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# A soft set theoretic approach to an AG-groupoid via ideal theory with applications

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**Abstract**
In this paper, we study the structural properties of a non-associative algebraic structure called an AG-groupoid by using soft set theory. We characterize a right regular class of an AG-groupoid in terms of soft intersection ideals and provide counter examples to discuss the converse part of various problems. We also characterize a weakly regular class of an AG***-groupoid by using generated ideals and soft intersection ideals. We investigate the relationship between SI-left-ideal, SI-right-ideal, SI-two-sided-ideal, and SI-interior-ideal of an AG-groupoid over a universe set by providing some practical examples.

**Keywords:** Left invertive law, Soft-sets, AG-groupoid, Right regularity, Weak regularity and SI-ideals

## Introduction

The concept of soft set theory was introduced by Molodtsov in [16]. This theory can be used as a generic mathematical tool for dealing with uncertainties. In soft set theory, the problem of setting the membership function does not arise, which makes the theory easily applied to many different fields [1, 2, 5–9]. At present, the research work on soft set theory in algebraic fields is progressing rapidly [19, 21–23] . A soft set is a parameterized family of subsets of the universe set. In the real world, the parameters of this family arise from the view point of fuzzy set theory. Most of the researchers of algebraic structures have worked on the fuzzy aspect of soft sets. Soft set theory is applied in the field of optimization by Kovkov in [12]. Several similarity measures have been discussed in [15], decision-making problems have been studied in [21], and reduction of fuzzy soft sets and its applications in decision-making problems have been analyzed in [13]. The notions of soft numbers, soft derivatives, soft integrals, and many more have been formulated in [14]. This concept have been used for forecasting the export and import volumes in international trade [28]. A. Sezgin have introduced the concept of a soft sets in non-associative semigroups in [24] and studied soft intersection left (right, two-sided) ideals, (generalized) bi-ideals, interior ideals, and quasi-ideals in an AG-groupoid. A lot of work has been done on the applications of soft sets to a non-associative rings by T. Shah et al. in [25, 26]. They have characterized the non-associative rings through soft M-systems and different soft ideals to get generalized results.

This paper is the continuation of the work carried out by F. Yousafzai et al. in [29] in which they define the smallest one-sided ideals in an AG-groupoid and use them to characterize a strongly regular class of an AG-groupoid along with its semilattices and soft intersection left (right, two-sided) ideals, and bi-ideals. The main motivation behind this paper is to study some structural properties of a non-associative structure as it has not attracted much attention compared to associative structures. We investigate the notions of SI-left-ideal, SI -right-ideal, SI-two-sided-ideal, and SI -interior-ideal in an AG-groupoid. We provide examples/counter examples for these SI-ideals and study the relationship between them in detail. As an application of our results, we get characterizations of a right regular AG-groupoid and weakly regular AG$^{***}$-groupoid in terms of SI-left-ideal, SI-right-ideal, SI-two-sided-ideal, and SI-interior-ideal.

## AG-groupoids

An AG-groupoid is a non-associative and a non-commutative algebraic structure lying in a gray area between a groupoid and a commutative semigroup. Commutative law is given by $abc = cba$ in ternary operations. By putting brackets on the left of this equation, i.e., $(ab)c = (cb)a$, in 1972, M. A. Kazim and M. Naseeruddin introduced a new algebraic structure called a left almost semigroup abbreviated as an LA-semigroup [10]. This identity is called the left invertive law. P. V. Protic and N. Stevanovic called the same structure an Abel-Grassmann's groupoid abbreviated as an AG-groupoid [20].

This structure is closely related to a commutative semigroup because a commutative AG-groupoid is a semigroup [17]. It was proved in [10] that an AG-groupoid $S$ is medial, that is, $ab \cdot cd = ac \cdot bd$ holds for all $a, b, c, d \in S$. An AG-groupoid may or may not contain a left identity. The left identity of an AG-groupoid permits the inverses of elements in the structure. If an AG-groupoid contains a left identity, then this left identity is unique [17]. In an AG-groupoid $S$ with left identity, the paramedial law $ab \cdot cd = dc \cdot ba$ holds for all $a, b, c, d \in S$. By using medial law with left identity, we get $a \cdot bc = b \cdot ac$ for all $a, b, c \in S$. We should genuinely acknowledge that much of the ground work has been done by M. A. Kazim, M. Naseeruddin, Q. Mushtaq, M. S. Kamran, P. V. Protic, N. Stevanovic, M. Khan, W. A. Dudek, and R. S. Gigon. One can be referred to [3, 4, 11, 17, 18, 20, 27] in this regard.

A nonempty subset $A$ of an AG-groupoid $S$ is called a left (right, interior) ideal of $S$ if $SA \subseteq A$ $(AS \subseteq A, SA \cdot S \subseteq A)$. Equivalently, a nonempty subset $A$ of an AG-groupoid $S$ is called a left (right, interior) ideal of $S$ if $SA \subseteq A$ $(AS \subseteq A, SA \cdot S \subseteq A)$. By two-sided ideal or simply ideal, we mean a nonempty subset of an AG-groupoid $S$ which is both left and right ideal of $S$.

## Soft sets

In [23], Sezgin and Atagun introduced some new operations on soft set theory and defined soft sets in the following way:

Let $U$ be an initial universe set, $E$ a set of parameters, $P(U)$ the power set of $U$, and $A \subseteq E$. Then, a *soft set* (briefly, a soft set) $f_A$ over $U$ is a function defined by:

$$f_A : E \to P(U) \text{ such that } f_A(x) = \emptyset, \text{ if } x \notin A.$$

Here, $f_A$ is called an *approximate function*. A soft set over $U$ can be represented by the set of ordered pairs as follows:

$$f_A = \left\{(x, f_A(x)) : x \in E, f_A(x) \in P(U)\right\}.$$

It is clear that a soft set is a parameterized family of subsets of $U$. The set of all soft sets is denoted by $S(U)$.

Let $f_A, f_B \in S(U)$. Then, $f_A$ is a soft subset of $f_B$, denoted by $f_A \mathrel{\widetilde{\subseteq}} f_B$ if $f_A(x) \subseteq f_B(x)$ for all $x \in S$. Two soft sets $f_A, f_B$ are said to be equal soft sets if $f_A \mathrel{\widetilde{\subseteq}} f_B$ and $f_B \mathrel{\widetilde{\subseteq}} f_A$ and is denoted by $f_A \mathrel{\widetilde{=}} f_B$. The union of $f_A$ and $f_B$, denoted by $f_A \mathbin{\widetilde{\cup}} f_B$, is defined by $f_A \mathbin{\widetilde{\cup}} f_B = f_{A\cup B}$, where $f_{A\cup B}(x) = f_A(x) \cup f_B(x)$, $\forall\, x \in E$. In a similar way, we can define the intersection of $f_A$ and $f_B$.

Let $f_A, f_B \in S(U)$. Then, the *soft product* [23] of $f_A$ and $f_B$, denoted by $f_A \circ f_B$, is defined as follows:

$$(f_A \circ f_B)(x) = \begin{cases} \bigcup\limits_{x=yz} \{f_A(y) \cap g_B(z)\} & \text{if } \exists\, y, z \in S \ni x = yz \\ \emptyset & \text{otherwise} \end{cases}.$$

Let $f_A$ be a soft set of an AG-groupoid $S$ over a universe $U$. Then, $f_A$ is called a *soft intersection left ideal*, *right ideal*, *interior ideal* (briefly, SI -left-ideal, SI-right-ideal, SI-interior-ideal) of $S$ over $U$ if it satisfies $f_A(xy) \supseteq f_A(y)$ $(f_A(xy) \supseteq f_A(x), f_A(xy \cdot z) \supseteq f_A(y))$, $\forall\, x, y \in S$. A soft set $f_A$ is called a *soft intersection two-sided ideal* (briefly, SI -two-sided-ideal) of $S$ over $U$ if $f_A$ is an SI -left-ideal and an SI-right-ideal of $S$ over $U$.

Let $A$ be a nonempty subset of $S$. We denote by $X_A$ the *soft characteristic function* of $A$ and define it as follows:

$$X_A = \begin{cases} U & \text{if } x \in A \\ \emptyset & \text{if } x \notin A \end{cases}.$$

Note that the soft characteristic mapping of the whole set $S$, denoted by $X_S$, is called an *identity soft mapping*.

## Basic results

**Lemma 1** [29] *For a nonempty subset $A$ of an AG-groupoid $S$, the following conditions are equivalent*:

$(i)$ *$A$ is a left ideal (right ideal, interior ideal) of $S$;*

$(ii)$ *A soft set $X_A$ of $S$ over $U$ is an SI-left-ideal (SI -right-ideal, SI-interior-ideal) of $S$ over $U$.*

**Lemma 2** [29] *Let $S$ be an AG-groupoid. For $\emptyset \neq A, B \subseteq S$, the following assertions hold*:

$(i)$ $X_A \mathbin{\widetilde{\cap}} X_B = X_{A\cap B}$;

$(ii)$ $X_A \circ X_B = X_{AB}$.

**Remark 1** [29] *The set $(S(U), \circ)$ forms an AG-groupoid and satisfies all the basic laws.*

**Remark 2** [29] *If $S$ is an AG-groupoid, then $X_S \circ X_S = X_S$.*

**Lemma 3** *Let $f_A$ be any soft set of a right regular AG-groupoid $S$ with left identity over $U$. Then, $f_A$ is an SI-right-ideal (SI-left-ideal, SI-interior-ideal) of $S$ over $U$ if and only if $f_A = f_A \circ X_S$ $(f_A = X_S \circ f_A, f_A = (X_S \circ f_A) \circ X_S)$ and $f_A$ is soft semiprime .*

*Proof* It is simple. □

**Lemma 4** *For every* SI*-interior-ideal* $f_A$ *of a right regular AG-groupoid* $S$ *with left identity over* $U$, $f_A = X_S \circ f_A = f \circ X_S$.

*Proof* Assume that $f_A$ is any SI-interior-ideal of $S$ with left identity over $U$. Then, by using Remark 2 and Lemma 3, we have $X_S \circ f_A = (X_S \circ X_S) \circ f_A = (f_A \circ X_S) \circ X_S = (f_A \circ X_S) \circ (X_S \circ X_S) = (X_S \circ X_S) \circ (X_S \circ f_A) = ((X_S \circ f_A) \circ X_S) \circ X_S = f_A \circ X_S$ and $X_S \circ f_A = (X_S \circ X_S) \circ f_A = (f_A \circ X_S) \circ X_S = (X_S \circ f_A) \circ X_S = f_A$. □

**Lemma 5** [29] *Let* $f_A$ *be any soft set of an AG-groupoid* $S$ *over* $U$. *Then,* $f_A$ *is an* SI*-right-ideal (* SI*-left-ideal) of* $S$ *over* $U$ *if and only if* $f_A \circ X_S \tilde{\subseteq} f_A$ $(X_S \circ f_A \tilde{\subseteq} f_A)$.

**Lemma 6** *A right (left, two-sided) ideal* $R$ *of an AG-groupoid* $S$ *is semiprime if and only if* $X_R$ *is soft semiprime over* $U$.

*Proof* Let $R$ be a right ideal of $S$. By Lemma 1, $X_R$ is an SI-right-ideal of $S$ over $U$. If $a \in S$, then by given assumption $(X_R)(a) \supseteq (X_R)(a^2)$. Now $a^2 \in R$, implies that $a \in R$. Thus every right ideal of $S$ is semiprime. The converse is simple. Similarly every left or two-sided ideal of $S$ is semiprime if and only if its identity soft mapping is *soft* semiprime over $U$. □

**Corollary 1** *If any* SI*-right-ideal (*SI *-left-ideal,* SI*-two-sided-ideal) of an AG-groupoid* $S$ *is S-semiprime, then any right (left, two-sided) ideal of* $S$ *is semiprime.*

The converse of Lemma 6 is not true in general which can be followed from the following example.

**Example 1** *Let us consider an initial universe set* $U$ *given by* $U = \mathbb{Z}$, *and* $S = \{1, 2, 3, 4, 5\}$ *be a set of parameters with the following binary operation.*

| $*$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 5 | 5 | 3 | 5 |
| 3 | 1 | 5 | 5 | 2 | 5 |
| 4 | 1 | 2 | 3 | 4 | 5 |
| 5 | 1 | 5 | 5 | 5 | 5 |

*It is easy to check that* $(S, *)$ *is an AG-groupoid with left identity* 4.

*Notice that the only left ideals of* $S$ *are* $\{1, 2, 5\}$, $\{1, 3, 5\}$, $\{1, 2, 3, 5\}$ *and* $\{1, 5\}$ *respectively which are semiprime. Clearly, the right and two-sided ideals of* $S$ *are* $\{1, 2, 3, 5\}$ *and* $\{1, 5\}$ *which are also semiprime. On the other hand, let* $A = S$ *and define* a soft set $f_A$ *of* $S$ *over* $U$ *as follows:*

$$f_A(x) = \left\{ \begin{array}{c} \mathbb{Z} \text{ if } x = 1 \\ 4\mathbb{Z} \text{ if } x = 2 \\ 4\mathbb{Z} \text{ if } x = 3 \\ 8\mathbb{Z} \text{ if } x = 4 \\ 2\mathbb{Z} \text{ if } x = 5 \end{array} \right\}.$$

*Then, $f_A$ is an* SI*-right-ideal* (SI*-left-ideal,* SI*-two-sided-ideal*) *of $S$ over $U$ but $f_A$ is not soft semiprime. Indeed $f_A(2) \supsetneq f_A(2^2)$.*

**Remark 3** *If any* SI*-interior-ideal of an AG-groupoid $S$ with left identity over $U$ is an $S$-semiprime over $U$, then any interior ideal of $S$ is semiprime. The converse inclusion is not true in general.*

The following lemma will be used frequently in upcoming section without mention in the sequel.

**Lemma 7** *Let $S$ be an AG-groupoid with left identity. Then, $Sa$ and $Sa^2$ are the left and interior ideals of $S$ respectively.*

*Proof* It is simple. □

## Right regular AG-groupoids

An element $a$ of an AG-groupoid $S$ is called a left (*right*) regular element of $S$ if there exists some $x \in S$ such that $a = a^2x$ ($a = xa^2$) and $S$ is called left (*right*) regular if every element of $S$ is left (*right*) regular.

**Remark 4** *Let $S$ be an AG-groupoid with left identity. Then, the concepts of left and right regularity coincide in $S$.*

*Indeed, for every $a \in S$ there exist some $x, y \in S$ such that $a = xa^2 = a^2y$. As $a = xa^2 = ex \cdot aa = aa \cdot xe = a^2y$, and $a = a^2y = xa^2$ also holds in a similar way.*

Let us give an example of an AG-groupoid which will be used for the converse parts of various problems in this section.

**Example 2** *Let us consider an AG-groupoid $S = \{1, 2, 3, 4, 5\}$ with left identity 4 defined in the following multiplication table.*

| $*$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 5 | 5 | 3 | 5 |
| 3 | 1 | 5 | 5 | 2 | 5 |
| 4 | 1 | 2 | 3 | 4 | 5 |
| 5 | 1 | 5 | 5 | 5 | 5 |

*It is easy to check that $S$ is non-commutative and non-associative.*

An AG-groupoid $S$ is called left (right) duo if every left (right) ideal of $S$ is a two-sided ideal of $S$ and is called duo if it is both left and right duo. Similarly an AG-groupoid $S$ is called an SI-left (SI-right) duo if every SI-left-ideal ( SI-right-ideal) of $S$ over $U$ is an SI-two-sided-ideal of $S$ over $U$, and $S$ is called an SI-duo if it is both an SI-left and an SI-right duo.

**Lemma 8** *If every* SI*-left-ideal of an AG-groupoid $S$ with left identity over $U$ is an* SI*-interior-ideal of $S$ over $U$, then $S$ is left duo.*

*Proof* Let $I$ be any left ideal of $S$ with left identity. Now by Lemma 1, the identity soft mapping $X_I$ is an SI-left-ideal of $S$ over $U$. Thus, by hypothesis, $X_I$ is an SI-interior-ideal of $S$ over $U$, and by using Lemma 1 again, $I$ is an interior ideal of $S$. Thus $IS = I \cdot SS = S \cdot IS = SS \cdot IS = SI \cdot SS = SI \cdot S \subseteq I$. This shows that $S$ is left duo. □

The converse part of Lemma 8 is not true in general. Let us consider an AG-groupoid $S$ (from Example 2). It is easy to see that $S$ is left duo because the only left ideal of $S$ is $\{1,5\}$ which is also a right ideal of $S$. Let $A = S$ and define a soft set $f_A$ of $S$ over $U = \{p_1, p_2, p_3, p_4, p_5, p_6\}$ as follows:

$$f_A(x) = \left\{ \begin{array}{c} U \text{ if } x = 1 \\ \{p_1, p_2, p_3, p_4\} \text{ if} x\ = 2 \\ \{p_2, p_3, p_{4,}, p_5\} \text{ if } x = 3 \\ \{p_3, p_{4,}, p_5\} \text{ if } x = 4 \\ \{p_1, p_2, p_3, p_{4,} p_5\} \text{ if } x = 5 \end{array} \right\}.$$

Then, it is easy to see that $f_A$ is an SI -left-ideal of $S$ over $U$ but it is not an SI -interior-ideal of $S$ over $U$ because $f_A(42 * 4) \not\supseteq f_A(2)$.

**Corollary 2** *Every interior ideal of an AG-groupoid S with left identity is a right ideal of S.*

**Theorem 1** *Every* SI*-right-ideal of an AG-groupoid S with left identity is an* SI*-interior-ideal of S over U if and only if S is right duo.*

*Proof* It is simple. □

**Theorem 2** *Let S be a right regular AG-groupoid with left identity. Then, S is left duo if and only if every* SI*-left-ideal of S over U is an* SI*-interior-ideal of S over U.*

*Proof* Necessity. Let a right regular $S$ with left identity be a left duo, and assume that $f_A$ is any SI-left-ideal of $S$ over $U$. Let $a, b, c \in S$, then $b \leq xb^2$ for some $x \in S$. Since $Sa$ is a left ideal of $S$, therefore by hypothesis, $Sa$ is a two-sided ideal of $S$. Thus, $ab \cdot c = a(x \cdot bb) \cdot c = a(b \cdot xb) \cdot c = b(a \cdot xb) \cdot c = c(a \cdot xb) \cdot b$. It follows that $ab \cdot c \in S(a \cdot SS) \cdot b \subseteq (S \cdot aS)b = (SS \cdot aS)b = (Sa \cdot SS)b \subseteq (Sa \cdot S)b \subseteq Sa \cdot b$. Thus, $ab \cdot c = ta \cdot b$ for some $t \in S$, and therefore $f_A(ab \cdot c) = f_A(ta \cdot b) \supseteq f_A(b)$, implies that $f_A$ is an SI-interior-ideal of $S$ over $U$.

Sufficiency. It can be followed from Lemma 8. □

By left-right dual of above Theorem, we have the following Theorem:

**Theorem 3** *Let S be a right regular AG-groupoid with left identity. Then ,S is right duo if and only if every* SI*-right-ideal of S over U is an* SI*-interior-ideal of S over U.*

**Lemma 9** *A non-empty subset A of a right regular AG-groupoid S with left identity is a two-sided ideal of S if and only if it is an interior ideal of S.*

*Proof* It is simple. □

**Lemma 10** *Every left ideal of an AG-groupoid S with left identity is an interior ideal of S if S is an* SI*-left duo.*

*Proof* It can be followed from Lemmas 1 and 9. □

The converse of Lemma 10 is not true in general. The only left ideal of $S$ (from Example 2) is $\{1, 2\}$ which is also an interior ideal of $S$. Let $A = \{2, 3, 4, 5\}$ and define a soft set $f_A$ of $S$ over $U = \mathbb{Z}$ as follows:

$$f_A(x) = \left\{ \begin{array}{c} 4\mathbb{Z} \text{ if } x = 2 \\ 8\mathbb{Z} \text{ if } x = 3 \\ 16\mathbb{Z} \text{ if } x = 4 \\ 2\mathbb{Z} \text{ if } x = 5 \end{array} \right\}.$$

Then, it is easy to see that $f_A$ is an SI -left-ideal of $S$ over $U$ but it is not an SI -right-ideal of $S$ over $U$ because $f_A(2 * 4) \not\supseteq f_A(2)$.

It is easy to see that every SI-right-ideal of $S$ with left identity over $U$ is an SI-left-ideal of $S$ over $U$.

**Remark 5** *Every* SI*-right-ideal of an AG-groupoid S with left identity is an* SI*-left-ideal of S over U, but the converse is not true in general.*

**Theorem 4** *Every right ideal of an AG-groupoid S with left identity is an interior ideal of S if and only if S is an* SI*-right duo.*

*Proof* It is straightforward. □

**Theorem 5** *Let S be a right regular AG-groupoid with left identity. Then ,S is an* SI*-left duo if and only if every left ideal of S is an interior ideal of S.*

*Proof* The direct part can be followed from Lemma 10. The converse is simple. □

By left-right dual of above Theorem, we have the following Theorem.

**Theorem 6** *Let S be a right regular AG-groupoid with left identity. Then ,S is an* SI*-right duo if and only if every right ideal of S is an interior ideal of S.*

**Theorem 7** *Let S be an AG-groupoid with left identity and* $E = \{x \in S : x = x^2\} \subseteq S$*. Then the following assertions hold*:

(*i*) *E forms a semilattice*;
(*ii*) *E is a singleton set, if* $a = ax \cdot a, \forall\, a, x \in S$*.*

*Proof* It is simple. □

**Theorem 8** *For an AG-groupoid S with left identity, the following conditions are equivalent*:

(*i*) *S is right regular*;
(*ii*) *For any interior ideal I of S*;
(*a*) $I \subseteq I^2$,
(*b*) *I is semiprime.*
(*iii*) *For any* SI*-interior-ideal* $f_A$ *of S over U*;
(*a*) $f_A \mathrel{\tilde{\subseteq}} f_A \circ f_A$,
(*b*) $f_A$ *is soft semiprime over U.*

*(iv) S is right regular and* $|E| = 1$, $(a = ax \cdot a, \forall\, a, x \in E)$;
*(v) S is right regular and* $\emptyset \neq E \subseteq S$ *is semilattice.*

*Proof* $(i) \Longrightarrow (v) \Longrightarrow (iv)$ can be followed from Theorem 7.

$(iv) \Longrightarrow (iii)$ : $(a)$. Let $f_A$ be any SI -interior-ideal of a right regular $S$ with left identity. Thus, for each $a \in S$, there exists some $x \in S$ such that $a = x \cdot aa = a \cdot xa = a \cdot x(x \cdot aa) = a \cdot (ex)(a \cdot xa) = a \cdot (xa \cdot a)(xe)$. Therefore,

$$\begin{aligned}(f_A \circ f_A)(a) &= \bigcup_{a = a\cdot(xa\cdot a)(xe)} \left\{ f_A(a) \cap f_A((xa \cdot a)(xe)) \right\} \\ &\supseteq f_A(a) \cap f_A((xa \cdot a)(xe)) \supseteq f_A(a) \cap f_A(a) = f_A(a).\end{aligned}$$

This shows that $f_A \,\widetilde{\subseteq}\, f_A \circ f_A$.

$(b)$. Also,

$$\begin{aligned} a &= x \cdot aa \leq ex \cdot aa = aa \cdot xe = (a \cdot xa^2)(xe) = (x \cdot aa^2)(xe) = x(ea \cdot aa) \cdot (xe) \\ &= x(aa \cdot ae) \cdot (xe) = (aa)(x \cdot ae) \cdot (xe) = (ae \cdot x)(aa) \cdot (xe) = (ae \cdot x)a^2 \cdot (xe). \end{aligned}$$

This implies that $f_A(a) = f_A((ae \cdot x)a^2 \cdot (xe)) \supseteq f_A(a^2)$. Hence, $f_A$ is *soft* semiprime.

$(iii) \Longrightarrow (ii)$ : $(a)$. Assume that $I$ is any interior ideal of $S$, then by using Lemma 1, $X_I$ is an SI-interior-ideal of $S$ over $U$. Let $i \in I$, then by using Lemma 2, we have $X_I(i) \subseteq (X_I \circ X_I)(i) = (X_I)(i) = U$. Hence, $I \subseteq I^2$.

$(b)$. Let $i^2 \in I$. Then, by given assumption, we have $X_I(i) \supseteq X_I(i^2) = U$. This implies that $i \in I$, and therefore $I$ is semiprime.

$(ii) \Longrightarrow (i)$ : Let $a \in S$ with left identity. Since $Sa^2$ is an interior ideals of $S$, and clearly $a^2 \in Sa^2$. Thus, by using given assumption, $a \in Sa^2$. Hence, $S$ is right regular. □

**Corollary 3** *Every* SI*-interior-ideal of a right regular AG-groupoid S with left identity is soft semiprime over U.*

*Proof* Let $f_I$ be any SI-interior-ideal of a right regular $S$ with left identity. Then, for each $a \in S$, there exists some $x \in S$ such that $f_I(a) = f_I(x \cdot aa) = f_I(a \cdot xa) = f_I(xa^2 \cdot xa) \supseteq f_I(a^2)$. □

**Corollary 4** *Let I be an interior ideal of an AG-groupoid S. Then, I is semiprime if and only if* $X_I$ *is soft semiprime over U.*

**Theorem 9** *Let S be an AG-groupoid with left identity. Then, S is right regular if and only if every* SI*-interior-ideal* $f_A$ *of S over U is soft idempotent and soft semiprime.*

*Proof* Necessity: Let $f_A$ be any SI-interior-ideal of a right regular $S$ with left identity over $U$. Then, clearly $f_A \circ f_A \,\widetilde{\subseteq}\, f_A$. Now for each $a \in S$, there exists some $x \in S$ such that $a = x \cdot aa = a \cdot xa = ea \cdot xa = ax \cdot ae = (ae \cdot x)a$. Thus,

$$\begin{aligned}(f_A \circ f_A)(a) &= \bigcup_{a=(ae\cdot x)a} \left\{ f_A(ae \cdot x) \cap f_A(a) \right\} \supseteq f_A(ae \cdot x) \cap f_A(a) \\ &\supseteq f_A(a) \cap f_A(a) = (f_A \cap f_A)(a) = f_A(a).\end{aligned}$$

This shows that $f_A$ is *soft* idempotent over $U$. Again $a = ex \cdot aa = aa \cdot xe = a^2 \cdot xe$. Therefore, $f_A(a) = f_A(a^2 \cdot xe) \supseteq f_A(a^2)$. Hence, $f_A$ is *soft* semiprime over $U$.

Sufficiency: Since $Sa^2$ is an interior ideal of $S$, therefore by Lemma 1, its soft characteristic function $X_{Sa^2}$ is an SI-interior-ideal of $S$ over $U$ such that $X_{Sa^2}$ is *soft* idempotent over $U$. Since by given assumption, $X_{Sa^2}$ is *soft* semiprime over $U$ so by Corollary 4, $Sa^2$ is semiprime. Since $a^2 \in Sa^2$, therefore, $a \in a^2S$. Thus, by using Lemma 2, we have $X_{Sa^2} \circ X_{Sa^2} = X_{Sa^2}$, and $X_{Sa^2} \circ X_{Sa^2} = X_{(Sa^2 \cdot Sa^2)}$. Thus, we get $X_{(Sa^2 \cdot Sa^2)} = X_{Sa^2}$. This implies that $X_{(Sa^2 \cdot Sa^2)}(a) = X_{Sa^2}(a) = U$. Therefore, $a \in Sa^2 \cdot Sa^2 = a^2S \cdot Sa^2 = (Sa^2 \cdot S)a^2 \subseteq Sa^2$. Hence ,$S$ is right regular. □

**Lemma 11** *Every* SI*-interior-ideal of a right regular AG-groupoid S with left identity over U is soft idempotent.*

*Proof* Let $f_A$ be any SI-interior-ideal of a right regular $S$ with left identity over $U$. Then, by using Lemma 4, $f_A \circ f_A \widetilde{\subseteq} f_A$. Since $S$ right regular, therefore for every $a \in S$ there exists some $x \in S$ such that $a = x \cdot aa = a \cdot xa = xa^2 \cdot xa = ax \cdot a^2x = (a^2x \cdot x)a = (xx \cdot aa)a = (aa \cdot x^2)a$. Therefore,

$$(f_A \circ f_A)(a) = \bigcup_{a=(aa \cdot x^2)a} \left\{f_A(aa \cdot x^2) \cap f_A(a)\right\} \supseteq f_A(aa \cdot x^2) \cap f_A(a)$$
$$\supseteq f_A(a) \cap f_A(a) = (f_A \cap f_A)(a).$$

Thus, $f_A \circ f_A = f_A$. □

**Theorem 10** *Let S be an AG-groupoid with left identity and $f_A$ be any* SI*-interior-ideal of S over U. Then ,S is right regular if and only if $f_A = (X_S \circ f_A)^2$ and $f_A$ is soft semiprime.*

*Proof* Necessity: Let $f_A$ be any SI-interior-ideal of a right regular $S$ with left identity over $U$. Then, by using Lemmas 4 and 2, we have

$$(X_S \circ (X_S \circ f_A)) \circ X_S = (X_S \circ f_A) \circ X_S = (f_A \circ X_S) \circ X_S = (X_S \circ X_S) \circ f_A = X_S \circ f_A.$$

This shows that $X_S \circ f_A$ is an SI -interior-ideal of $S$ over $U$. Now by using Lemmas 11 and 4, we have $(X_S \circ f_A)^2 = X_S \circ f_A = f_A$. It is easy to see that $f_A$ is *soft* semiprime.

Sufficiency: Let $f_A = (X_S \circ f_A)^2$ holds for any SI-interior-ideal $f_A$ of $S$ over $U$. Then, by given assumption and Lemma 14, we get, $f_A = (X_S \circ f_A)^2 = f_A^2$. Thus, by using Theorem 9, $S$ is right regular. □

**Corollary 5** *Let S be an AG-groupoid with left identity and $f_A$ be any* SI*-interior-ideal of S over U. Then, S is right regular if and only if $f_A = X_S \circ f_A^2$ and $f_A$ is soft semiprime.*

*Proof* From above theorem, $f_A = (X_S \circ f_A)^2 = (X_S \circ f_A)(X_S \circ f_A) = (X_S \circ f_A) \circ f_A = (f_A \circ f_A) \circ X_S = (f_A \circ f_A) \circ (X_S \circ X_S) = (X_S \circ X_S) \circ (f_A \circ f_A) = X_S \circ f_A^2$. □

**Lemma 12** *Let S be an AG-groupoid with left identity and $f_A$ be any* SI*-left-ideal (*SI *-right-ideal,* SI*-two-sided-ideal) of S over U. Then, S is right regular if and only if $f_A$ is soft idempotent.*

*Proof* Necessity: Let $f_A$ be an SI-left-ideal of a right regular $S$ with left identity over $U$. Then, it is easy to see that $f_A \circ f_A \mathrel{\tilde{\subseteq}} f_A$. Let $a \in S$, then there exists $x \in S$ such that $a = aa \cdot x = xa \cdot a$. Thus

$$(f_A \circ f_A)(a) = \bigcup_{a=xa\cdot a} \{f_A(xa) \cap f_A(a)\} \supseteq f_A(a) \cap f_A(a) = f_A(a),$$

which implies that $f_A$ is *soft* idempotent.

Sufficiency: Assume that $f_A \circ f_A = f_A$ holds for all SI-left-ideal of $S$ with a left identity over $U$. Since $Sa$ is a left ideal of $S$, therefore by Lemma 1, it follows that $X_{Sa}$ is an SI-left-ideal of $S$ over $U$. Since $a \in Sa$, it follows that $(X_{Sa})(a) = U$. By hypothesis and Lemma 2, we obtain $(X_{Sa}) \circ (X_{Sa}) = X_{Sa}$ and $(X_{Sa}) \circ (X_{Sa}) = X_{Sa\cdot Sa}$. Thus, we have $(X_{Sa\cdot Sa})(a) = X_{Sa}(a) = U$, which implies that $a \in Sa \cdot Sa$. Therefore, $a \in Sa \cdot Sa = S^2a^2 = Sa^2$. This shows that $S$ is right regular. □

**Theorem 11** *Let S be an AG-groupoid with left identity and $f_A$ be any* SI*-left-ideal* (SI *-right-ideal,* SI*-two-sided-ideal*) *of S over U. Then, S is right regular if and only if $f_A = (X_S \circ f_A) \circ (X_S \circ f_A)$.*

*Proof* Necessity: Let $S$ be a right regular $S$ with left identity and let $f_A$ be any SI-left-ideal of $S$ over $U$. It is easy to see that $X_S \circ f_A$ is also an SI -left-ideal of $S$ over $U$. By Lemma 12, we obtain $(X_S \circ f_A) \circ (X_S \circ f_A) = (X_S \circ f_A) \mathrel{\tilde{\subseteq}} f_A$. Let $a \in S$, then there exists $x \in S$ such that $a = aa \cdot x = xa \cdot a = (xa)(aa \cdot x) = (xa)(xa \cdot a)$. Therefore,

$$\begin{aligned}\big((X_S \circ f_A) \circ (X_S \circ f_A)\big)(a) &\supseteq (X_S \circ f_A)(xa) \cap (X_S \circ f_A)(xa \cdot a)\\ &\supseteq X_S(x) \cap f_A(a) \cap X_S(xa) \cap f_A(a) = f_A(a),\end{aligned}$$

which is what we set out to prove.

Sufficiency: Suppose that $f_A = (X_S \circ f_A) \circ (X_S \circ f_A)$ holds for all SI-left-ideal $f_A$ of $S$ over $U$. Then $f_A = (X_S \circ f_A) \circ (X_S \circ f_A) \mathrel{\tilde{\subseteq}} f_A \circ f_A \mathrel{\tilde{\subseteq}} X_S \circ f_A \mathrel{\tilde{\subseteq}} f_A$. Thus, by Lemma 12, it follows that $S$ is right regular. □

**Lemma 13** *Let $f_A$ be any* SI*-interior-ideal of a right regular AG-groupoid S with left identity over U. Then, $f_A(a) = f_A(a^2)$, for all $a \in S$.*

*Proof* Let $f_A$ be any SI-interior-ideal of a right regular $S$ with left identity over $U$. For $a \in S$, there exists some $x$ in $S$ such that $a = ex\cdot aa = aa\cdot xe = (xe\cdot a)a = (xe\cdot a)(ex\cdot aa) = (xe\cdot a)(aa\cdot xe) = aa\cdot(xe\cdot a)(xe) = ea^2\cdot(xe\cdot a)(xe)$. Therefore $f_A(a) = f_A(ea^2\cdot(xe\cdot a)(xe)) \supseteq f_A(a^2) = f_A(aa) = f_A(a(ex \cdot aa)) = f_A(a(aa \cdot xe)) = f_A((aa)(a \cdot xe)) = f_A((xe \cdot a)(aa)) \supseteq f_A(a)$. That is, $f_A(a) = f_A(a^2)$, $\forall\, a \in S$ □

The converse of Lemma 13 is not true in general. Let us consider an AG-groupoid $S$ (from Example 2). Let $A = \{1, 2, 4, 5\}$ and define a soft set $f_A$ of $S$ over $U = \left\{\begin{bmatrix} 0 & 0 \\ x & x \end{bmatrix} / x \in \mathbb{Z}_3\right\}$ (the set of all $2 \times 2$ matrices with entries from $\mathbb{Z}_3$) as follows:

$$f_A(x) = \left\{ \begin{array}{c} \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix} \right\} \text{ if } x = 1 \\ \left\{ \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix} \right\} \text{ if } x = 2 \\ \left\{ \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix} \right\} \text{ if } x = 4 \\ \left\{ \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 2 & 2 \end{bmatrix} \right\} \text{ if } x = 5 \end{array} \right\}.$$

It is easy to see that $f_A$ is an SI -interior-ideal of $S$ such that $f_A(x) \supseteq f_A(x^2)$, $\forall\, x \in S$ but $S$ is not right regular.

On the other hand, it is easy to see that every SI -two-sided-ideal of $S$ over $U$ is an SI -interior-ideal of $S$ over $U$.

**Remark 6** *Every SI-two-sided-ideal of a right regular AG-groupoid S with left identity over U is an SI-interior-ideal of S over U but the converse is not true in general.*

**Theorem 12** *For an AG-groupoid S with left identity, the following conditions are equivalent*:

$(i)$ *S is right regular*;
$(ii)$ *Every interior ideal of S is semiprime*;
$(iii)$ *Every SI-interior-ideal of S over U is soft semiprime*;
$(iv)$ *For every SI-interior-ideal $f_A$ of S over $U$, $f_A(a) = f_A(a^2)$, $\forall\, a \in S$.*

*Proof* $(i) \Rightarrow (iv)$ can be followed from Lemma 13.

$(iv) \Rightarrow (iii)$ and $(iii) \Rightarrow (ii)$ are obvious.

$(ii) \Rightarrow (i)$ : Since $Sa^2$ is an interior ideal of $S$ with left identity such that $a^2 \in Sa^2$, therefore by given assumption, we have $a \in Sa^2$. Thus, $S$ is right regular. □

## Weakly regular AG$^{***}$-groupoids

An AG-groupoid $S$ is called an *AG$^{***}$-groupoid* [29] if the following conditions are satisfied:

$(i)$ For all $a, b, c \in S$, $a \cdot bc = b \cdot ac$;

$(ii)$ For all $a \in S$, there exist some $b, c \in S$ such that $a = bc$.

An AG-groupoid satisfying $(i)$ is called an AG$^{**}$-groupoid. The condition $(ii)$ for an AG$^{**}$-groupoid to become an AG$^{***}$-groupoid is equivalent to $S = S^2$.

Let $S = \{1, 2, 3, 4\}$ be an AG-groupoid define in the following multiplication table.

| · | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 4 | 3 | 4 |
| 3 | 1 | 2 | 4 | 4 |
| 4 | 1 | 4 | 4 | 4 |

It is easy to verify that $(S, \cdot)$ is an AG$^{***}$ -groupoid.

Note that every AG-groupoid with left identity is an AG$^{***}$ -groupoid but the converse is not true in general. An AG-groupoid in the above example is an AG$^{***}$-groupoid, but it does not contains a left identity. Hence, we can say that an AG$^{***}$-groupoid is the generalization of an AG-groupoid with left identity.

An element $a$ of an AG-groupoid $S$ is called a weakly regular element of $S$ if there exist some $x, y \in S$ such that $a = ax \cdot ay$ and $S$ is called weakly regular if every element of $S$ is weakly regular.

**Remark 7** *Let S be an AG$^{***}$-groupoid. Then, the concepts of weak and right regularity coincide in S.*

Let $S$ be an AG$^{***}$-groupoid. From now onward, $R$ (*resp.* $L$) will denote any right (*resp.* left) ideal of $S$; $\langle R\rangle_{a^2}$ will denote a right ideal $Sa^2 \cup a^2$ of $S$ containing $a^2$ and $\langle L\rangle_a$ will denote a left ideal $Sa \cup a$ of $S$ containing $a$; $f_A$ (*resp.* $g_B$) will denote any SI-right-ideal of $S$ (*resp.* SI-left-ideal of $S$) over $U$ unless otherwise specified.

**Theorem 13** *Let S be an AG$^{***}$-groupoid . Then, S is weakly regular if and only if* $\langle R\rangle_{a^2} \cap \langle L\rangle_a = \langle R\rangle_{a^2}^2 \langle L\rangle_a^2$ *and* $\langle R\rangle_{a^2}$ *is semiprime.*

*Proof* Necessity: Let $S$ be weakly regular. It is easy to see that $\langle R\rangle_{a^2}^2 \langle L\rangle_a^2 \subseteq \langle R\rangle_{a^2} \cap \langle L\rangle_a$. Let $a \in \langle R\rangle_{a^2} \cap \langle L\rangle_a$. Then, there exist some $x, y \in S$ such that

$$\begin{aligned} a &= ax \cdot ay = (ax \cdot ay)x \cdot (ax \cdot ay)y = (x \cdot ay)(ax) \cdot (y \cdot ay)(ax) \\ &= (a \cdot xy)(ax) \cdot (ay^2)(ax) = (a \cdot xy)(ax) \cdot (xa)(y^2a) \\ &\in (\langle R\rangle_{a^2} S \cdot \langle R\rangle_{a^2} S)(S \langle L\rangle_a \cdot S \langle L\rangle_a) \subseteq \langle R\rangle_{a^2}^2 \langle L\rangle_a^2, \end{aligned}$$

which shows that $\langle R\rangle_{a^2} \cap \langle L\rangle_a = \langle R\rangle_{a^2}^2 \langle L\rangle_a^2$. It is easy to see that $\langle R\rangle_{a^2}$ is semiprime.

Sufficiency: Since $Sa^2 \cup a^2$ and $Sa \cup a$ are the right and left ideals of $S$ containing $a^2$ and $a$ respectively. Thus, by using given assumption, we get

$$\begin{aligned} a &\in \left(Sa^2 \cup a^2\right) \cap (Sa \cup a) = \left(Sa^2 \cup a^2\right)^2 (Sa \cup a)^2 \\ &= \left(Sa^2 \cup a^2\right)\left(Sa^2 \cup a\right) \cdot (Sa \cup a)(Sa \cup a) \\ &\subseteq S\left(Sa^2 \cup a\right) \cdot S(Sa \cup a) = \left(S \cdot Sa^2 \cup Sa\right)(S \cdot Sa \cup Sa) \\ &= \left(a^2S \cdot S \cup Sa\right)(aS \cdot S \cup Sa) = \left(Sa^2 \cup Sa\right)(Sa \cup Sa) \\ &= \left(a^2S \cup Sa\right)(Sa \cup Sa) = (Sa \cdot a \cup Sa)(Sa \cup Sa) \\ &\subseteq (Sa \cup Sa)(Sa \cup Sa) = Sa \cdot Sa = aS \cdot aS. \end{aligned}$$

This implies that $S$ is weakly regular. □

**Corollary 6** *Let S be an AG$^{***}$-groupoid . Then ,S is weakly regular if and only if* $\langle R\rangle_{a^2} \cap \langle L\rangle_a = \langle L\rangle_a^2 \langle R\rangle_{a^2}^2$ *and* $\langle R\rangle_{a^2}$ *is semiprime.*

**Theorem 14** *Let S be an AG$^{***}$-groupoid. Then, the following conditions are equivalent*:

(*i*) *S is weakly regular*;
(*ii*) $\langle R\rangle_{a^2} \cap \langle L\rangle_a = \langle L\rangle_a^2 \langle R\rangle_{a^2}^2$ *and* $\langle R\rangle_{a^2}$ *is semiprime*;
(*iii*) $R \cap L = L^2R^2$ *and R semiprime* ;
(*iv*) $f_A \tilde{\cap} g_B = (f_A \circ g_B) \circ (f_A \circ g_B)$ *and* $f_A$ *is soft semiprime*;
(*v*) *S is weakly regular and* $|E| = 1$, $(a = ax \cdot a, \forall\, a, x \in E)$;
(*vi*) *S is weakly regular and* $\emptyset \neq E \subseteq S$ *is semilattice.*

*Proof* $(i) \Longrightarrow (vi) \Longrightarrow (v)$ : It can be followed from Theorem 7.

$(v) \Longrightarrow (iv)$ : Let $f_A$ and $g_B$ be any SI -right-ideal and SI-left-ideal of a weakly regular $S$ over $U$ respectively. From Lemma 5, it is easy to show that $(f_A \circ g_B) \circ (f_A \circ g_B) \,\widetilde{\subseteq}\, f_A \,\widetilde{\cap}\, g_B$. Now for $a \in S$, there exist some $x, y \in S$ such that

$$\begin{aligned} a &= ax \cdot ay = (ax \cdot ay)x \cdot (ax \cdot ay)y = (ax \cdot ay) \cdot ((ax \cdot ay)x)y \\ &= (ax \cdot ay) \cdot (yx)(ax \cdot ay) = (ax \cdot ay) \cdot (ax)(yx \cdot ay) \\ &= (ax \cdot ay) \cdot (ay \cdot yx)(xa) = (ax \cdot ay) \cdot ((yx \cdot y)a)(xa) \\ &= (ax)((yx \cdot y)a) \cdot (ay)(xa) = (ax)(ba) \cdot (ay)(xa), \text{ where } yx \cdot y = b. \end{aligned}$$

Therefore,

$$\begin{aligned} ((f_A \circ g_B) \circ (f_A \circ g_B))(a) &= \bigcup_{a=(ax)(ba)\cdot(ay)(xa)} \{(f_A \circ g_B)(ax \cdot ba) \\ &\quad \cap (f_A \circ g_B)(ay \cdot xa)\} \\ &\supseteq \bigcup_{ax \cdot ba = ax \cdot ba} \{f_A(ax) \cap g_B(ba)\} \\ &\quad \cap \bigcup_{ay \cdot xa = ay \cdot xa} \{f_A(ay) \cap g_B(xa)\} \\ &\supseteq f_A(ax) \cap g_B(ba) \cap f_A(ay) \cap g_B(xa) \\ &\supseteq f_A(a) \cap g_B(a), \end{aligned}$$

which shows that $(f_A \circ g_B) \circ (f_A \circ g_B) \,\widetilde{\supseteq}\, f_A \,\widetilde{\cap}\, g_B$. Hence, $f_A \,\widetilde{\cap}\, g_B = (f_A \circ g_B) \circ (f_A \circ g_B)$. Also by using Lemma 3, $f_A$ is *soft* semiprime.

$(iv) \Longrightarrow (iii)$ : Let $R$ and $L$ be any left and right ideals of $S$. Then, by using Lemma 1, $X_R$ and $X_L$ are the SI-right-ideal and SI-left-ideal of $S$ over $U$ respectively. Now by using Lemma 2, we get $X_{R\cap L} = X_R \,\widetilde{\cap}\, X_L = (X_R \circ X_L) \circ (X_R \circ X_L) = (X_R \circ X_R) \circ (X_L \circ X_L) = X_{R^2} \circ X_{L^2} = X_{R^2L^2} = X_{L^2R^2}$, which implies that $R \cap L = L^2R^2$.

$(iii) \Longrightarrow (ii)$ : It is simple.

$(ii) \Longrightarrow (i)$ : It can be followed from Corollary 6. □

**Lemma 14** *Let R be a right ideal and L be a left ideal of a unitary AG-groupoid S with left identity respectively. Then ,RL is a left ideal of S.*

*Proof* It is simple. □

**Theorem 15** *Let S be an* $AG^{***}$*-groupoid. Then, the following conditions are equivalent:*

*(i) S is weakly regular;*
*(ii)* $\langle R\rangle_{a^2} \cap \langle L\rangle_a = \langle R\rangle_{a^2} \langle L\rangle_a \cdot \langle R\rangle_{a^2}$ *and* $\langle R\rangle_{a^2}$ *is semiprime;*
*(iii)* $R \cap L = RL \cdot R$ *and R is semiprime;*
*(iv)* $f_A \,\widetilde{\cap}\, g_B = (f_A \circ g_B) \circ f_A$ *and* $f_A$ *is soft semiprime;*
*(v) S is weakly regular and* $|E| = 1$, $(a = ax \cdot a, \forall\, a, x \in E)$;
*(vi) S is weakly regular and* $\emptyset \neq E \subseteq S$ *is semilattice.*

*Proof* $(i) \Longrightarrow (vi) \Longrightarrow (v)$ : It can be followed from Theorem 7.

$(v) \Longrightarrow (iv)$ : Let $f_A$ and $g_B$ be any SI -left-ideals of a weakly regular $S$ over $U$. Now, for $a \in S$, there exist some $x, y \in S$ such that $a = ax \cdot ay = ax \cdot (ax \cdot ay)y = ((ax \cdot ay)y \cdot x)a = (xy \cdot (ax \cdot ay))a = (ax \cdot (xy \cdot ay))a = (ax \cdot (a \cdot (xy)y))a$.

Therefore,

$$\begin{aligned} ((f_A \circ g_B) \circ f_A)(a) &= \bigcup_{a=(ax\cdot(a\cdot(xy)y))a} \{(f_A \circ g_B)(ax \cdot (a \cdot (xy)y)) \cap g_B(a)\} \\ &\supseteq \bigcup_{ax\cdot(a\cdot(xy)y)=ax\cdot(a\cdot(xy)y} \{f_A(ax) \cap g_B(a \cdot (xy)y)\} \cap g_B(a) \\ &\supseteq f_A(ax) \cap g_B(a \cdot (xy)y) \cap g_B(a) \supseteq f_A(a) \cap g_B(a), \end{aligned}$$

which shows that $(f_A \circ g_B) \circ f_A \mathrel{\widetilde{\supseteq}} f_A \mathbin{\widetilde{\cap}} g_B$. By using Lemmas 5 and 3, it is easy to show that $(f_A \circ g_B) \circ f_A \mathrel{\widetilde{\subseteq}} f_A \mathbin{\widetilde{\cap}} g_B$. Thus, $f_A \mathbin{\widetilde{\cap}} g_B = (f_A \circ g_B) \circ f_A$. Also, by using Lemma 3, $f_A$ is *soft* semiprime.

$(iv) \Longrightarrow (iii)$ : Let $R$ and $L$ be any left and right ideals of $S$ respectively. Then, by Lemma 1, $X_R$ and $X_L$ are the SI-right-ideal and SI -left-ideal of $S$ over $U$ respectively. Now, by using Lemmas 2, 14, we get $X_{R\cap L} = X_R \mathbin{\widetilde{\cap}} X_L = (X_R \circ X_L) \circ X_L = X_{RL\cdot R}$, which shows that $R \cap L = RL \cdot R$. Also, by using Lemma 6, $R$ is semiprime.

$(iii) \Longrightarrow (ii)$ : It is obvious.

$(ii) \Longrightarrow (i)$ : Since $Sa^2 \cup a^2$ and $Sa \cup a$ are the right and left ideals of $S$ containing $a^2$ and $a$ respectively. Thus, by using given assumption and Lemma, we get

$$\begin{aligned} a &\in (Sa^2 \cup a^2) \cap (Sa \cup a) = (Sa^2 \cup a^2)(Sa \cup a) \cdot (Sa^2 \cup a^2) \\ &\subseteq S(Sa \cup a) \cdot (Sa^2 \cup a^2) = (S^2a \cup Sa)(Sa^2 \cup a^2) \\ &= (S^2a \cdot Sa^2) \cup (S^2a \cdot a^2) \cup (Sa \cdot Sa^2) \cup (S^2a \cdot a^2) \\ &\subseteq (Sa \cdot a^2S) \cup (Sa \cdot Sa) \cup (Sa \cdot a^2S) \cup (Sa \cdot Sa) \\ &\subseteq (Sa \cdot Sa) \cup (Sa \cdot Sa) \cup (Sa \cdot Sa) \cup (Sa \cdot Sa) \\ &= Sa \cdot Sa = aS \cdot aS. \end{aligned}$$

Hence, $S$ is weakly regular. □

## Comparison of SI-left (right, two-sided, interior) ideals

A very major and an abstract conclusion from this section is that SI-left-ideal, SI-right-ideal and SI-interior-ideal need not to be coincide in an AG-groupoid $S$ even if $S$ has a left identity, but they will coincide in a right regular class of an AG-groupoid $S$ with left identity.

**E-1.** Take a collection of 8 chemicals as an initial universe set $U$ given by $U = \{s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8\}$.

Let a set of parameters $S = \{1, 2, 3, 4, 5\}$ be a set of particular properties of each chemical in $U$ with the following type of natures:

1 stands for the parameter "density",

2 stands for the parameter "melting point",

3 stands for the parameter "combustion",

4 stands for the parameter "enthalpy",

5 stands for the parameter "toxicity".

Let us define the following binary operation on a set of parameters $S$ as follows.

| $*$ | 1 | 2 | 3 | 4 | 5 |
|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 1 | 2 | 2 | 2 | 2 |
| 3 | 1 | 2 | 4 | 5 | 3 |
| 4 | 1 | 2 | 3 | 4 | 5 |
| 5 | 1 | 2 | 5 | 3 | 4 |

It is easy to check that $(S, *)$ is non-commutative and non-associative. Also, by routine calculation, one can easily verify that $(S, *)$ forms an AG-groupoid with left identity 4. Note that $S$ is left (*right*) regular. Indeed, for $a \in S$ there does exists some $x \in S$ such that $a = xa^2$ $(a = a^2x)$.

Let $A = S$ and define a soft set $f_A$ of $S$ over $U$ as follows:

$$f_A(x) = \left\{ \begin{array}{c} \{s_1, s_2, s_3, s_{4,}s_5, s_6\} \text{ if } x = 1 \\ \{s_2, s_3, s_{4,}\} \text{ if } x = 2 \\ \{s_2, s_3 \text{ if } x = 3 = 4 = 5 \end{array} \right\}.$$

Then, it is easy to verify that $f_A$ is an SI -interior-ideal of $S$ over $U$.

**E-2.** There are seven civil engineers in an initial universe set $U$ given by $U = \{s_1, s_2, s_3, s_4, s_5, s_6, s_7\}$.

Let a set of parameters $S = \{1, 2, 3\}$ be a set of status of each civil engineer in $U$ with the following type of attributes:

1 stands for the parameter "critical thinking",

2 stands for the parameter "decision making",

3 stands for the parameter "project management".

Let us define the following binary operation on a set of parameters $S$ as follows.

| $*$ | 1 | 2 | 3 |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| 2 | 3 | 3 | 3 |
| 3 | 1 | 1 | 1 |

It is easy to check that $(S, *)$ is non-commutative and non-associative. One can easily verify that $(S, *)$ forms an AG-groupoid. Note that $S$ is not left (*right*) regular. Indeed for $3 \in S$ there does not exists some $x \in S$ such that $3 = x * 3^2$ $(3 = 3^2 * x)$. Let $A = S$ and define a soft set $f_A$ of $S$ over $U$ as follows:

$$f_A(x) = \left\{ \begin{array}{c} \{s_1, s_2, s_3, s_4\} \text{ if } x = 1 \\ \{s_1, s_2, s_3\} \text{ if } x = 2 \\ \{s_2, s_3\} \text{ if } x = 3 \end{array} \right\}.$$

Then, it is easy to verify that $f_A$ is an SI -interior-ideal of $S$ over $U$ but it is not an SI -left-ideal, SI-right-ideal, and SI -interior-ideal of $S$ which can be seen from the following:

$$f_A(2 * 2) \supsetneq f_A(2) \text{ and } f_A(3 * 2) \supsetneq f(2).$$

**Lemma 15** *Every* SI*-right-ideal of an AG-groupoid S with left identity over U is an* SI*-left-ideal of S over U*.

*Proof* It is simple. □

The converse of above Lemma is not true in general which can be seen from the following example.

**E-3.** Let us consider an AG-groupoid $S$ with left identity 4 given in an Example 1 with an initial universe set $U = \{s_1, s_2, ..., s_{12}\}$. Let a set of parameters $S = \{1, 2, 3, 4, 5\}$ be a set of status of houses in which,

1 stands for the parameter "beautiful" ,

2 stands for the parameter "cheap" ,

3 stands for the parameter "in good location" ,

4 stands for the parameter "in green surroundings" ,

5 stands for the parameter "secure" .

It is important to note that $S$ is not right regular because for $3 \in S$ there does not exists some $x \in S$ such that $3 = x * 3^2$.

Let $A = S$ and define a soft set $f_A$ of $S$ over $U$ as follows:

$$f_A(x) = \left\{ \begin{array}{c} U \text{ if } x = 1 \\ \{s_2, s_3, s_4, s_5, s_6, s_7, s_8\} \text{ if } x = 2 \\ \{s_2, s_3, s_4, s_5, s_6\} \text{ if } x = 3 \\ \{s_2, s_3, s_4, s_5\} \text{ if } x = 4 \\ \{s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8, s_9, s_{10}\} \text{ if } x = 5 \end{array} \right\}.$$

It is easy to verify that $f_A$ is an SI-left-ideal of $S$ over $U$, but it is not an SI-right-ideal of $S$ over $U$, because $f_A(2 * 4) \not\supseteq f_A(2)$. Also, one can easily see that $f_A$ is an SI-interior-ideal of $S$ over $U$ but it is not an SI-two-sided-ideal of $S$ over $U$.

Note that every SI-two-sided-ideal of an AG-groupoid $S$ with left identity over $U$ is an SI-interior-ideal of $S$ over $U$.

**Theorem 16** *Let $f_A$ be any soft set of a right regular AG-groupoid $S$ with left identity over $U$. Then, $f_A$ is an SI-left-ideal of $S$ over $U$ if and only if $f_A$ is an SI-right-ideal of $S$ over $U$ if and only if $f_A$ is an SI-two-sided-ideal of $S$ over $U$ if and only if $f_A$ is an SI-interior-ideal of $S$ over $U$.*

*Proof* Assume that $f_A$ is any SI-left-ideal of a right regular $S$ with left identity over $U$. Let $a, b \in S$. For $a \in S$, there exists some $x \in S$ such that $a = xa^2$. Thus, $ab = xa^2 \cdot b = (a \cdot xa)b = (b \cdot xa)a$. Therefore, $f_A((b \cdot xa)a) \supseteq f_A(a)$. Now, by using Lemma 15, $f_A$ is an SI-left-ideal of $S$ over $U$ if and only if $f_A$ is an SI-right-ideal of $S$ over $U$. Let $f_A$ is any SI-right-ideal of a right regular with left identity over $U$. Let $a, b, c \in S$, then $f_A(ab \cdot c) = f_A((xa^2 \cdot b)c) = f_A(cb \cdot xa^2) = f_A(a^2x \cdot bc) = f_A(b(a^2x \cdot c)) \supseteq f_A(b)$. Again assume that $f_A$ is any SI-interior-ideal of a right regular $S$ with left identity over $U$. Thus, $f_A(ab) \supseteq f_A(xa^2 \cdot b) \supseteq f_A(a^2) = f_A(xa^2 \cdot xa^2) = f_A(a^2x \cdot a^2x) = f_A((aa)(a^2x \cdot x)) \supseteq f_A(a)$, which is what we set out to prove. □

## Conclusions

Every AG-groupoid with left identity can be considered as an AG***-groupoid, but the converse is not true in general. This leads us to the fact that an AG***-groupoid can be seen as the generalization of an AG-groupoid with left identity. Thus, the results of "Right regular AG-groupoids" section can be trivially followed for an AG***-groupoid.

The idea of soft sets in an AG-groupoid will help us in verifying the existing characterizations and to achieving new and generalized results in future works. Some of them are as under:

1. To generalize the results of a semigroups using soft sets.

2. To characterize a newly developed substructure called an AG***-groupoid through soft sets.

3. To study the structural properties of an AG-hypergroupoid by using soft sets.

4. To introduce and examine the concept of a $\Gamma$-AG-groupoid in terms of soft sets.

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