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Differential subordination applications to a class of meromorphic multivalent functions associated with Mittag-Leffler function

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Abstract
In this paper, using the principal of differential subordination, we obtain some properties of certain class of p-valent meromorphic functions, which are defined by Mittag-Leffler function.

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2010 mathematics subject classification: 30C45, 30C50, 30C55

Introduction

Denote by $\Sigma_{p,m}$ the class of analytic meromorphic multivalent functions of the form:

$$f(z) = \frac{1}{z^p} + \sum_{k=m}^{\infty} a_k z^k \ \ (p \in \mathbb{N} = \{1, 2, \ldots\}; m > -p)\,, \tag{1}$$

where $\mathbb{U}^* = \{z \in \mathbb{C} \text{ and } 0 < |z| < 1\} = \mathbb{U}\backslash\{0\}$. We note that $Sigma_{p,1-p} = \Sigma_p$.

For two functions $f(z)$ and $g(z)$, analytic in $\mathbb{U}$, $f(z)$ is subordinate to $g(z)$ $\big(f(z) \prec g(z)\big)$ in $\mathbb{U}$, if there exists a function $\omega(z)$, analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1, f(z) = g(\omega(z))(z \in \mathbb{U})$ and if $g(z)$ is univalent in $\mathbb{U}$, then (see for details [1] and also [2])

$$f(z) \prec g(z) \Longleftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

The Hadamard product of $f(z)$ and $g(z)$ given by

$$g(z) = \frac{1}{z^p} + \sum_{k=m}^{\infty} b_k z^k$$

is defined by

$$\big(f * g\big)(z) = \frac{1}{z^p} + \sum_{k=m}^{\infty} a_k b_k z^k = \big(g * f\big)(z)\,. \tag{2}$$

The Mittag-Leffler function $E_{\alpha}(z)$ $(z \in \mathbb{U}^{*})$ ([3] and [4]) see also ([5, 6] and [7]) is defined by

$$E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha + 1)} z^{k}, \alpha \in \mathbb{C}, \Re(\alpha) > 0.$$

For $\alpha, \beta, \gamma \in \mathbb{C}, \Re(\alpha) > 0$, max $\{0, \Re(c) - 1\}$ and $\Re(c) > 0$, Srivastava and Tomovski [8] generalized Mittag-Leffler function by the function

$$E_{\alpha,\beta}^{\gamma,c}(z) = \sum_{k=0}^{\infty} \frac{(\gamma)_{kc}}{\Gamma(k\alpha + \beta)k!} z^{nk} \tag{3}$$

and proved that it is an entire function in the complex z-plane, where

$$(\gamma)_{\theta} = \frac{\Gamma(\gamma + \theta)}{\Gamma(\gamma)} \begin{cases} 1, & \theta = 0 \\ \gamma(\gamma + 1) \ldots (\gamma + \theta - 1), & \theta \neq 0 \end{cases}.$$

Mostafa and Aouf [9] (see also [10]) used the function $E_{\alpha,\beta}^{\gamma,c}(z)$ and defined the meromorphic function

$$\begin{aligned} \mathcal{M}_{p,\alpha,\beta}^{\gamma,c}(z) &= z^{-p}\Gamma(\beta)E_{\alpha,\beta}^{\gamma,c}(z) \\ &= z^{-p} + \sum_{k=m}^{\infty} \frac{\Gamma(\beta)\Gamma[\gamma + (k+p)c]}{\Gamma(\gamma)\Gamma[\beta + (k+p)\alpha]\,(k+p)!} z^{k}, \\ &\quad (\Re(\alpha) = 0 \text{ when } \Re(c) = 1 \text{ with } \beta \neq 0), \end{aligned} \tag{4}$$

and for $f(z) \in \Sigma_{p,m}$, they defined the operator

$$\begin{aligned} \mathcal{H}_{p,\alpha,\beta}^{\gamma,c} f(z) &= \mathcal{M}_{p,\alpha,\beta}^{\gamma,c}(z) * f(z) \\ &= z^{-p} + \sum_{k=m}^{\infty} \frac{\Gamma(\beta)\Gamma[\gamma + (k+p)c]}{\Gamma(\gamma)\Gamma[\beta + (k+p)\alpha]\,(k+p)!} a_k z^{k}. \end{aligned} \tag{5}$$

From (5) it is easy to have

$$cz(\mathcal{H}_{p,\alpha,\beta}^{\gamma,c} f(z))' = \gamma \mathcal{H}_{p,\alpha,\beta}^{\gamma+1,c} f(z) - (\gamma + pc)\mathcal{H}_{p,\alpha,\beta}^{\gamma,c} f(z) \; (c > 0) \tag{6}$$

and

$$\alpha z \left(\mathcal{H}_{p,\alpha,\beta+1}^{\gamma,c} f(z)\right)' = \beta\, \mathcal{H}_{p,\alpha,\beta}^{\gamma,c} f(z) - (p\alpha + \beta)\, \mathcal{H}_{p,\alpha,\beta+1}^{\gamma,c} f(z), \alpha \neq 0. \tag{7}$$

We note that:
(i) $\mathcal{H}_{p,0,\beta}^{1,1} f(z) = f(z)$;
(ii) $\mathcal{H}_{p,0,\beta}^{2,1} f(z) = (p+1)f(z) + zf'(z)$;
(iii) $\mathcal{H}_{1,0,\beta}^{2,1} f(z) = 2f(z) + zf'(z)$.
Using the operator $\mathcal{H}_{p,\alpha,\beta}^{\gamma,c} f(z)$, we have the following definition.

Definition 1. For fixed A and B $(-1 \leq B < A \leq 1)$, we say that a function $f \in \Sigma_{p,m}$ is in the class $\Sigma_{p,m}^{\gamma,c}(\alpha, \beta; A, B)$ if it satisfies

$$-\frac{z^{p+1}\left(\mathcal{H}_{p,\alpha,\beta}^{\gamma,c} f(z)\right)'}{p} \prec \frac{1 + Az}{1 + Bz}. \tag{8}$$

In view of the definition of differential subordination, (8) is equivalent to

$$\left| \frac{z^{p+1}\left(\mathcal{H}_{p,\alpha,\beta}^{\gamma,c} f(z)\right)^{'} + p}{Bz^{p+1}\left(\mathcal{H}_{p,\alpha,\beta}^{\gamma,c} f(z)\right)^{'} + pA} \right| < 1. \tag{9}$$

We note that:

(i)

$$\begin{aligned} \Sigma_{p,1}^{1,1}(0,1;A,B) &= \Sigma_p(A,B)\left(-1 \le B < A \le 1; \mathbb{U}^*\right) \\ &= \left\{ f \in \Sigma_p : -\frac{z^{p+1}f^{'}(z)}{p} \prec \frac{1+Az}{1+Bz} \right\}, \end{aligned}$$

the class $\Sigma_p(A,B)$ was introduced and studied by Mogra [11].

(ii)

$$\begin{aligned} \Sigma_{p,m}^{\gamma,c}\left(\alpha,\beta;1-\frac{2\eta}{p},-1\right) &= \Sigma_{p,m}^{\gamma,c}(\alpha,\beta,\eta)\,(0 \le \eta < p) \\ &= \left\{ f \in \Sigma_{p,m} : \Re\{-z^{p+1}\left(\mathcal{H}_{p,\alpha,\beta}^{\gamma,c} f(z)\right)^{'}\} > \eta \right\}. \end{aligned} \tag{10}$$

Preliminary results

The following lemmas will be required in our investigation.

Lemma 1 *[12]. Let h be a convex (univalent) function in $\mathbb{U}$ with $h(0) = 1$. Also let*

$$\phi(z) = 1 + d_{p+m}z^{p+m} + d_{p+m+1}z^{p+m+1} + \ldots, \tag{11}$$

be analytic in $\mathbb{U}$. If

$$\phi(z) + \frac{z\phi^{'}(z)}{\tau} \prec h(z)\ \ (\Re(\tau) \ge 0;\ \tau \ne 0;\ z \in \mathbb{U}), \tag{12}$$

then

$$\phi(z) \prec \Psi(z) = \frac{\tau}{p+m} z^{-\frac{\tau}{p+m}} \int_0^z t^{\frac{\tau}{p+m}-1} h(t)dt. \tag{13}$$

Lemma 2 *[13]. Let μ be a positive measure on the unit interval $[0,1]$. Let $g(z,t)$ be a complex valued function defined on $\mathbb{U} \times [0,1]$ such that $g(.,t)$ is analytic in $\mathbb{U}$ for each $t \in [0,1]$ and such that $g(z,.)$ is μ integrable on $[0,1]$ for all $z \in \mathbb{U}$. In addition, suppose that $\Re\{g(z,t)\} > 0$, $g(-r,t)$ is real and*

$$\Re\left\{\frac{1}{g(z,t)}\right\} \ge \frac{1}{g(-r,t)}\ (|z| \le r < 1; t \in [0,1]).$$

If the function G is defined by

$$G(z) = \int_0^1 g(z,t)d\mu(t),$$

then

$$\Re\left\{\frac{1}{G(z)}\right\} \ge \frac{1}{G(-r)}\ (|z| \le r < 1).$$

Each of the identities (asserted by Lemma 2) is fairly well known (cf., e.g., [[8], ch. 14]).

Lemma 3 *[14]. For real or complex numbers a, b, and c* $(c \neq 0, -1, -2, \ldots)$

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,dt = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}\,{}_2F_1(a,b;c;z)\ (\Re(c) < \Re(b) > 0);$$

$${}_2F_1(a,b;c;z) = (1-z)^{-a}{}_2F_1\left(a, c-b; c; \frac{z}{z-1}\right)(z \neq 1) \tag{14}$$

and

$${}_2F_1(a,b;c;z) = {}_2F_1(b,a;c;z). \tag{15}$$

Lemma 4 *[15]. Let* Φ *be analytic in* $\mathbb{U}$ *with*

$$\Phi(0) = 1 \text{ and } \Re\{\Phi(z)\} > \frac{1}{2}.$$

Then, for any function F, analytic in $\mathbb{U}$, $(\Phi * F)(\mathbb{U})$ *is contained in the convex hull of* $F(\mathbb{U})$.

We used the technique used by ([16–18] and [19]).

Main inclusion relationships

Unless otherwise mentioned, we assume throughout this paper that $-1 \leq B < A \leq 1, \alpha, \beta, \gamma \in \mathbb{C}, \Re(\alpha) > 0, \max\{0, \Re(c) - 1\}, \Re(c) > 0, \delta > 0, f(z)$ given by (1) and $z \in \mathbb{U}^*$.

Theorem 1 *Let* $\gamma \neq 0$ *and* $f(z)$ *satisfy:*

$$-\frac{(1-\delta)z^{p+1}\left(\mathcal{H}^{\gamma,c}_{p,\alpha,\beta}f(z)\right)' + \delta z^{p+1}\left(\mathcal{H}^{\gamma+1,c}_{p,\alpha,\beta}f(z)\right)'}{p} \prec \frac{1+Az}{1+Bz}, \tag{16}$$

then

$$-\frac{z^{p+1}\left(\mathcal{H}^{\gamma,c}_{p,\alpha,\beta}f(z)\right)'}{p} \prec \Psi(z) \prec \frac{1+Az}{1+Bz}, \tag{17}$$

where

$$\Psi(z) = \begin{cases} \frac{A}{B} + \left(1-\frac{A}{B}\right)(1+Bz)^{-1}{}_2F_1\left(1, 1; \frac{\gamma}{\delta c(p+m)} + 1; \frac{Bz}{1+Bz}\right), & B \neq 0 \\ 1 + \frac{\gamma}{\gamma+\delta c(p+m)}Az, & B = 0. \end{cases} \tag{18}$$

is the best dominant of (17). Furthermore,

$$\Re\left\{-\frac{z^{p+1}\left(\mathcal{H}^{\gamma,c}_{p,\alpha,\beta}f(z)\right)'}{p}\right\} > \rho\ (0 \leq \rho < 1), \tag{19}$$

where

$$\rho = \begin{cases} \frac{A}{B} + \left(1-\frac{A}{B}\right)(1-B)^{-1}{}_2F_1\left(1, 1; \frac{\gamma}{\delta c(p+m)} + 1; \frac{B}{B-1}\right), & B \neq 0, \\ 1 - \frac{\gamma}{\gamma+\delta c(p+m)}A, & B = 0. \end{cases} \tag{20}$$

The result is the best possible.

Proof Let

$$\phi(z) = -\frac{z^{p+1}\left(\mathcal{H}^{\gamma,c}_{p,\alpha,\beta}f(z)\right)'}{p}, \tag{21}$$

where ϕ is given by (11). Differentiating (21) and using (6), we get

$$-\frac{(1-\delta)\,z^{p+1}\left(\mathcal{H}^{\gamma,c}_{p,\alpha,\beta}f(z)\right)' + \delta z^{p+1}\left(\mathcal{H}^{\gamma+1,c}_{p,\alpha,\beta}f(z)\right)'}{p} = \phi(z) + \frac{\delta cz\phi'(z)}{\gamma} \prec \frac{1+Az}{1+Bz}.$$

Now, by using Lemma 1 for $\tau = \frac{\gamma}{\delta c}$, we get

$$-\frac{z^{p+1}\left(\mathcal{H}^{\gamma,c}_{p,\alpha,\beta}f(z)\right)'}{p} \prec \Psi(z) = \frac{\gamma}{\delta c\,(p+m)} z^{-\frac{\gamma}{\delta c(p+m)}} \int_0^z t^{\frac{\gamma}{\delta c(p+m)}-1}\left(\frac{1+At}{1+Bt}\right)dt$$

$$= \begin{cases} \frac{A}{B} + \left(1-\frac{A}{B}\right)(1+Bz)^{-1}{}_2F_1\left(1,1;\frac{\gamma}{\delta c(p+m)}+1;\frac{Bz}{1+Bz}\right), & B \neq 0 \\ 1 + \frac{\gamma}{\gamma+\delta c(p+m)}Az, & B = 0. \end{cases}$$

This proves (17) of Theorem 1. In order to prove (20), we need to show that

$$\inf_{|z|<1}\{\Re(\Psi(z))\} = \Psi(-1). \tag{22}$$

We have

$$\Re\left\{\frac{1+Az}{1+Bz}\right\} \geq \frac{1-Ar}{1-Br}\;(|z| \leq r < 1)\,.$$

Putting

$$G(z,\zeta) = \frac{1+A\zeta z}{1+B\zeta z} \text{ and } dv(\zeta) = \frac{\gamma}{\delta c\,(p+m)}\zeta^{\frac{\gamma}{\delta c(p+m)}-1}d\zeta\;(0 \leq \zeta \leq 1)\,,$$

which is a positive measure on $[0,1]$, we obtain

$$\Psi(z) = \int_0^1 G(z,\zeta)\,dv(\zeta).$$

Then

$$\Re(\Psi(z)) \geq \int_0^1 \frac{1-A\zeta r}{1-B\zeta r}dv(\zeta) = \Psi(-r)\;(|z| \leq r < 1)\,.$$

Assuming $r \to 1^-$ in the above inequality, we obtain (22). The result in (19) is the best possible and Ψ is the best dominant of (17). This completes the proof of Theorem 1. □

Theorem 2 *Let* $f(z) \in \Sigma^{\gamma,c}_{p,m}(\alpha,\beta,\eta)\;(0 \leq \eta < p)$, *then*

$$\Re\left\{-z^{p+1}\left[(1-\delta)\left(\mathcal{H}^{\gamma,c}_{p,\alpha,\beta}f(z)\right)' + \delta\left(\mathcal{H}^{\gamma+1,c}_{p,\alpha,\beta}f(z)\right)'\right]\right\} > \eta\;(|z| < R)\,, \tag{23}$$

where

$$R = \left\{\frac{\sqrt{c^2\delta^2\,(p+m)^2+\gamma^2} - c\delta\,(p+m)}{\gamma}\right\}^{\frac{1}{p+m}}. \tag{24}$$

The result is the best possible.

Proof Since $f(z) \in \Sigma_{p,m}^{\gamma,c}(\alpha,\beta,\eta)$, let

$$-z^{p+1}\left(\mathcal{H}_{p,\alpha,\beta}^{\gamma,c}f(z)\right)^{'} = \eta + (p-\eta)\,u(z), \tag{25}$$

where $u(z)$ in the form (11) and $\Re\{u(z)\} > 0$. Differentiating (25) and using (6), we get

$$-\frac{z^{p+1}\left[(1-\delta)\left(\mathcal{H}_{p,\alpha,\beta}^{\gamma,c}f(z)\right)^{'} + \delta\left(\mathcal{H}_{p,\alpha,\beta}^{\gamma+1,c}f(z)\right)^{'}\right] + \eta}{p-\eta} = u(z) + \frac{c\delta z u^{'}(z)}{\gamma}. \tag{26}$$

Applying the following estimate [20],

$$\frac{\left|zu^{'}(z)\right|}{\Re\{u(z)\}} \le \frac{2(p+m)r^{p+m}}{1-r^{2(p+m)}}\ (|z| = r < 1);$$

in (26), we get

$$\Re\left\{-\frac{z^{p+1}\left[(1-\delta)\left(\mathcal{H}_{p,\alpha,\beta}^{\gamma,c}f(z)\right)^{'} + \delta\left(\mathcal{H}_{p,\alpha,\beta}^{\gamma+1,c}f(z)\right)^{'}\right] + \eta}{p-\eta}\right\}$$

$$\ge \Re\left(u(z)\right)\left(1 - \frac{2c\delta(p+m)r^{p+m}}{\gamma\left(1-r^{2(p+m)}\right)}\right). \tag{27}$$

It is easily seen that the right-hand side of (27) is positive, if $r < R$, where R is given by (24). In order to show that the bound R is the best possible, we consider the function $f \in \Sigma_{p,m}$ defined by

$$-z^{p+1}\left(\mathcal{H}_{p,\alpha,\beta}^{\gamma,c}f(z)\right)^{'} = \eta + (p-\eta)\frac{1+z^{p+m}}{1-z^{p+m}}.$$

Noting that

$$-\frac{z^{p+1}\left[(1-\delta)\left(\mathcal{H}_{p,\alpha,\beta}^{\gamma,c}f(z)\right)^{'} + \delta\left(\mathcal{H}_{p,\alpha,\beta}^{\gamma+1,c}f(z)\right)^{'}\right] + \eta}{p-\eta}$$

$$= \frac{\gamma\left(1-z^{2(p+m)}\right) + 2c\delta(p+m)z^{p+m}}{\gamma\left(1-z^{p+m}\right)^{2}} = 0,$$

for

$$z = R\,\Re\left(\frac{i\pi}{p+m}\right).$$

This completes the proof of Theorem 2. □

Putting $\delta = 1$ in Theorem 2, we obtain the following result.

Corollary 1 *If* $f(z) \in \Sigma_{p,m}^{\gamma,c}(\alpha,\beta,\eta)\ (0 \le \eta < p)$, *then* $f(z) \in \Sigma_{p,m}^{\gamma+1,c}(\alpha,\beta,\eta)$ *for* $|z| < R^{*}$, *where*

$$R^{*} = \left\{\frac{\sqrt{c^2(p+m)^2+\gamma^2} - c(p+m)}{\gamma}\right\}^{\frac{1}{p+m}}.$$

The result is the best possible.

Theorem 3 *If the function* $f(z) \in \Sigma_{p,m}$ *satisfies*

$$z^p\left[(1-\delta)\mathcal{H}_{p,\alpha,\beta}^{\gamma,c}f(z)+\delta\mathcal{H}_{p,\alpha,\beta}^{\gamma+1,c}f(z)\right] \prec \frac{1+Az}{1+Bz},$$

then

$$z^p\mathcal{H}_{p,\alpha,\beta}^{\gamma,c}f(z) \prec \Psi_1(z) \prec \frac{1+Az}{1+Bz}$$

and

$$\Re\left\{z^p\mathcal{H}_{p,\alpha,\beta}^{\gamma,c}f(z)\right\} > \rho,$$

where $\Psi_1(z)$ *is in the form (18) and* ρ *given by (20). The result is the best possible.*

Proof The proof follows by taking the same lines as in the proof of Theorem 1 and taking $\phi(z) = z^p\mathcal{H}_{p,\alpha,\beta}^{\gamma,c}f(z)$ in (21). □

For the function $f(z)$ in the class $\Sigma_{p,m}$, Kumar and Shukla [21] defined the integral operator $F_{\mu,p} : \Sigma_{p,m} \to \Sigma_{p,m}$ as follows:

$$\begin{aligned} F_{\mu,p}(f)(z) &= \frac{\mu}{z^{\mu+p}}\int_0^z t^{\mu+p-1}f(t)dt \\ &= z^{-p} + \sum_{k=m}^{\infty}\frac{\mu}{k+p+\mu}a_kz^k \ (\mu > 0)\,. \end{aligned} \tag{28}$$

From (28), we get

$$z\left(\mathcal{H}_{p,\alpha,\beta}^{\gamma,c}F_{\mu,p}(f)(z)\right)' = \mu\mathcal{H}_{p,\alpha,\beta}^{\gamma,c}f(z) - (\mu+p)\,\mathcal{H}_{p,\alpha,\beta}^{\gamma,c}F_{\mu,p}(f)(z). \tag{29}$$

Theorem 4 *Let the function* $f(z)$ *given by (1) be in the class* $\Sigma_{p,m}^{\gamma,c}(\alpha,\beta;A,B)$ *and* $F_{\mu,p}(f)(z)$ *defined by (28). Then*

$$-\frac{z^{p+1}\left(\mathcal{H}_{p,\alpha,\beta}^{\gamma,c}F_{\mu,p}(f)(z)\right)'}{p} \prec \Theta(z) \prec \frac{1+Az}{1+Bz}, \tag{30}$$

where

$$\Theta(z) = \begin{cases} \frac{A}{B} + \left(1-\frac{A}{B}\right)(1+Bz)^{-1}{}_2F_1\left(1,1;\frac{\mu}{(p+m)}+1;\frac{Bz}{1+Bz}\right), & B \neq 0 \\ 1+\frac{\mu}{\mu+p+m}Az, & B = 0. \end{cases} \tag{31}$$

is the best dominant of (31). Furthermore,

$$\Re\left\{-\frac{z^{p+1}\left(\mathcal{H}_{p,\alpha,\beta}^{\gamma,c}F_{\mu,p}(f)(z)\right)'}{p}\right\} > \sigma \ (0 \le \sigma < 1)\,, \tag{32}$$

where

$$\sigma = \begin{cases} \frac{A}{B} + \left(1-\frac{A}{B}\right)(1-B)^{-1}{}_2F_1\left(1,1;\frac{\mu}{(p+m)}+1;\frac{B}{B-1}\right), & B \neq 0 \\ 1-\frac{\mu}{\mu+p+m}A, & B = 0. \end{cases} \tag{33}$$

The result is the best possible.

Proof Let

$$L(z) = -\frac{z^{p+1}\left(\mathcal{H}^{\gamma,c}_{p,\alpha,\beta}F_{\mu,p}(f)(z)\right)'}{p}, \tag{34}$$

where L in the form (11). Differentiating (34) and using (29), we get

$$-\frac{z^{p+1}\left(\mathcal{H}^{\gamma,c}_{p,\alpha,\beta}f(z)\right)'}{p} = L(z) + \frac{z}{\mu}L'(z) \prec \frac{1+Az}{1+Bz}.$$

Now the remaining part of Theorem 4 follows by using the technique used in proving Theorem 1. □

Theorem 5 *Let the function $F_{\mu,p}(f)(z)$ defined by (28) satisfy:*

$$z^p\left[(1-\delta)\,\mathcal{H}^{\gamma,c}_{p,\alpha,\beta}F_{\mu,p}(f)(z) + \delta\mathcal{H}^{\gamma,c}_{p,\alpha,\beta}f(z)\right] \prec \frac{1+Az}{1+Bz}, \tag{35}$$

then

$$\Re\left\{z^p\mathcal{H}^{\gamma,c}_{p,\alpha,\beta}F_{\mu,p}(f)(z)\right\} > \theta, \tag{36}$$

where

$$\theta = \begin{cases} \frac{A}{B} + \left(1-\frac{A}{B}\right)(1-B)^{-1}{}_2F_1\left(1,1;\frac{\mu}{\delta(p+m)}+1;\frac{B}{B-1}\right), & B \neq 0 \\ 1 - \frac{\mu}{\mu+\delta(p+m)}A, & B = 0. \end{cases}$$

The result is the best possible.

Proof Let

$$K(z) = z^p\mathcal{H}^{\gamma,c}_{p,\alpha,\beta}F_{\mu,p}(f)(z), \tag{37}$$

where K in the form (11). Differentiating (37) and using (29) and (35), we get

$$K(z) + \frac{\delta z}{\mu}K'(z) \prec \frac{1+Az}{1+Bz}.$$

Now the remaining part of Theorem 5 follows by using the technique used in proving Theorem 1. □

Theorem 6 *Let the function $f(z) \in \Sigma_{p,m}$ satisfy:*

$$-\frac{z^{p+1}\left[(1-\delta)\left(\mathcal{H}^{\gamma,c}_{p,\alpha,\beta}F_{\mu,p}(f)(z)\right)' + \delta\left(\mathcal{H}^{\gamma,c}_{p,\alpha,\beta}f(z)\right)'\right]}{p} \prec \frac{1+Az}{1+Bz},$$

then

$$\Re\left\{-\frac{z^{p+1}\left(\mathcal{H}^{\gamma,c}_{p,\alpha,\beta}F_{\mu,p}(f)(z)\right)'}{p}\right\} > \theta,$$

where $F_{\mu,p}(f)(z)$ is given by (28) and θ is given as in Theorem 5. The result is the best possible.

Proof The proof follows by taking the same lines as in Theorem 5. □

Theorem 7 *Let $f(z)$ be in the class $\Sigma_{p,m}$. Also, let $g(z) \in \Sigma_{p,m}$ satisfy:*

$$\Re\left\{z^p \mathcal{H}^{\gamma,c}_{p,\alpha,\beta} g(z)\right\} > 0.$$

If

$$\left|\frac{\mathcal{H}^{\gamma,c}_{p,\alpha,\beta} f(z)}{\mathcal{H}^{\gamma,c}_{p,\alpha,\beta} g(z)} - 1\right| < 1,$$

then

$$\Re\left\{-\frac{z\left(\mathcal{H}^{\gamma,c}_{p,\alpha,\beta} f(z)\right)'}{\mathcal{H}^{\gamma,c}_{p,\alpha,\beta} f(z)}\right\} > 0 \ \left(|z| < R_0\right), \tag{38}$$

where

$$R_0 = \frac{\sqrt{9(p+m)^2 + 4p(2p+m)} - 3(p+m)}{2(2p+m)}. \tag{39}$$

Proof Let

$$\phi(z) = \frac{\mathcal{H}^{\gamma,c}_{p,\alpha,\beta} f(z)}{\mathcal{H}^{\gamma,c}_{p,\alpha,\beta} g(z)} - 1 = e_{p+m} z^{p+m} + e_{p+m+1} z^{p+m+1} + \ldots, \tag{40}$$

we note that ϕ is analytic in $\mathbb{U}$, with $\phi(0) = 0$ and $|\phi(z)| \leq |z|^{p+m}$. Then, by applying the familiar Schwarz Lemma [22], we have $\phi(z) = z^{p+m}\Psi(z)$ is analytic in $\mathbb{U}$ and $|\Psi(z)| \leq 1$ $(z \in \mathbb{U})$. Therefore, (40) leads to

$$\mathcal{H}^{\gamma,c}_{p,\alpha,\beta} f(z) = \mathcal{H}^{\gamma,c}_{p,\alpha,\beta} g(z)\left(z^{p+m}\Psi(z) + 1\right). \tag{41}$$

Differentiating (41), we obtain

$$\frac{z\left(\mathcal{H}^{\gamma,c}_{p,\alpha,\beta} f(z)\right)'}{\mathcal{H}^{\gamma,c}_{p,\alpha,\beta} f(z)} = \frac{z\left(\mathcal{H}^{\gamma,c}_{p,\alpha,\beta} g(z)\right)'}{\mathcal{H}^{\gamma,c}_{p,\alpha,\beta} g(z)} + \frac{z^{p+m}\left[(p+m)\,\Psi(z) + z\Psi'(z)\right]}{1 + z^{p+m}\Psi(z)}. \tag{42}$$

Letting $\chi(z) = z^p \mathcal{H}^{\gamma,c}_{p,\alpha,\beta} g(z)$, we see that the function χ is of the form (11) and is analytic in $\mathbb{U}$, $\Re\{\chi(z)\} > 0$ and

$$\frac{z\left(\mathcal{H}^{\gamma,c}_{p,\alpha,\beta} g(z)\right)'}{\mathcal{H}^{\gamma,c}_{p,\alpha,\beta} g(z)} = \frac{z\chi'(z)}{\chi(z)} - p,$$

so, we find from (42) that

$$\Re\left\{-\frac{z\left(\mathcal{H}^{\gamma,c}_{p,\alpha,\beta} f(z)\right)'}{\mathcal{H}^{\gamma,c}_{p,\alpha,\beta} f(z)}\right\} \geq p - \left|\frac{z\chi'(z)}{\chi(z)}\right| - \left|\frac{z^{p+m}\left[(p+m)\Psi(z) + z\Psi'(z)\right]}{1 + z^{p+m}\Psi(z)}\right|. \tag{43}$$

Using the following known estimates [23] (see also [20]),

$$\left|\frac{\chi'(z)}{\chi(z)}\right| \leq \frac{2(p+m)r^{p+m-1}}{1 - r^{2(p+m)}} \text{ and } \left|\frac{(p+m)\,\Psi(z) + z\Psi'(z)}{1 + z^{p+m}\Psi(z)}\right| \leq \frac{p+m}{1 - r^{p+m}} \ \left(|z| = r < 1\right),$$

in (43), we have

$$\Re\left\{-\frac{z\left(\mathcal{H}^{\gamma,c}_{p,\alpha,\beta} f(z)\right)'}{\mathcal{H}^{\gamma,c}_{p,\alpha,\beta} f(z)}\right\} \geq \frac{p - 3(p+m)r^{p+m} - (2p+m)r^{2(p+m)}}{1 - r^{2(p+m)}},$$

which is certainly positive, provided that $r < R_0$, R_0 given by (39). □

Theorem 8 *Let the function* $f(z) \in \Sigma_{p,m}$ *satisfy:*

$$z^p\left[(1-\delta)\,\mathcal{H}_{p,\alpha,\beta}^{\gamma,c}f(z)+\delta\mathcal{H}_{p,\alpha,\beta}^{\gamma+1,c}f(z)\right]\prec\frac{1+Az}{1+Bz},$$

then

$$\Re\left\{\left(z^p\mathcal{H}_{p,\alpha,\beta}^{\gamma,c}f(z)\right)^{\frac{1}{q}}\right\}>\epsilon^{\frac{1}{q}}\ \ (q\in\mathbb{N}),$$

where ϵ *in the form (20). The result is the best possible.*

Proof Let

$$\phi(z)=z^p\mathcal{H}_{p,\alpha,\beta}^{\gamma,c}f(z), \tag{44}$$

where ϕ in the form (11). Differentiating (44) and using (6), we have

$$z^p\left[(1-\delta)\,\mathcal{H}_{p,\alpha,\beta}^{\gamma,c}f(z)+\delta\mathcal{H}_{p,\alpha,\beta}^{\gamma+1,c}f(z)\right]=\phi(z)+\frac{\delta cz\phi'(z)}{\gamma}\prec\frac{1+Az}{1+Bz}.$$

Now the remaining part of Theorem 8 follows by using the technique used in proving Theorem 1, and using the inequality:

$$\Re(w^{\frac{1}{q}})\geq(\Re(w))^{\frac{1}{q}}\ \ (\Re(w)>0;q\in\mathbb{N}),$$

we have the result asserted by Theorem 8. □

Theorem 9 *Let the function* $f(z)\in\Sigma_{p,m}^{\gamma,c}(\alpha,\beta;A,B)$ *and let* $g(z)\in\Sigma_{p,m}$ *satisfy:*

$$\Re\left(z^pg(z)\right)>\frac{1}{2}.$$

Then,

$$\left(f*g\right)(z)\in\Sigma_{p,m}^{\gamma,c}(\alpha,\beta;A,B).$$

Proof We have

$$-\frac{z^{p+1}\left(\mathcal{H}_{p,\alpha,\beta}^{\gamma,c}\left(f*g\right)(z)\right)'}{p}=-\frac{z^{p+1}\left(\mathcal{H}_{p,\alpha,\beta}^{\gamma,c}f(z)\right)'}{p}*z^pg(z).$$

Since

$$\Re\left(z^pg(z)\right)>\frac{1}{2}$$

and $\frac{1+Az}{1+Bz}$ is convex in $\mathbb{U}$, it follows from (8) and Lemma 4 that $\left(f*g\right)(z)\in\Sigma_{p,m}^{\gamma,c}(\alpha,\beta;A,B)$, which completes the proof of Theorem 9. □

Remark 1 *For different value of* γ, c, α, β, *and* p *in the above results, we obtain results corresponding to the functions given in the introduction.*

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