta-1}(1-t)^{\gamma-\delta-1}(1-t\,z)^{-\alpha}\mathrm{d}t; \tag{3.10}$$
$$(\beta \text{ is Beta function } \mathrm{Re}\gamma > \mathrm{Re}\delta > 0),$$

the formula (3.9), yields

$$K_n(u,v) = \frac{e^{2n}}{\pi(u+v)} \int_0^1 z^{n-\frac{1}{2}}(1-z)^{n-\frac{1}{2}}(1-e^2 z)^{-n-\frac{1}{2}}\mathrm{d}z$$

We combine the relation between elliptic form and the Bessel function, then we have

$$K_n(u,v) = \sqrt{uv} \int_0^\infty J_n(u\tau)J_n(v\tau)\mathrm{d}\tau \tag{3.11}$$

where $J_n(\cdot)$ is the Bessel function of order n.

In the ring stamp, the spectral relationships that equivalent to (3.6) takes the form:

$$\int_0^1 K_n(u,v)\frac{v^{n+1}}{\sqrt{1-v^2}}P_k^{(n,-\frac{1}{2})}(1-2v^2)\mathrm{d}v = \begin{cases} \pi\vartheta\,\frac{\Gamma(k+n+\frac{1}{2})\Gamma(k+\frac{1}{2})}{n!\,(k+n)!}u^n P_k^{(n,-\frac{1}{2})}(1-2u^2), & u<1 \\ \pi\vartheta(-1)^k\frac{\Gamma(k+n+\frac{1}{2})\Gamma(k+\frac{1}{2})}{n!\,(k+n)!}u^n P_k^{(-\frac{1}{2},n)}(2u^2-1), & u>1 \end{cases} \tag{3.12}$$

Moreover, the complete solution of the problem reduces to:

$$\Phi(u;t) = \sum_{k=1}^{\infty} A_k(t) \begin{cases} \pi\vartheta\,\frac{\Gamma k+n+\frac{1}{2})\Gamma(k+\frac{1}{2})}{n!\,(k+n)!}u^n P_k^{(n,-\frac{1}{2})}(1-2u^2), & u<1 \\ \pi\vartheta(-1)^k\frac{\Gamma(k+n+\frac{1}{2})\Gamma(k+\frac{1}{2})}{n!\,(k+n)!}u^n P_k^{(-\frac{1}{2},n)}(2u^2-1), & r>1 \end{cases} \tag{3.13}$$

An integral equation with logarithmic kernel

We consider the integral equation

$$\int_{-1}^{1} \ln\left|x-y\right|\Phi(y)\mathrm{d}y = f^*(x) \tag{3.14}$$

For obtaining the spectral relationships of (3.14), we use the orthogonal polynomial method.

Therefore, we assume that the Chebyshev polynomials of the first kind are
$T_n(x) = \cos(n\cos^{-1}x),\ x \in [-1,1],\ n \geq 0$. With weight function $(1-x^2)^{-\frac{1}{2}}$.

And the Chebyshev polynomials of the second kind are given by
$U_n(x) = \frac{\sin[(n+1)\cos^{-1}x]}{\sin(\cos^{-1}x)},\ n \geq 0$, with weight function $(1 - x^2)^{\frac{1}{2}}$.

After using, the orthogonal polynomials method with some well-known algebraic and integral relations associated with Chebyshev polynomials see [28, 29], we have

$$\int_{-1}^{1} \frac{\ln|x-y|}{\sqrt{1-y^2}} T_n(y)dy = \begin{cases} \pi \ln 2 & n = 0, \\ \pi \frac{T_n(x)}{n}, & n \geq 1. \end{cases} \tag{3.15}$$

Finally, the general solution takes the form

$$\Phi(x;t) = \sum_{n=1}^{\infty} A_n(t) \begin{cases} \pi \ln 2 & n = 0, \\ \pi \dfrac{T_n(x)}{n}, & n \geq 1. \end{cases} \tag{3.16}$$

Also, for logarithmic kernel, we can have the solution

$$\Phi(r;t) = \sum_{k=1}^{\infty} A_k(t) \begin{cases} \frac{2\pi\vartheta}{2n+1}\sqrt{r}P_k^{(\frac{1}{2},-\frac{1}{2})}(1-2r^2), & r < 1 \\ \frac{2(-1)^k\vartheta\pi}{2n+1}\sqrt{r}P_k^{(-\frac{1}{2},\frac{1}{2})}(2r^2-1), & r > 1 \end{cases} \tag{3.17}$$

Many special cases can be derived from the generalized kernel (3.4) as the following (Figs. 1, 2).

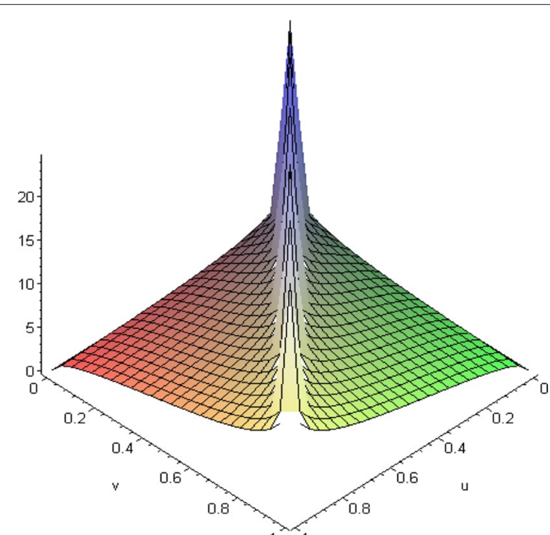

Fig. 1 *h* = 0.5, *n* = 0 (elliptic kernel)

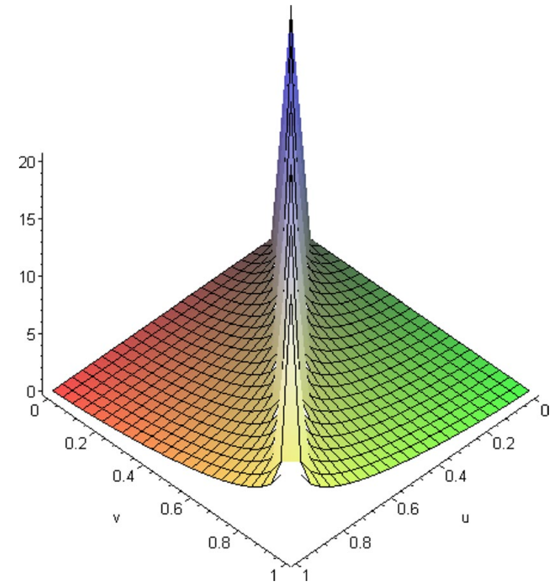

Fig. 2 $h = 0.5$, $n = 1$ (potential kernel)

Conclusions

From the above results and discussions, the following conclusions may be deduced:

1 The integral Eq. (1.1) under the conditions (1.2) is established from the semi-symmetric Hertz contact problem of frictionless impression of a rigid surface (G, ν) having an elastic material in the domain $\Omega = \left\{\sqrt{x^2+y^2} \le a,\, z = 0\right\}$, where $f_1(x,y) \in L_2(\Omega)$ describes the surface of stamp, which is impressed into an elastic layer surface. The variable pressure $P(t)$ is eccentricity of application $e\,(t)$ and the variable momentum $M(t), 0 \le t \le T < 1$, cause rigid displacements $\gamma\,(t)$ and $\beta(t)x$. Respectively the force $F(t,\tau)\,t,\ \tau \in [0,T]$ is called the characterized resistance function of the material, G is the displacement magnitude, ν is Poisson's coefficient, and $\Phi\,(x,y,t)$ is the unknown function.

2 The displacement problem of anti-plane deformation of an infinite rigid strip has width $2a$, after putting $t = 1, F(t,\tau) = 1, f(x,t) = H$ and $\phi(x,1) = \psi(x)$.
Here, H is the displacement magnitude and $\psi(x)$ is the unknown function, see [8, 30].

3 Moreover, we can derive the following special relations

$$\int_0^1 K_n^h(r,\rho) P_m^n(\rho)\,(1-\rho^2)^{h-1}\,\rho^{n+1} \mathrm{d}\rho = \begin{cases} A_{n\,m}^h r^n P_n^h(r) & r < 1 \\ B_{n\,m}^h Q_n^{(h-1,\,n)}(2r^2-1) & r > 1 \end{cases} \quad (4.1)$$

where

$$A_{n\,k}^h = 2^{2(h-1)}\pi\,\vartheta\,\Gamma(k+n+h)\Gamma(k+h)[\Gamma(k+n+1)\,m!]^{-1},\ \left(P_k^h(r) = P_k^{(n,h-1)}(1-2r^2)\right)$$

$$B^h_{nk} = (-1)^k\, 2^{2h-1}\,(\sin\pi h)\Gamma(k+n+h)\Gamma(k+h)[\pi\Gamma(k+n+1)m!]^{-1}, \quad k=1,2,\ldots$$

Here $\Gamma(.)$ is the Gamma function while $P_m^{(\alpha,\beta)}(x)$, $Q_n^{(\alpha,\beta)}(x)$ are the Jacobi polynomial of the first and second type, respectively.

3.1 The spectral relations for the elliptic kernel in the Jacobi polynomials form are

$$\int_0^1 K_0(r,\rho)\,\frac{\rho}{\sqrt{1-\rho^2}}P_k^{(0,-\frac{1}{2})}(1-2\rho^2)\mathrm{d}\rho = \begin{cases} \pi\vartheta\left[\frac{\Gamma(k+\frac{1}{2})}{k!}\right]^2 P_k^{(0,-\frac{1}{2})}(1-2r^2), & r<1, \\ \pi\vartheta(-1)^k\left[\frac{\Gamma(k+\frac{1}{2})}{k!}\right]^2 P_k^{(-\frac{1}{2},0)}(2r^2-1), & r>1, \end{cases} \tag{4.2}$$

4.1 The spectral relations, with Carleman functions, when $n=\frac{1}{2}$ are

$$\int_0^1 K_{\frac{1}{2}}(u,v)\,v^{\frac{3}{2}}(1-v^2)^{\mu-\frac{1}{2}}\,P_k^{(\frac{1}{2},\mu-\frac{1}{2})}(1-2v^2)\mathrm{d}v = \begin{cases} D_k\sqrt{u}P_k^{(\frac{1}{2},\mu-\frac{1}{2})}(1-2u^2), & u<1 \\ (-1)^k D_k\sqrt{u}P_k^{(\mu-\frac{1}{2},\frac{1}{2})}(2u^2-1), & u>1 \end{cases} \tag{4.3}$$

$$D_k = \pi\vartheta\,\frac{\Gamma(\mu+k+1)\Gamma(k+\mu+\frac{1}{2})\Gamma(\frac{1}{2}-\mu)}{k!\,\Gamma(k+\frac{3}{2})\Gamma(\frac{1}{2}+\mu)}.$$

5.1 In addition, for $n=-\frac{1}{2}$ in the Jacobi polynomial form, are

$$\int_0^1 K_{-\frac{1}{2}}(u,v)v^{\frac{1}{2}}(1-v^2)^{\mu-\frac{1}{2}}P_k^{(-\frac{1}{2},\mu-\frac{1}{2})}(1-2v^2)\mathrm{d}v = \begin{cases} D_k^* u^{-\frac{1}{2}}P_k^{(-\frac{1}{2},\mu-\frac{1}{2})}(1-2u^2), & u<1 \\ (-1)^k D_k^* u^{-\frac{1}{2}}P_k^{(\mu-\frac{1}{2},-\frac{1}{2})}(2u^2-1), & u>1 \end{cases} \tag{4.4}$$

$$D_k^* = \pi\vartheta\,\frac{\Gamma(\mu+k)\,\Gamma(k+\mu+\frac{1}{2})\,\Gamma(\frac{1}{2}-\mu)}{k!\,\Gamma(k+\frac{1}{2})\Gamma(\frac{1}{2}+\mu)}.$$

6.1 The spectral relations for the potential kernel in the Jacobi polynomial form, when $\mu=0$, $n=\frac{1}{2}$ are

$$\int_0^1 K_{\frac{1}{2}}(r,\rho)\,\frac{\rho^{\frac{3}{2}}}{\sqrt{1-\rho^2}}P_n^{(\frac{1}{2},-\frac{1}{2})}(1-2\rho^2)\mathrm{d}\rho = \begin{cases} A_n\sqrt{r}P_n^{(\frac{1}{2},-\frac{1}{2})}(1-2r^2), & r<1 \\ B_n\sqrt{r}\,P_n^{(-\frac{1}{2},\frac{1}{2})}(2r^2-1), & r>1 \end{cases} \tag{4.5}$$

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Author details
[1] Department of Mathematics, Faculty of Education, Alexandria University, Alexandria, Egypt. [2] Department of Mathematics, Faculty of Science, Damanhour University, Damanhour, Egypt.

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References

1. Abdou, M.A., Mahmoud, S.A., Darwish, M.A.: A numerical method for solving the Fredholm integral equation of the second kind. Korean J. Comput. Appl. Math. **5**, 251–258 (1998)
2. Abdou, M.A.: Fredholm-Volterra integral equation of the first kind and contact problem. Appl. Math. Comput. **125**, 177–193 (2002)
3. Aleksandrov, V.M., Covalence, E.V.: Problems in Mechanics Media with Mixed Boundary Conditions. Nuka, Moscow (1986)
4. Aleksandrov, V.M., Covalence, E.V.: Mathematical method in the displacement problem. Inzh. Zh. Mekh. Tverd. Tela. **2**, 77–89 (1989)
5. Constanda, C.: Integral equations of the first kind in plane elasticity. Q. Appl. Math. **53**, 783–793 (1995)
6. Fahmy, M.H., Abdou, M.A., Darwish, M.A.: On the solution of an integral equation appearing in displacement problems. Southwest J. Pure Appl. Math. **1**, 84–88 (2000)
7. Fahmy, M.H., Abdou, M.A., Darwish, M.A.: Integral equations and potential-theoretic type integrals of orthogonal polynomials. J. Comput. Appl. Math. **106**, 245–254 (1999)
8. Liu, Z., Kang, S.M., Ume, J.S.: Solvability and asymptotic stability of a nonlinear functional-integral equation. Appl. Math. Lett. **24**, 911–917 (2011)
9. Ngoc, L.T.P., Long, N.T.: Solving a system of nonlinear integral equations and existence of asymptotically stable solutions. Differ. Equ. Appl. **4**, 233–255 (2012)
10. Aziz, I., ul-Islam, S.: New algorithms for the numerical solution of nonlinear Fredholm and Volterra integral equations using Haar wavelets. J. Comput. Appl. Math. **239**, 333–345 (2013)
11. Abdou, M.A., Raad, S., Al-Hazmi, S.: Fundamental contact problem and singular mixed integral equation. Life Sci. J. **8**, 323–329 (2014)
12. Al-Jarrah, Y., Lin, E.: Wavelet interpolation method for solving singular integral equations. Appl. Math. **4**, 1–4 (2013)
13. Anastassiou, G.A., Aral, A.: Generalized Picard singular integrals. Comput. Math. Appl. **57**, 821–830 (2009)
14. Bazm, S., Babolian, E.: Numerical solution of nonlinear two-dimensional Fredholm integral equations of the second kind using Gauss product quadrature rules. Commun. Nonlinear Sci. Numer. Simul. **17**, 1215–1223 (2012)
15. Ngoc, L.T.P., Long, N.T.: On a nonlinear Volterra-Hammerstein integral equation in two variables. Acta Math. Sci. Ser. B (Engl. Ed.) **33**, 484–494 (2013)
16. Popov, G.Y.: Contact Problems for a Linearly Deformable Base. Kiev, Odessa (1982)
17. Atkinson, K.E.: The Numerical Solution of Integral Equations of the Second Kind, Cambridge Monographs on Applied and Computational Mathematics, vol. 4. Cambridge University Press, Cambridge (1997)
18. Bai, Z.: Eigenvalue estimates for saddle point matrices of Hermitian and indefinite leading blocks. J. Comput. Appl. Math. **237**, 295–306 (2013)
19. Cao, Y., Yao, L., Jiang, M., Niu, Q.: A relaxed HSS preconditioner for saddle point problems from mesh free discretization. J. Comput. Math. **31**, 398–421 (2013)
20. Delves, L.M., Mohamed, J.L.: Computational Methods for Integral Equations. Cambridge University Press, Cambridge, UK (1985)
21. Diogo, T., Lima, P.: Superconvergence of collocation methods for class of weakly singular integral equations. J. Comput. Appl. Math. **218**, 307–316 (2008)
22. Kangro, R., Oja, P.: Convergence of spline collocation for Volterra integral equations. Appl. Numer. Math. **58**, 1434–1447 (2008)
23. ul-Islam, S., Aziz, I., Fayyaz, M.: A new approach for numerical solution of integro-differential equations via Haar wavelets. Int. J. Comput. Math. **90**, 1971–1989 (2013)
24. Abdou, M.A.: Fredholm integral equation with potential kernel and its structure resolvent. Appl. Math. Comput. **107**, 169–180 (2000)
25. Abdou, M.A.: Fredholm-Volterra integral equation and generalized potential kernel. Appl. Math. Comput. **131**, 81–94 (2002)
26. Abdou, M.A., Salama, F.A.: Volterra-Fredholm integral equation of the first kind and spectral relationships. Appl. Math. Comput. **153**, 141–153 (2004)
27. Bateman, G.: Higher Trancendental Functions II. Mc-Graw Hill, New York (1989)
28. Gradshteyn, I.S., Ryzhik, I.M.: Tables of Integrals, Series, and Products. Columbia University Press, New York (1951)
29. Mkhitarian, S.M., Abdou, M.A.: On different method of solution of the integral equation for the planer contact problem of elasticity. Dakl. Acad. Nauk Arm. SSR **89**, 59–74 (1990)
30. Mkhitarian, S.M., Abdou, M.A.: On various method for the solution of the Carleman integral equation. Dakl. Acad. Nauk Arm. SSR **89**, 125–129 (1990)

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