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ORIGINAL ARTICLE

L'Hospital rule for matrix functions

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Abstract In this paper, the L'Hospital rule for evaluating limits of complex matrix functions is introduced. We present some specific examples on certain matrix functions showing the applicability of our approach.

MATHEMATICS SUBJECT CLASSIFICATION (2010): 11C, 15B, 30B

1. Introduction

Matrix functional calculus is a fundamental area of mathematics with wide applications not only to many branches of mathematics but also to science and engineering. It is a connection to many different branches of mathematics (see e.g. [1–5] and elsewhere). In [7], Kratz derived a limit theorem for matrices from L'Hospital's rule. Some applications of this theorem were given to linear algebra and to differential equations. In this paper we derive a limit theorem for complex matrix functions from the L'Hospital's rule. Using this theorem, applied examples on some complex matrix functions are given. Further investigations and extensions of this topic will be reported in a forthcoming paper.

Throughout this paper, we consider the complex space $\mathbb{C}^{N\times N}$ of complex matrices of common order N. The matrices I and $\mathbf{0}$ stand for the identity matrix and the null matrix in $\mathbb{C}^{N\times N}$, respectively. A matrix X is a positive stable matrix in $\mathbb{C}^{N\times N}$ if $Re(\lambda) > 0$ for all $\lambda \in \sigma(X)$, where $\sigma(X)$ is the set of all eigenvalues of X. If $a_0, a_1, \ldots, a_n$ are elements of $\mathbb{C}$ and $a_n \neq 0$, then we call

$$P_n(X) = a_n X^n + a_{n-1}X^{n-1} + a_{n-2}X^{n-2} + \cdots + a_0 I_n,$$

a matrix polynomial of degree n in X. The exponential matrix function and other matrix functions are defined in [1,3,5].

If $f(z)$ and $g(z)$ are holomorphic functions of the complex variable z which are defined in an open set $\Omega \subset \mathbb{C}$ and A is a matrix in $\mathbb{C}^{N\times N}$ such that $\sigma(A) \subset \Omega$, then (see [1,2,6])

$$f(A)g(A) = g(A)f(A).$$

Hence, if B in $\mathbb{C}^{N\times N}$ is a matrix for which $\sigma(B) \subset \Omega$ and if $AB = BA$, then

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$f(A)g(B) = g(B)f(A)$.

The symbol for the quotient of two matrices $\frac{A}{B}$ does not have a definite meaning. We interpret it as in [8] by AB^{-1} or $B^{-1}A$. These two products are in general distinct, it is only in an exact significance and this can be obtained when $AB = BA$ with B non-singular.

Definition 1.1. We define $X = [x_{ij}(z)]$, $\frac{dX}{dz}$, $F(X)$ and $G(X)$ to be commutative complex matrices for all z in Ω with the following properties:

(i) $[x_{ij}(z)]$, $F(X) = [f_{ij}(z)]$ and $G(X) = [g_{ij}(z)]$ are analytic functions of complex variables for all $i,j = 1, 2, \ldots, N$ in Ω;
(ii) $\frac{d}{dz}F(X) = \frac{dF}{dX}\cdot\frac{dX}{dz}$ and $\frac{d}{dz}G(X) = \frac{dG}{dX}\cdot\frac{dX}{dz}$ for all z in Ω;
(iii) $\lim_{z\to z_0}F(X)\ G^{-1}(X) = \lim_{z\to z_0}G^{-1}(X)\ F(X) = \lim_{X\to X_0}\frac{F(X)}{G(X)}$.

The progress made and connection to the preceding Definition 1.1 can be formulated in the following interesting result, which is considered to be the L'Hospital rule for matrix functions. Our main theorem is stated as follows:

Theorem 1.1. *Let $X = [x_{ij}(z)]$, $\frac{dX}{dz}$, $F(X)$ and $G(X)$ be defined as in Definition 1.1. Suppose further that $\frac{dX}{dz}$, $G(X)$ and $\frac{dG}{dz}$ are non-singular for all $z \neq z_0$ in Ω and as well as $f_{ij}(z_0) = g_{ij}(z_0) = \mathbf{0}$, $\forall i,j = 1, 2, \ldots, N$. It follows that*

$$\lim_{X\to X_0}\frac{F(X)}{G(X)} = \lim_{z\to z_0}\frac{\frac{d\ F(X)}{d\ X}}{\frac{d\ G(X)}{d\ X}} = \lim_{X\to X_0}\frac{F'(X)}{G'(X)} = \frac{A}{B} = B^{-1}A = AB^{-1}.$$

To proceed with the proof of Theorem 1.1, some definitions and facts should be adopted as indicated through the following sections.

2. Preliminaries

Definition 2.1 (*see 1,2*). Let $X = [x_{ij}(z)]$ be a square matrix of order n ($i,j = 1, 2, \ldots, N$). Then, its determinant, represented by $\det(X)$, is defined as follows:

$$\det(X) = \sum_{i_1}\sum_{i_2}\cdots\sum_{i_n}(-1)^{\rho(i_1,\ldots,i_n)}x_{1i_1}x_{2i_2}\ldots x_{ni_n}$$

where $i_1, \ldots, i_n$, represent the column numbers and $\rho(i_1, \ldots, i_n)$ stands for the number of transpositions which are needed to bring $(i_1, \ldots, i_n)$ to the natural order $(1, 2, \ldots, n)$.

Also, the determinant of the matrix X can be defined in the form

$$\det(X) = \sum_{v=1}^{N}x_{iv}x^*_{iv},$$

where x^*_{iv} is the cofactor of the element x_{iv}. A matrix X is said to be non-singular if $\det(X) \neq 0$, otherwise, it is termed as a singular matrix.

Definition 2.2. For a non-singular square matrix $X = [x_{ij}(z)]_{n\times n}$, the inverse of X is given by

$$X^{-1} = \frac{1}{\det(X)}\widetilde{X},$$

where $\widetilde{X}$ is, as usual, the adjoint matrix of the matrix X.

Fact 2.1 (*see 1,2*). *If the $n \times n$ matrices A_i, $i = 1, 2, \ldots, r$, all have inverses, then their product $\prod_{i=1}^{r}A_i$ has the inverse $\prod_{i=0}^{r-1}A^{-1}_{r-i}$; that is, the inverse of a product is the product of the inverses in the reverse order.*

Fact 2.2 (*see 1,8*). *Let the matrix function $F(X)$; $X = [x_{ij}(z)]$ is a square complex matrix whose its elements are functions of the complex variable z. the limit of this function is defined as follows:*

$$\lim_{X\to X_0}F(X) = [\lim_{z\to z_0}f_{ij}(z)],$$

we write

$$\lim_{X\to X_0}f(X) = A \Longleftrightarrow \lim_{z\to z_0}f_{ij}(z) = a_{ij},$$

where $X_0 = [x_{ij}(z_0)]$ and $A = [a_{ij}]$ is constant matrix.

Fact 2.3 (*see 1*). *Let $f(X)$ be matrix function of the square complex matrix $X = \{x_{ij}(z)\}$, we say that $f(X)$ is continuous in a region Ω if*

$$\lim_{h\to 0}\|f(X + I\ h) - f(X)\| = \mathbf{0}.$$

Fact 2.4. *Suppose that $X = [x_{ij}(z)]$ is a square complex matrix of finite order N, whose elements are functions of the complex variable z. The derivative $\frac{d}{d\ X}f$ of the matrix function $f(X)$ will be defined as follows (cf. [1,8]):*

$$\frac{d}{d\ X}f = \lim_{h\to 0}\frac{f(X + I\ h) - f(X)}{h}; \quad h = h_1 + ih_2;\ h_1, h_2 \in \mathbb{R}. \tag{2.1}$$

Fact 2.5 (*see 1*). *We say that the matrix function $f(X)$ is differentiable in Ω if the limit (2.1) exists for all $z \in \Omega$.*

3. Limits of matrix functions

Suppose that $F(X)$ and $G(X)$ are two commutative complex matrix functions defined in Ω. Also, let $G(X)$ be non-singular for all $z \neq z_0$ in Ω, on the assumption that $X = [x_{ij}(z)]$, $F(X) = [f_{ij}(z)]$ and $G(X) = [g_{ij}(z)]$, $i,j = 1, 2, \ldots, N$.

Then

$$F(X) = G^{-1}(X)F(X)G(X); \quad z \neq z_0,$$

$$F(X)G^{-1}(X) = G^{-1}(X)F(X) = \frac{F(X)}{G(X)}; \quad z \neq z_0.$$

Using the algebraic properties of limits and matrices it follows that

$$\begin{aligned}\lim_{z\to z_0}F(X)G^{-1}(X) &= \lim_{z\to z_0}\frac{1}{\det G(X)}F(X)\ adj\ G(X)\\ &= \lim_{z\to z_0}\left\{\frac{\sum_{s=1}^{N}f_{is}(z)\tilde{g}_{si}(z)}{\sum_{v=1}^{N}g_{iv}(z)g^{\star}_{iv}(z)}\right\}\\ &= \left\{\frac{\lim_{z\to z_0}\sum_{s=1}^{N}f_{is}(z)\tilde{g}_{sj}(z)}{\lim_{z\to z_0}\sum_{s=1}^{N}g_{is}(z)g^{\star}_{is}(z)}\right\}\\ &= \frac{\sum_{s=1}^{N}a_{is}\tilde{b}_{sj}}{\sum_{v=1}^{N}b_{iv}b^{\star}_{iv}} = \frac{1}{\det B}\ A\ adjB = A\ B^{-1}.\end{aligned} \tag{3.1}$$

Similarly, we get

$$\lim_{z\to z_0} G^{-1}(X)f(X) = \lim_{z\to z_0} \frac{adjG(X)}{\det G(X)}F(X)$$
$$= \lim_{z\to z_0}\left\{\frac{\sum_{s=1}^{N}\widetilde{G}_{is}(z)f_{sj}(z)}{\sum_{v=1}^{N}g_{iv}(z)g_{iv}^{\star}(z)}\right\}$$
$$= \left\{\frac{\lim_{z\to z_0}\sum_{s=1}^{N}\tilde{g}_{is}(z)f_{sj}(z)}{\lim_{z\to z_0}\sum_{v=1}^{N}g_{iv}(z)g_{iv}^{\star}(z)}\right\} = \frac{\sum_{s=1}^{N}\tilde{b}_{is}a_{sj}}{\sum_{v=1}^{N}b_{iv}b_{iv}^{\star}}$$
$$= B^{-1}A, \tag{3.2}$$

where

$$G^{-1}(X) = \frac{1}{\det G(X)} adj\ G(X) = \frac{[\tilde{g}_{ij}(z)]}{\sum_{v=1}^{N}g_{iv}(z)g_{iv}^{\star}(z)};$$
$$\det G(X) = \sum_{v=1}^{N}g_{iv}(z)g_{iv}^{\star}(z), \quad \text{and } adjG(X) = [\tilde{g}_{ij}(z)].$$

Eqs. (3.1) and (3.2) lead to

$$AB^{-1} = \lim_{z\to z_0}F(X)G^{-1}(X) = \lim_{z\to z_0}G^{-1}(X)F(X) = B\ A^{-1}$$
$$= \lim_{X\to X_0}\frac{F(X)}{G(X)} = \frac{A}{B},$$

where $X_0 = [x_{ij}(z_0)]$ and $B = [b_{ij}]$ is a non-singular matrix.

4. Proof of Theorem 1.1

Let X, $\frac{dX}{dz}$, $F(X)$ and $G(X)$ be commutative complex matrices for all z in Ω. In addition, consider

$$\lim_{z\to z_0} f_{ij}(z) = \lim_{z\to z_0} g_{ij}(z) = 0; \quad i,j = 1,2,\ldots,N,$$
$$\lim_{z\to z_0}\frac{d}{dz}f_{ij}(z) = a_{ij} \text{ and } \lim_{z\to z_0}\frac{d}{dz}g_{ij}(z) = b_{ij}.$$

Then

$$\lim_{z\to z_0}f_{is}(z)\tilde{g}_{sj}(z) = \lim_{z\to z_0}g_{is}(z)g_{is}^{\star}(z) = 0; \quad i,j,s = 1,2,\ldots,N.$$

Applying L'Hospital rule in the case of functions of the complex variables z, we have

$$\lim_{z\to z_0}F(X)G^{-1}(X) = \lim_{z\to z_0}\left\{\frac{\sum_{s=1}^{N}f_{is}(z)\tilde{g}_{sj}(z)}{\sum_{v=1}^{N}g_{iv}(z)g_{iv}^{\star}(z)}\right\}$$
$$= \lim_{z\to z_0}\left\{\frac{\sum_{s=1}^{N}(f_{is}(z)\tilde{g}'_{sj}(z)+f'_{is}(z)\tilde{g}_{sj})}{\sum_{v=1}^{N}(g_{iv}(z)g_{iv}'^{\star}(z)+g'_{iv}(z)g_{iv}^{\star}(z))}\right\}$$
$$= \lim_{z\to z_0}\left\{\frac{\sum_{s=1}^{N}\left(f_{is}(z)\tilde{g}''_{sj}(z)+2f'_{is}(z)\tilde{g}'_{sj}(z)+f''_{is}(z)\tilde{g}_{sj}(z)\right)}{\sum_{v=1}^{N}g_{iv}(z)g_{iv}''^{\star}(z)+2g'_{iv}(z)g_{iv}'^{\star}(z)+g''_{iv}(z)g_{iv}(z)}\right\}$$
$$= \lim_{z\to z_0}\left\{\frac{2\sum_{s=1}^{N}f'_{is}(z)\tilde{g}'_{sj}(z)}{2\sum_{v=1}^{N}g'_{iv}(z)g_{iv}'^{\star}(z)}\right\}$$
$$= \frac{[\sum_{s=1}^{N}\lim_{z\to z_0}f'_{is}(z)\tilde{g}'_{sj}(z)]}{[\sum_{v=1}^{N}\lim_{z\to z_0}g'_{iv}(z)g_{iv}'^{\star}(z)]} = \frac{[\sum_{s=1}^{N}a_{is}\tilde{b}_{sj}]}{[\sum_{v=1}^{N}b_{iv}b_{iv}^{\star}]}$$
$$= \frac{[\sum_{s=1}^{N}a_{is}\tilde{b}_{sj}]}{\det B} = AB^{-1}. \tag{4.1}$$

Similarly, one can get

$$\lim_{z\to z_0}g^{-1}(X)F(X) = \lim_{z\to z_0}\left\{\frac{\sum_{s=1}^{N}\tilde{g}_{is}(z)f_{sj}(z)}{\sum_{v=1}^{N}g_{iv}(z)g_{iv}^{\star}(z)}\right\}$$
$$= \lim_{z\to z_0}\left\{\frac{\sum_{s=1}^{N}\left(\tilde{g}'_{is}(z)f_{sj}(z)+\tilde{g}_{is}f'_{sj}(z)\right)}{\sum_{v=1}^{N}(g_{iv}(z)g_{iv}'^{\star}(z)+g'_{iv}(z)g_{iv}^{\star}(z))}\right\}$$
$$= \lim_{z\to z_0}\left\{\frac{\sum_{s=1}^{N}\left(\tilde{g}''_{is}(z)f_{sj}(z)+2\tilde{g}'_{is}(z)f'_{sj}(z)+\tilde{g}_{is}(z)f''_{sj}(z)\right)}{\sum_{v=1}^{N}g_{iv}(z)g_{iv}''^{\star}(z)+2g'_{iv}(z)g_{iv}'^{\star}(z)+g''_{iv}(z)g_{iv}(z)}\right\}$$
$$= \lim_{z\to z_0}\left\{\frac{2\sum_{s=1}^{N}\tilde{g}'_{is}(z)f'_{sj}(z)}{2\sum_{v=1}^{N}g'_{iv}(z)g_{iv}'^{\star}(z)}\right\} = \frac{[\sum_{s=1}^{N}\lim_{z\to z_0}\tilde{g}'_{is}(z)f'_{sj}(z)]}{[\sum_{v=1}^{N}\lim_{z\to z_0}g'_{iv}(z)g_{iv}'^{\star}(z)]}$$
$$= \frac{[\sum_{s=1}^{N}\tilde{b}_{is}a_{sj}]}{[\sum_{v=1}^{N}b_{iv}b_{iv}^{\star}]} = \frac{[\sum_{s=1}^{N}\tilde{b}_{is}a_{sj}]}{\det B} = B^{-1}A. \tag{4.2}$$

From (4.1) and (4.2) it follows immediately that

$$AB^{-1} = \lim_{z\to z_0}F(X)G^{-1}(X) = \lim_{z\to z_0}G^{-1}(X)F(X) = BA^{-1}.$$

That is to say

$$\lim_{X\to X_0}\frac{F(X)}{G(X)} = \lim_{z\to z_0}\frac{\frac{dF(X)}{dX}\cdot\frac{dX}{dz}}{\frac{dG(X)}{dX}\cdot\frac{dX}{dz}} = \lim_{X\to X_0}\frac{F'(X)}{G'(X)} = \frac{A}{B}. \tag{4.3}$$

It is worth observing that this main equality (4.3) can be obtained by using Taylor expansions for matrix functions (cf.[1]) as follows: For

$$F(X) = \sum_{n=1}^{\infty}A_n(z-z_0)^n, \quad \text{and } G(X) = \sum_{n=1}^{\infty}B_n(z-z_0)^n,$$

with

$$A_0 = B_0 = F(X_0) = G(X_0) = \mathbf{0}, \quad \text{and}$$
$$A_n = \{\frac{d^n}{d\,z^n}F(X)\}_{z=z_0}, \quad B_n = \{\frac{d^n}{d\,z^n}G(X)\}_{z=z_0}; \quad n = 1,2,\ldots.$$

Thus

$$A_1 = \left\{\frac{d\,F}{d\,X}\cdot\frac{d\,X}{d\,z}\right\}_{z=z_0}, \quad B_1 = \left\{\frac{d\,G}{d\,X}\frac{d\,X}{d\,z}\right\}_{z=z_0}.$$

Therefore

$$\lim_{z\to z_0}\frac{F(X)}{G(X)} = \lim_{X\to X_0}F(X)G^{-1}(X) = \lim_{X\to X_0}G^{-1}(X)F(X)$$
$$= \lim_{z\to z_0}\left\{A_1B_1^{-1}+B_1^{-1}\sum_{n=2}^{\infty}A_n(z-z_0)^{n-1}\right\}\left\{I+B_1^{-1}\sum_{n=2}^{\infty}B_n(z-z_0)^{n-1}\right\}^{-1}$$
$$= \left\{A_1B_1^{-1}+B_1^{-1}\lim_{z\to z_0}\sum_{n=2}^{\infty}A_n(z-z_0)^{n-1}\right\}\left\{I+B_1^{-1}\lim_{z\to z_0}\sum_{n=2}^{\infty}B_n(z-z_0)^{n-1}\right\}^{-1}$$
$$= A_1B_1^{-1} = \left\{\left(\frac{d\,F}{d\,X}\cdot\frac{d\,X}{d\,z}\right)\cdot\left(\frac{d\,G}{d\,X}\frac{d\,X}{d\,z}\right)^{-1}\right\}_{z=z_0} = \lim_{X\to X_0}\frac{F'(X)}{G'(X)} = \frac{A}{B},$$

which ensures our main result. Therefore, we have proved the theorem.

5. Examples

This section presents some application examples to show the usability of our approach.

Example 5.1. Consider

$$\lim_{X\to X_0}\frac{e^X-I}{X}=\lim_{X\to X_0}\frac{\sin X}{X}=\lim_{X\to X_0}\frac{\sinh X}{X}=I,$$

where

$$X=\begin{pmatrix} e^z-1 & 3\sin z & 5\sinh z & \cosh z-1\\ 3\sin z & e^z-1 & \cosh z-1 & 5\sinh z\\ 5\sinh z & \cosh z-1 & e^z-1 & 3\sin z\\ \cosh z-1 & 5\sinh z & 3\sin z & e^z-1\end{pmatrix};$$

$X_0 = [x_{ij}(0)] = \mathbf{0}$,

$$\frac{d\,X}{d\,z}=\begin{pmatrix} e^z & 3\cos z & 5\cosh z & \sinh z\\ 3\cos z & e^z & \sinh z & 5\cosh z\\ 5\cosh z & \sinh z & e^z & 3\cos z\\ \sinh z & 5\cosh z & 3\cos z & e^z\end{pmatrix}.$$

Example 5.2. Consider

$$\lim_{X\to A}\frac{X^n-A^n}{X-A}=nA^{n-1};\quad A=[x_{ij}(z_0)]=\left[x_{ij}\left(\frac{\pi}{4}\right)\right],$$

where

$$X=\begin{pmatrix}\cos z & \sin z\\ -\sin z & \cos z\end{pmatrix},\quad A=\begin{pmatrix}\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}}\end{pmatrix}.$$

Example 5.3. Consider

$$\lim_{X\to A}\frac{X^n-A^n}{X^m-A^m}=\frac{n}{m}\;A^{n-m};$$

$$X=\begin{pmatrix}e^z & e^{-z}\\ e^z & -e^{-z}\end{pmatrix},\quad A=\begin{pmatrix}e^a & e^{-a}\\ e^a & -e^{-a}\end{pmatrix}=[x_{ij}(a)];\quad a\in\mathbb{C}.$$

It is clear that the above examples give a direct generalization to the standard complex case of L'Hospital rule. Therefore our result of the previous sections can be exploited to establish further consequences regarding other several problems in this area.

Remark 5.1. The case both limits $\lim_{X\to X_0}\frac{F(X)}{G(X)}=\lim_{X\to X_0}\frac{F'(X)}{G'(X)}$ diverge to $\pm\infty$, should be manipulated separately. It is still an open question to be answered in a forthcoming work.

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