Journal of the Egyptian Mathematical Society (2013) **21**, 97–102

Egyptian Mathematical Society

Journal of the Egyptian Mathematical Society

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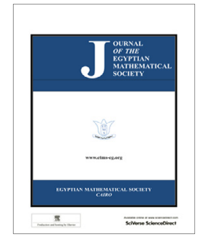

ORIGINAL ARTICLE

Interpolation on non-uniformly separated sequences in a weighted Bergman space

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Received 2 April 2012; revised 4 January 2013; accepted 8 January 2013
Available online 27 March 2013

KEYWORDS

Bergman space;
Interpolation;
Uniformly separated

Abstract We consider a definition of interpolation, called O-interpolation, that includes the possibility of sequences that are not uniformly separated. We prove that the density condition used to describe classical interpolation sequences is actually sufficient to give O-interpolation.

2010 MATHEMATICS SUBJECT CLASSIFICATION: 30C, 30H20, 46E20

1. Introduction

Interpolation sequences in the weighted Bergman spaces were characterized by Seip [5], Berndtsson and Ortega-Cerdà [2], and Ortega-Cerdà and Seip [3]. In all of these papers, the definition of interpolation sequence was such that every interpolation sequence has to be uniformly separated, that is, there is a positive lower bound on the distance between pairs of points in the sequence. Ostrovsky [4] considered a notion of interpolation that does not require uniform separation. He proved that the density condition given in the above articles is sufficient to give this generalized interpolation in the setting of the Bargmann-Fock space. The purpose of this note is to show how this can be done in the weighted Bergman space.

Let φ be C^2 and subharmonic in $\mathbb{D}$ satisfying $0 < m \leqslant \widetilde{\Delta}\varphi \leqslant M$ where $\widetilde{\Delta} = (1-|z|^2)^2 \frac{\partial^2}{\partial z \partial \bar{z}}$. Denote by A^2_φ the set of functions analytic in $\mathbb{D}$ satisfying

$$\|f\| = \left(\int_{\mathbb{D}} |f(z)|^2 \frac{e^{-\varphi(z)}}{1-|z|^2} \, d\sigma(z) \right)^{\frac{1}{2}} < \infty,$$

where $d\sigma = dA/\pi$ is normalized Lebesgue measure.

For $z, \zeta \in \mathbb{D}$, let $\varphi_\zeta(z) = \frac{\zeta - z}{1 - \bar{\zeta} z}$. We define the pseudohyperbolic metric $\rho(z,\zeta) = |\varphi_\zeta(z)|$ and the pseudohyperbolic disk $D(z,r) = \{\zeta \in \mathbb{D} : \rho(z,\zeta) < r\}$.

Let Γ be a sequence of points in $\mathbb{D}$. For any $\gamma \in \Gamma$, we define $\sigma_\gamma = \inf_{\gamma' \neq \gamma} \rho(\gamma, \gamma')$, and we let $\rho_\gamma = \frac{\sigma_\gamma}{2}$. The sequence Γ is said to be uniformly separated if there is a positive constant σ such that $\sigma_\gamma \geqslant \sigma$ for all $\gamma \in \Gamma$. Let $n_\gamma = \#\left(\Gamma \cap D\left(\gamma, \frac{1}{2}\right)\right)$.

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Peer review under responsibility of Egyptian Mathematical Society.

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http://dx.doi.org/10.1016/j.joems.2013.01.004

We say that a sequence $\Gamma = \{\gamma\}$ is an O-interpolation sequence for A^2_φ if for any sequence $\{a_\gamma\}$ of complex numbers satisfying

$$\sum_{\gamma\in\Gamma}|a_\gamma|^2 \frac{e^{-\varphi(\gamma)}}{\rho_\gamma^{2n_\gamma}}(1-|\gamma|^2) < \infty, \tag{1}$$

there is a function $f \in A^2_\varphi$ such that $f(\gamma) = a_\gamma$.

Note that if Γ is uniformly separated, then the denominator in (1) is bounded above and below by constants. Thus (1) holds if and only if $\sum_{\gamma\in\Gamma}|a_\gamma|^2 e^{-\varphi(\gamma)}(1-|\gamma|^2) < \infty$.

The invariant convolution of a measure μ and a measurable function g is defined by the formula

$$(\mu * g)(\zeta) = \int_{\mathbb{D}} g(\varphi_\zeta(z))\frac{d\mu(z)}{(1-|z|^2)^2}$$

whenever the integral exists.

For a sequence Γ in the unit disk, define the measure $\nu = \pi\sum_{\gamma\in\Gamma}(1-|\gamma|^2)^2\delta_\gamma$, where δ_γ is the Dirac delta measure at the point γ.

For $\frac{1}{2} < r < 1$, define the function

$$\xi_r(\zeta) = \begin{cases} c_r \ \log\frac{1}{|\zeta|^2} & \text{if } \frac{1}{2} < |\zeta| < r, \\ 0 & \text{otherwise,} \end{cases}$$

where the constant c_r is chosen so that

$$\int_{\mathbb{D}} \xi_r(\zeta)\frac{d\sigma(\zeta)}{(1-|\zeta|^2)^2} = 1. \tag{2}$$

It can be shown that $c_r \ \log\frac{1}{1-r} \to 1$ as $r \to 1$. Define also $\varphi^r = \varphi * \xi_r$.

Our goal in this paper is to prove

Theorem 1 (*Main Theorem*). *Let Γ be a sequence of points in $\mathbb{D}$. Suppose that $\widetilde{\Delta}\varphi$ is uniformly bounded in $\mathbb{D}$. If there exists $r < 1$ and $\delta > 0$ such that*

$$(\nu * \xi_r)(z) < \widetilde{\Delta}\varphi d\sigma * \xi_r(z) - \delta \tag{3}$$

for all $z \in \mathbb{D}$, then Γ is an O-interpolation sequence for A^2_φ.

2. Interpolation

O-interpolation is closely related to the classical notion of interpolation. A sequence Γ is said to be interpolating for A^2_φ if whenever

$$\sum_{\gamma\in\Gamma}|a_\gamma|^2 e^{-\varphi(\gamma)}(1-|\gamma|^2) < \infty,$$

there is a function $f \in A^2_\varphi$ such that $f(\gamma) = a_\gamma$ for all γ. It follows from an argument involving the closed graph theorem that every interpolation sequence for A^2_φ is uniformly separated. On the other hand, by the remark in the previous section, we see that when Γ is uniformly separated, interpolation is equivalent to O-interpolation.

It is proved in [1] and [2] that uniformly separated sequences satisfying the condition in Theorem 1 are interpolating for A^2_φ. It is also shown that the condition is necessary, although technically this was proved only in the setting of the Bargmann-Fock space in [3]. The authors note that similar arguments should work in the Bergman space.

Note that we are making no claim regarding the necessity direction for O-interpolation, although we do not believe that the condition is necessary for O-interpolation..

Similar theorems regarding interpolation in the setting of the Bargmann-Fock space were proved in [5,6,2] and [3]. Ostrovsky [4] introduced O-interpolation in the Bargmann-Fock space and proved an analoge of our Main Theorem.

Versions of interpolation involving non-uniformly separated sequences exist in the setting of the Hardy space, the set of bounded analytic functions and the Payley-Wiener space. (See [7] for references.) In these cases interpolation sequences are not described by density conditions and are qualitatively different from interpolating sequences in the Bergman and Bargmann-Fock spaces.

Luecking [8] considers interpolation on non-uniformly separated sequences in the Bergman spaces with standard radial weights. We do not know whether his methods could be modified to work in the more general setting.

Our proof is modeled after the proofs of Theorems 3 and 4 in [2], as well as the proof in [4]. For the sake of clarity, we will use notation similar to the one employed in [2].

3. Preliminaries

The following result appears in one form or another in many of the papers cited above. We do not know if it is available in precisely the form we require, so we provide a proof here.

Lemma 3.1. *Let φ be as above, $z \in \mathbb{D}$, and $0 < r < 1$. Then there exists a holomorphic function H_z defined in $D(z,r)$, with $H_z(z) = 0$, and a constant C, independent of z, such that*

$$|\varphi(z) - \varphi(w) + 2\Re H_z(w)| \leqslant C$$

for all $w \in D(z,r)$.

Proof. Define $h_z(w)$ in $D(z,r)$ to be

$$h_z(w) = \varphi(w) - \varphi(z) + \int_{D(z,r)}\left(\log\left|\frac{z-\xi}{1-\bar{z}\xi}\right| - \log\left|\frac{w-\xi}{1-\bar{w}\xi}\right|\right)\frac{\widetilde{\Delta}\varphi(\xi)}{(1-|\xi|^2)^2}dA(\xi). \tag{4}$$

Note that $h_z(z) = 0$. To see that h_z is harmonic, it is enough to show that

$$\widetilde{\Delta}\left(\int_{D(z,r)}\log\left|\frac{w-\xi}{1-\bar{w}\xi}\right|\frac{\widetilde{\Delta}\varphi(\xi)}{(1-|\xi|^2)^2}dA(\xi)\right) = \widetilde{\Delta}\varphi(w).$$

Now,

$$\begin{aligned}\int_{D(z,r)}\log\left|\frac{w-\xi}{1-\bar{w}\xi}\right|\frac{\widetilde{\Delta}\varphi(\xi)}{(1-|\xi|^2)^2}dA(\xi) &= \int_{D(z,r)}\log\left|\frac{w-\xi}{1-\bar{w}\xi}\right|\Delta\varphi(\xi)dA(\xi)\\ &= \int_{D(z,r)}\log|w-\xi|\Delta\varphi(\xi)dA(\xi) - \int_{D(z,r)}\log|1-\bar{w}\xi|\Delta\varphi(\xi)dA(\xi)\\ &= \int_{D(z,r)}[\psi(w,\xi) - G(w,\xi)]\Delta\varphi(\xi)dA(\xi) - \int_{D(z,r)}\log|1-\bar{w}\xi|\Delta\varphi(\xi)dA(\xi),\end{aligned}$$

where ψ is harmonic in $D(z,r)$ and G is a Green's function for this disk. Then, since ψ and $\log|1-\bar{w}\xi|$ are harmonic in w, and by the reproducing property of Green's functions, we have

$$\widetilde{\Delta}\left(\int_{D(z,r)}[\psi(w,\xi)-G(w,\xi)]\Delta\varphi(\xi)dA(\xi)-\int_{D(z,r)}\log|1-\bar{w}\xi|\Delta\varphi(\xi)dA(\xi)\right)$$
$$=\widetilde{\Delta}\left(\int_{D(z,r)}-G(w,\xi)\Delta\varphi(\xi)dA(\xi)\right)=\widetilde{\Delta}\varphi(w).$$

Thus h_z is harmonic, and so there exists a holomorphic function H_z such that $h_z = 2\Re H_z$.

We now seek to establish bounds on the integral in (4). To do this, we will consider the terms in this integral separately. Since φ is subharmonic, and $\log\left|\frac{z-\xi}{1-\bar{z}\xi}\right| \leqslant 0$ for $\xi \in D(z,r)$, we have

$$\int_{D(z,r)}\log\left|\frac{z-\xi}{1-\bar{z}\xi}\right|\frac{\widetilde{\Delta}\varphi(\xi)}{(1-|\xi|^2)^2}dA(\xi)\leqslant 0.$$

Moreover,

$$\int_{D(z,r)}\log\left|\frac{z-\xi}{1-\bar{z}\xi}\right|\frac{\widetilde{\Delta}\varphi(\xi)}{(1-|\xi|^2)^2}dA(\xi)$$
$$\geqslant M\int_{D(z,r)}\log\left|\frac{z-\xi}{1-\bar{z}\xi}\right|\frac{dA(\xi)}{(1-|\xi|^2)^2}.$$

Transform this last integral by applying the change of variables $\xi=\frac{z-w}{1-\bar{z}w}$ (we also have $dA(\xi)=\left(\frac{1-|z|^2}{1-|w|^2}\right)^2dA(w)$), and our integral becomes

$$\int_{D(z,r)}\log\left|\frac{z-\xi}{1-\bar{z}\xi}\right|\frac{\widetilde{\Delta}\varphi(\xi)}{(1-|\xi|^2)^2}dA(\xi)$$
$$\geqslant M\int_{D(0,r)}\log|w|\left(\frac{1}{1-|w|^2}\right)^4dA(w)$$
$$=M\int_0^r\frac{\log|\rho|}{(1-\rho^2)^4}\rho\, d\rho=-C_rM.$$

We now consider the second term in the integral (4). Let $D_1 = D(z,r)\cap D(w,r)$ and $D_2 = D(z,r) - D(w,r)$. Then

$$-\int_{D(z,r)}\log\left|\frac{w-\xi}{1-\bar{w}\xi}\right|\frac{\widetilde{\Delta}\varphi(\xi)}{(1-|\xi|^2)^2}dA(\xi)$$
$$=-\int_{D_1}\log\left|\frac{w-\xi}{1-\bar{w}\xi}\right|\frac{\widetilde{\Delta}\varphi(\xi)}{(1-|\xi|^2)^2}dA(\xi)-\int_{D_2}$$
$$\times\log\left|\frac{w-\xi}{1-\bar{w}\xi}\right|\frac{\widetilde{\Delta}\varphi(\xi)}{(1-|\xi|^2)^2}dA(\xi).$$

Note that the preceding calculations also give us

$$0\leqslant-\int_{D_1}\log\left|\frac{w-\xi}{1-\bar{w}\xi}\right|\frac{\widetilde{\Delta}\varphi(\xi)}{(1-|\xi|^2)^2}dA(\xi)$$
$$\leqslant-\int_{D(w,r)}\log\left|\frac{w-\xi}{1-\bar{w}\xi}\right|\frac{\widetilde{\Delta}\varphi(\xi)}{(1-|\xi|^2)^2}dA(\xi)\leqslant C_rM.$$

Furthermore, for $\xi\in D_2, r\leqslant\left|\frac{w-\xi}{1-\bar{w}\xi}\right|\leqslant\frac{2r}{1+r^2}$ since $\xi\notin D(z,r)$ and $w\in D(z,r)$. So

$$-\int_{D_2}\log\left|\frac{w-\xi}{1-\bar{w}\xi}\right|\frac{\widetilde{\Delta}\varphi(\xi)}{(1-|\xi|^2)^2}dA(\xi)$$
$$\leqslant-\int_{D_2}\log r\frac{\widetilde{\Delta}\varphi(\xi)}{(1-|\xi|^2)^2}dA(\xi)\leqslant-M\cdot\log r\int_{D(z,r)}\frac{dA(\xi)}{(1-|\xi|^2)^2}$$
$$=-M\frac{r^2\log r}{1-r^2},$$

and

$$-\int_{D_2}\log\left|\frac{w-\xi}{1-\bar{w}\xi}\right|\frac{\widetilde{\Delta}\varphi(\xi)}{(1-|\xi|^2)^2}dA(\xi)$$
$$\geqslant-\int_{D_2}\log\frac{2r}{1+r^2}\frac{\widetilde{\Delta}\varphi(\xi)}{(1-|\xi|^2)^2}dA(\xi)\geqslant-m\cdot\frac{r^2\log\frac{2r}{1+r^2}}{1-r^2}.\quad\square$$

Recall that invariant Laplacian $\widetilde{\Delta}$ is defined for functions $f\in C^2(\mathbb{D})$ by

$$\widetilde{\Delta}f(z)=(1-|z|^2)^2f_{z\bar{z}}(z)=\frac{1}{4}(1-|z|^2)^2\Delta f(z),$$

where Δ is the standard Laplacian. The definition of the invariant Laplacian can be extended using the theory of distributions.

In particular, if h is subharmonic, then there is a positive measure μ such that

$$\int_{\mathbb{D}}\widetilde{\Delta}\psi(z)h(z)\frac{d\sigma(z)}{(1-|z|^2)^2}=\int_{\mathbb{D}}\psi(z)\frac{d\mu(z)}{(1-|z|^2)^2}$$

for all $\psi\in C_0^\infty(\mathbb{D})$. It is customary to express this relationship by the equation $\widetilde{\Delta}h=\mu$. When $h\in C^2(\mathbb{D})$, that notation conflicts with the ordinary interpretation of the invariant Laplacian as a function $f=\widetilde{\Delta}h$. The conflict is resolved by identifying f with the absolutely continuous measure μ for which $d\mu = f d\sigma$. Then we will write, with abuse of notation, $\widetilde{\Delta}h=f\,d\sigma$.

We let $E(z) = 2\log|z|$. It follows from the reproducing property of Green's function, as well as Fubini's Theorem, that if μ is a measure for which $\mu * E$ is well-defined, then

$$\widetilde{\Delta}(\mu*E)=\mu. \tag{5}$$

Note also that if h is a function that is harmonic on a disk $\mathbb{D}(z,r)$, then

$$\int_{\mathbb{D}}\xi_r(\varphi_z(\zeta))h(\zeta)\frac{d\sigma(\zeta)}{(1-|\zeta|^2)^2}=\int_{\mathbb{D}}\xi_r(\zeta)h(\varphi_z(\zeta))\frac{d\sigma(\zeta)}{(1-|\zeta|^2)^2}$$
$$=\int_{D(0,r)}\xi_r(\zeta)h(\varphi_z(\zeta))\frac{d\sigma(\zeta)}{(1-|\zeta|^2)^2}$$
$$=h(z).$$

This follows from the mean value property of harmonic functions and the fact that $\frac{\xi_r(\zeta)d\sigma(\zeta)}{(1-|\zeta|^2)^2}$ is a radial unit measure.

Recall that $\varphi^r(z) = \varphi * \xi_r(z)$.

Lemma 3.2. *There is a constant C such that $|\varphi(z)-\varphi^r(z)|\leqslant C$ for all $z\in\mathbb{D}$.*

Proof.

$$\varphi(z)-\varphi^r(z)=\varphi(z)-\varphi d\sigma*\xi_r(z)$$
$$=\varphi(z)-\int_{\mathbb{D}}\xi_r(\varphi_z(\zeta))\varphi(\zeta)\frac{d\sigma(\zeta)}{(1-|\zeta|^2)^2}$$
$$\leqslant\varphi(z)+C\int_{\mathbb{D}}\xi_r(\varphi_z(\zeta))\frac{d\sigma(\zeta)}{(1-|\zeta|^2)^2}-\varphi(z)$$
$$\times\int_{\mathbb{D}}\xi_r(\varphi_z(\zeta))\frac{d\sigma(\zeta)}{(1-|\zeta|^2)^2}-2$$
$$\times\int_{\mathbb{D}}\xi_r(\varphi_z(\zeta))\Re H_z(\zeta)\frac{d\sigma(\zeta)}{(1-|\zeta|^2)^2}$$
$$=C.$$

The inequality follows from Lemma 3.1, and the last line follows from (2). The other direction of the inequality is proved similarly. □

It follows immediately from Lemma 3.2 that a sequence Γ is an O-interpolation sequence for A^2_φ if and only if it is an O-interpolation sequence for $A^2_{\varphi^r}$.

We will first prove:

Theorem 2. *Let Γ be a sequence of points in $\mathbb{D}$. Suppose that $\widetilde{\Delta}\varphi$ is uniformly bounded in $\mathbb{D}$. If there exists $r < 1$ and $\delta > 0$ such that*

$$(\nu * \xi_r)(z) < \widetilde{\Delta}\varphi(z) - \delta \tag{6}$$

for all $z \in \mathbb{D}$, then Γ is an O-interpolation sequence for A^2_φ.

Assume now that we have proven Theorem 2 and that the hypothesis of Theorem 1 holds, i.e. that there exists $r < 1$ and $\delta > 0$ such that

$$(\nu * \xi_r)(z) < \widetilde{\Delta}\varphi d\sigma * \xi_r(z) - \delta \tag{7}$$

for all $z \in \mathbb{D}$. Since $\widetilde{\Delta}\varphi d\sigma * \xi_r(z) = \widetilde{\Delta}(\varphi d\sigma * \xi_r)(z) = \widetilde{\Delta}\varphi^r(z)$, we see by Theorem 2 that Γ is O-interpolating for $A^2_{\varphi^r}$, which implies by the above that Γ is O-interpolating for A^2_φ. (Here we have used the fact that $\widetilde{\Delta}\varphi^r$ is uniformly bounded, a fact that follows from the identity $\widetilde{\Delta}(\varphi d\sigma * \xi_r) = \widetilde{\Delta}\varphi d\sigma * \xi_r$.)

For $0 < r < 1$, define the function

$$v_r = (\nu_\Gamma - (\nu_\Gamma * \xi_r)d\sigma) * E.$$

It is shown in [9] that v_r is well defined and has the property $v_r(z) \leqslant 0$ in $\mathbb{D}$. It follows from (5) that

$$\widetilde{\Delta} v_r = \nu_\Gamma - (\nu_\Gamma * \xi_r)d\sigma. \tag{8}$$

It is also shown in [9] that

$$\begin{aligned} v_r(z) &= 2\sum_{\gamma \in D(z,r)} \{\log|\varphi_\gamma(z)| - I(\gamma,r,z)\} \\ &= 2\sum_{\gamma} \{\log|\varphi_\gamma(z)| - I(\gamma,r,z)\}, \end{aligned}$$

where

$$I(\gamma,r,z) = \int_{\mathbb{D}} \xi_r(\varphi_\gamma(\zeta)) \log|\varphi_z(\zeta)| \frac{d\sigma(\zeta)}{(1-|\zeta|^2)^2}.$$

For $\gamma \in \Gamma$, define $T_\gamma = \left\{ z \in \mathbb{D} : \frac{\rho_\gamma}{4} \leqslant \left|\frac{z-\gamma}{1-\bar{z}\gamma}\right| \leqslant \frac{3\rho_\gamma}{4} \right\}$.

Lemma 3.3. *There is a constant C_r such that*

$$v_r(z) \geqslant n_\gamma \log \rho_\gamma^2 - C_r n_\Gamma(z,r)$$

for all $z \in T_\gamma$.

Proof. Let $z \in T_\gamma$. Then

$$\begin{aligned} \sum_{\tilde\gamma \in \Gamma\cap D(z,r)} \log\left|\frac{z-\tilde\gamma}{1-\bar{z}\tilde\gamma}\right|^2 &= \sum_{\tilde\gamma \in (\Gamma\cap D(z,r))\cap\left(\Gamma\cap D\left(\gamma,\frac12\right)\right)} \log\left|\frac{z-\tilde\gamma}{1-\bar{z}\tilde\gamma}\right|^2 \\ &\quad + \sum_{\tilde\gamma \in (\Gamma\cap D(z,r))-\left(\Gamma\cap D\left(\gamma,\frac12\right)\right)} \log\left|\frac{z-\tilde\gamma}{1-\bar{z}\tilde\gamma}\right|^2 \\ &\geqslant n_\gamma \log\left(\frac{\rho_\gamma}{4}\right)^2 + \log\left(\frac{1}{64}\right)(n_\Gamma(z,r) - n_\gamma) \\ &\geqslant n_\gamma \log \rho_\gamma^2 - C n_\Gamma(z,r), \end{aligned}$$

where C is a positive constant. Note that the first part of the first inequality holds since $\rho(z,\gamma) \leqslant \frac{3\rho_\gamma}{4}$ and $\rho(\gamma,\tilde\gamma) \geqslant \rho_\gamma$ imply that

$$\rho(z,\tilde\gamma) \geqslant \rho_\gamma - \frac{3\rho_\gamma}{4} = \frac{\rho_\gamma}{4}.$$

Here we have used the fact that $\rho(\gamma,\tilde\gamma) \geqslant \rho_\gamma$. The second part of the first inequality holds since $\rho(\gamma,\tilde\gamma) \geqslant \frac12$ and $\rho(z,\gamma) \leqslant \frac{3\rho_\gamma}{4}$, which implies

$$\rho(z,\tilde\gamma) \geqslant \frac12 - \frac{3\rho_\gamma}{4} \geqslant \frac18.$$

Here we have used the fact that $\rho_\gamma < \frac12$.

Since $I(\tilde\gamma, r, z) \leqslant 0$, we have for $z \in T_\gamma$

$$v_r(z) \geqslant n_\gamma \log \rho_\gamma^2 - C_r n_\Gamma(z,r). \quad \square$$

4. Main result

We proceed now to prove Theorem 2 and assume that there exists $r < 1$ and $\delta > 0$ such that

$$(\nu * \xi_r)(z) < \widetilde{\Delta}\varphi(z) - \delta \tag{9}$$

for all $z \in \mathbb{D}$.

Let

$$\sum_i |a_i|^2 \frac{e^{-\varphi(\gamma_i)}}{\rho_{\gamma_i}^{2n_{\gamma_i}}}(1-|\gamma_i|^2) < \infty.$$

For each $\gamma \in \Gamma$, let H_γ be a function satisfying the conclusion of Lemma 3.1, with $r = \rho_\gamma$. Define $F_\gamma : D(\gamma,\rho_\gamma) \to \mathbb{C}$ by $F_\gamma(z) = a_\gamma e^{H_\gamma(z)}$. Then F_γ is holomorphic in $D(\gamma,\rho_\gamma)$, $F_\gamma(\gamma) = a_\gamma$, and

$$\begin{aligned} \int_{D(\gamma,\rho_\gamma)} |F_\gamma(\zeta)|^2 e^{-\varphi(\zeta)} \frac{d\sigma(\zeta)}{1-|\zeta|^2} &= \int_{D(\gamma,\rho_\gamma)} |a_\gamma|^2 e^{2\Re H_\gamma(\zeta)-\varphi(\zeta)} \\ &\quad \times \frac{d\sigma(\zeta)}{1-|\zeta|^2} \\ &\leqslant C|a_\gamma|^2 e^{-\varphi(\gamma)} \int_{D(\gamma,\rho_\gamma)} \frac{d\sigma(\zeta)}{1-|\zeta|^2} \\ &\leqslant C|a_\gamma|^2 e^{-\varphi(\gamma)} \\ &\quad \times \frac{1}{1-|\gamma|^2}|D(\gamma,\rho_\gamma)| \\ &= C|a_\gamma|^2 e^{-\varphi(\gamma)} \frac{1}{1-|\gamma|^2} \\ &\quad \times \frac{\rho_\gamma^2(1-|\gamma|^2)^2}{1-\rho_\gamma^2|\gamma|^2} \\ &\leqslant C|a_\gamma|^2 e^{-\varphi(\gamma)} \rho_\gamma^2 (1-|\gamma|^2). \end{aligned} \tag{10}$$

The first and second lines follow from Lemma 3.1. The third line follows from the fact that for each $r > 0$, there exists constants C_1 and C_2 such that

$$C_1 \leqslant \frac{1-|w|^2}{1-|\zeta|^2} \leqslant C_2$$

for all $w, \zeta \in D(z,r)$, where $z \in \mathbb{D}$ is arbitrary. The fourth line follows from the formula for the area of a pseudohyperbolic disk, found in [9], for example. The fifth line is a consequence of the inequality $\rho_\gamma \leqslant \frac12$.

Similarly,

$$\begin{aligned}\int_{D(\gamma,\rho_\gamma)} |F_\gamma(\zeta)|^2 e^{-\varphi(\zeta)}(1-|\zeta|^2)d\sigma(\zeta) &= \int_{D(\gamma,\rho_\gamma)} |a_\gamma|^2 e^{2\Re H_\gamma(\zeta)-\varphi(\zeta)}(1-|\zeta|^2)d\sigma(\zeta)\\ &\leqslant C|a_\gamma|^2 e^{-\varphi(\gamma)}\int_{D(\gamma,\rho_\gamma)}(1-|\zeta|^2)d\sigma(\zeta)\\ &\leqslant C|a_\gamma|^2 e^{-\varphi(\gamma)}(1-|\gamma|^2)|D(\gamma,\rho_\gamma)|\\ &= C|a_\gamma|^2 e^{-\varphi(\gamma)}(1-|\gamma|^2)\frac{\rho_\gamma^2(1-|\gamma|^2)^2}{1-\rho_\gamma^2|\gamma|^2}\\ &\leqslant C|a_\gamma|^2 e^{-\varphi(\gamma)}\rho_\gamma^2(1-|\gamma|^2)^3 \leqslant C|a_\gamma|^2 e^{-\varphi(\gamma)}(1-|\gamma|^2)^3\rho_\gamma^2. \qquad (11)\end{aligned}$$

Let $\eta:[0,\infty)\to[0,1]$ be a smooth function that is identically 1 on $\left[0,\frac{1}{4}\right]$ and identically 0 on $\left[\frac{3}{4},\infty\right)$. Define $\widehat{F}:\mathbb{D}\to\mathbb{C}$ by

$$\widehat{F}(z) = \sum_\gamma F_\gamma(z)\eta_\gamma(z),$$

where $\eta_\gamma = \eta\left(\left(\frac{\varphi_\gamma(z)}{\rho_\gamma}\right)^2\right)$. Note that if $\rho(z,\gamma)\geqslant\rho_\gamma$ for all $\gamma\in\Gamma$, then $\eta_\gamma(z)=0$ for all γ. Thus $\widehat{F}$ is supported on $\cup_{\gamma\in\Gamma}D(\gamma,\rho_\gamma)$. Moreover, if $\gamma'\neq\gamma$ and $z\in D(\gamma,\rho_\gamma)$, then, since $\rho_\gamma\leqslant\frac{\rho(\gamma,\gamma')}{2}$ and $\rho_{\gamma'}\leqslant\frac{\rho(\gamma,\gamma')}{2}$, we have

$$\begin{aligned}\frac{\rho(z,\gamma')}{\rho_{\gamma'}} &\geqslant \frac{\rho(\gamma,\gamma')}{\rho_{\gamma'}} - \frac{\rho(\gamma,z)}{\rho_{\gamma'}} \geqslant \frac{\rho(\gamma,\gamma')}{\rho_{\gamma'}} - \frac{\rho_\gamma}{\rho_{\gamma'}} \geqslant \frac{\rho(\gamma,\gamma')}{\rho_{\gamma'}} - \frac{\rho(\gamma,\gamma')}{2\rho_{\gamma'}}\\ &= \frac{\rho(\gamma,\gamma')}{2\rho_{\gamma'}} \geqslant 1,\end{aligned}$$

which implies, in particular, that $\eta_{\gamma'}(z)=0$ if $z\in D(\gamma,\rho_\gamma)$. Thus, if $z\in D(\gamma,\rho_\gamma)$, $\widehat{F}(z)=F_\gamma(z)n_\gamma(z)$, and in particular, $\widehat{F}(\gamma)=a_\gamma$.

We have

$$\begin{aligned}\int_{\mathbb{D}} |\widehat{F}(\zeta)|^2 e^{-\varphi(\zeta)}\frac{d\sigma(\zeta)}{(1-|\zeta|^2)} &= \int_{\cup_{\gamma\in\Gamma}D(\gamma,\rho_\gamma)} \Big|\sum_{\gamma'\in\Gamma}F_{\gamma'}(\zeta)\eta_{\gamma'}(\zeta)\Big|^2 e^{-\varphi(\zeta)} \times\frac{d\sigma(\zeta)}{(1-|\zeta|^2)}\\ &= \sum_\gamma\int_{D(\gamma,\rho_\gamma)}|F_\gamma(\zeta)n_\gamma(\zeta)|^2 e^{-\varphi(\zeta)}\times\frac{d\sigma(\zeta)}{(1-|\zeta|^2)}\\ &\leqslant C\sum_\gamma|a_\gamma|^2 e^{-\varphi(\gamma)}\rho_\gamma^2(1-|\gamma|^2)\\ &\leqslant C\sum_\gamma|a_\gamma|^2 e^{-\varphi(\gamma)}\rho_\gamma^{-2n_\gamma}(1-|\gamma|^2)<\infty.\end{aligned}$$

The first inequality follows from (10).

Therefore $\widehat{F}$ is a smooth solution to this problem. Our last step is to correct $\widehat{F}$ to produce a holomorphic solution. Let

$$\psi_r = v_r + \varphi.$$

By Lemma 3.3, we have that

$$e^{-\psi_r(z)} \leqslant \frac{1}{\rho_\gamma^{2n_\gamma}} e^{C_r n_\Gamma(z,r)} e^{-\varphi(z)}$$

for all $z\in T_\gamma$. It follows from arguments in Chapter 6 of [9] that if (9) holds, then there is a constant C such that $n_\Gamma(z,r)\leqslant C(1-r)^{-1}$ for all $z\in\mathbb{D}$ and $r<1$. We thus have

$$e^{-\psi_r(z)} \leqslant C\rho_\gamma^{-2n_\gamma}e^{-\varphi(z)} \qquad (12)$$

for all $z\in T_\gamma$.

Note that $\bar\partial\widehat{F}$ is supported on $\cup_\gamma T_\gamma$, from which we have

$$\int_{\mathbb{D}}|\bar\partial\widehat{F}(\zeta)|^2 e^{-\psi_r(\zeta)}\frac{d\sigma(\zeta)}{1-|\zeta|^2} = \sum_\gamma\int_{T_\gamma}|\bar\partial\eta_\gamma(\zeta)|^2|\widehat{F}(\zeta)|^2 e^{-\psi_r(\zeta)}\times\frac{d\sigma(\zeta)}{1-|\zeta|^2}.$$

For $z\in D(\gamma,\rho_\gamma)$, we have

$$\begin{aligned}|\bar\partial\eta_\gamma(z)|^2 &= \left|\bar\partial\eta\left(\left(\frac{\varphi_\gamma(z)}{\rho_\gamma}\right)^2\right)\frac{2\varphi_\gamma(z)\varphi'_\gamma(z)}{\rho_\gamma}\right|^2 \leqslant C\frac{|\varphi'_\gamma(z)|^2}{\rho_\gamma^2}\\ &\leqslant C\frac{1}{\rho_\gamma^2}\frac{1}{(1-|\gamma|^2)^2}. \qquad (13)\end{aligned}$$

The first line follows from the chain rule and the third follows from the identity

$$|\varphi'_\zeta(z)|^2 = \frac{(1-|\varphi_\zeta(z)|^2)^2}{(1-|\zeta|^2)^2}.$$

Therefore,

$$\begin{aligned}\int_{\mathbb{D}}|\bar\partial\widehat{F}(\zeta)|^2 e^{-\psi_r(\zeta)}(1-|\zeta|^2)d\sigma(\zeta) &= \sum_\gamma\int_{T_\gamma}|\bar\partial\eta_\gamma(\zeta)|^2|\widehat{F}(\zeta)|^2 e^{-\psi_r(\zeta)}(1-|\zeta|^2)d\sigma(\zeta)\\ &\leqslant C\sum_\gamma\frac{1}{\rho_\gamma^2(1-|\gamma|^2)^2}\times\int_{T_\gamma}|F_\gamma(\zeta)|^2 e^{-\psi_r(\zeta)}(1-|\zeta|^2)d\sigma(\zeta)\\ &\leqslant C_r\sum_\gamma\frac{1}{\rho_\gamma^{2+2n_\gamma}(1-|\gamma|^2)^2}\times\int_{T_\gamma}|F_\gamma(\zeta)|^2 e^{-\varphi(\zeta)}(1-|\zeta|^2)d\sigma(\zeta)\\ &\leqslant C_r\sum_\gamma\frac{1}{\rho_\gamma^{2+2n_\gamma}(1-|\gamma|^2)^2}\times\int_{D(\gamma,\rho_\gamma)}|F_\gamma(\zeta)|^2 e^{-\varphi(\zeta)}(1-|\zeta|^2)d\sigma(\zeta)\\ &\leqslant C\sum_\gamma\frac{1}{\rho_\gamma^{2+2n_\gamma}(1-|\gamma|^2)^2}|a_\gamma|^2 e^{-\varphi(\gamma)}(1-|\gamma|^2)^3\rho_\gamma^2\\ &= \sum_\gamma|a_\gamma|^2 e^{-\varphi(\gamma)}(1-|\gamma|^2)\rho_\gamma^{-2n_\gamma}<\infty.\end{aligned}$$

The first inequality follows from (13), the second from (12) and the fourth from (11).

By (9) and (8) we have

$$\begin{aligned}\widetilde{\Delta}\psi_r &= \widetilde{\Delta}v_r + \widetilde{\Delta}\varphi = v_\Gamma - (v_\Gamma * \xi_r)(z) + \widetilde{\Delta}\varphi\\ &> v_\Gamma - (v_\Gamma * \xi_r)(z) + (v_\Gamma * \xi_r)(z) + \delta \geqslant \delta.\end{aligned}$$

This allows us to apply the following variant of Hörmander's Theorem, which is due to Ohsawa [10].

Theorem 3. *Let ψ be any subharmonic function in the disk such that $\widetilde{\Delta}\psi > \delta > 0$. Then there is a solution U to the equation $\bar{\partial}U = g$ such that*

$$\int_{\mathbb{D}} |U(\zeta)|^2 \frac{e^{-\psi(\zeta)}}{1-|\zeta|^2} d\sigma(\zeta) \leqslant C \int_{\mathbb{D}} |g|^2 e^{-\psi(\zeta)} (1-|\zeta|^2)\ d\sigma(\zeta).$$

So there is a function U such that $\bar{\partial}U = \bar{\partial}\widehat{F}$ and

$$\int_{\mathbb{D}} |U(\zeta)|^2 e^{-\psi_r(\zeta)} \frac{d\sigma(\zeta)}{1-|\zeta|^2} \leqslant C \int_{\mathbb{D}} |\bar{\partial}\widehat{F}(\zeta)|^2 e^{-\psi_r(\zeta)} (1-|\zeta|^2) d\sigma(\zeta) < \infty.$$

Since $e^{-\psi_r(\zeta)}$ is not locally integrable at any $\gamma \in \Gamma$, we must have that $U(\gamma) = 0$. Moreover,

$$\int_{\mathbb{D}} |U(\zeta)|^2 e^{-\varphi(\zeta)} \frac{d\sigma(\zeta)}{1-|\zeta|^2} \leqslant C \int_{\mathbb{D}} |U(\zeta)|^2 e^{-\psi_r(\zeta)} \frac{d\sigma(\zeta)}{(1-|\zeta|^2)} < \infty.$$

Now, define the function $F = \widehat{F} - U$. Then $F(\gamma) = a_\gamma$ and F is holomorphic. Since both $\widehat{F}$ and U have finite L^2_φ norms, so does F.

Acknowledgement

The authors would like to thank the referee for carefully reading the manuscript and making several valuable suggestions.

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