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Best proximity points for asymptotic proximal pointwise weaker Meir–Keeler-type ψ-contraction mappings

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Abstract In this paper, we study the new class of an asymptotic proximal pointwise weaker Meir–Keeler-type ψ-contraction and prove the existence of solutions for the minimization problem in a uniformly convex Banach space. Also, we give some an example for support our main result.

2000 MATHEMATICS SUBJECT CLASSIFICATION: 47H09, 47H10

1. Introduction and preliminaries

The best proximity theorem furnishes sufficient conditions for the existence of an optimal approximate solution x, known as the best proximity point of the non-self mapping T, satisfying the condition that $d(x, Tx) = dist(A, B)$. Interestingly, the best proximity theorems also serve as a natural generalization of fixed point theorems. Indeed, the best proximity point becomes a fixed point if the mapping under consideration is a self-mapping. On the other hand, though the best proximity theorems ensure the existence of approximate solutions, such results need not yield optimal solutions. But, best proximity point theorems furnish sufficient conditions that assure the existence of approximate solutions which are optimal as well.

The classical and well-known Banach's contraction principle states that if a self-mapping T of a complete metric space X is a contraction mapping (i.e., $d(Tx, Ty) \leqslant \alpha d(x, y)$ for all $x, y \in X$, where $\alpha \in [0, 1)$), then T has a unique fixed point. This principle has been extended in several ways such as [1–6]. In 2003, Kirk, Srinivasan, and Veeramani [7] extended the Banach's contraction principle to case of cyclic mappings. Let (X, d) be a metric space and let A, B, be a non-empty subset of X. A mapping T: $A \cup B \to A \cup B$ is called a *cyclic mapping* if $T(A) \subset B$ and $T(B) \subset A$. A point $x \in A$ is called a *best proximity point* of T in A if $d(x, Tx) = \text{dist}(A, B)$, where $\text{dist}(A, B) = \inf\{d(x, y): \; x \in A, y \in B\}$. A cyclic mapping T: $A \cup B \to A \cup B$ is said to be a *relatively non-expansive* if $\|Tx - Ty\| \leqslant \|x - y\|$ for all $x \in A$ and $y \in B$ (notice that a relatively non-expansive mapping need not be a continuous in general). In 2005, Eldred, Kirk and Veeramani [8] proved the existence of a best proximity point for relatively non-expansive

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mappings by using the notion of proximal normal structure. In 2006, Eldred and Veeramani [9] introduced the notion called *cyclic contraction* and gave sufficient condition for the existence of a best proximity point for a cyclic contraction mapping T on a uniformly convex Banach space. In 2009, Suzuki et al. [10] introduced the notion of the property UC as follow :

Definition 1.1 10. Let A and B be non-empty subsets of a metric space (X, d). Then (A, B) is said to be satisfy the property UC, if the following holds: If $\{x_n\}$ and $\{\acute{x}_n\}$ are sequences in A and $\{y_n\}$ is a sequence in B such that

$$\lim_{n\to\infty} d(x_n, y_n) = dist(A, B) \text{ and } \lim_{n\to\infty} d(\acute{x}_n, y_n) = dist(A, B),$$

then $\lim_{n\to\infty} d(x_n, \acute{x}_n) = 0$.

Also, they extended the result in [9] to metric spaces with the property UC. The following lemma plays an important role in next sections;

Lemma 1.2 10. *Let A and B be subsets of a metric space (X,d). Assume that (A,B) has the property UC. Let $\{x_n\}$ and $\{y_n\}$ be sequences in A and B, respectively, such that either of the following holds:*

$$\lim_{m\to\infty} \sup_{n\geqslant m} d(x_m, y_n) = dist(A, B) \text{ or } \lim_{n\to\infty} \sup_{m\geqslant n} d(x_m, y_n) = dist(A, B).$$

Then $\{x_n\}$ is Cauchy.

On the other hand, in 2003, Kirk [11], introduced the notion of an asymptotic contraction mapping as follows:

Definition 1.3 11. Let (X, d) be a metric space. A mapping T: $X \to X$ is said to be an asymptotic contraction if

$$d(T^n(x), T^n(y)) \leqslant \phi_n(d(x, y)) \quad \text{for all } x, y \in X,$$

where ϕ_n: $[0,\infty) \to [0,\infty)$ and $\phi_n \to \phi$ uniformly on the range of d in which ϕ: $[0,\infty) \to [0,\infty)$ is continuous and $\phi(s) < s$ for all $s > 0$.

In 2007, Kirk [12], introduced the notion of an asymptotic pointwise contraction mapping as follows:

Definition 1.4 12. Let (X, d) be a metric space. A mapping T: $X \to X$is said to be an asymptotic pointwise contraction if there exists a sequence of functions $\alpha_n : X \to \mathbb{R}^+$ such that $\alpha_n \to \alpha$ pointwise on X and for each integer $n \geqslant 1$,

$$d(T^n(x), T^n(y)) \leqslant \alpha_n(x)(d(x, y)) \quad \text{for all } x, y \in X.$$

In 2008, Kirk and Xu [13], introduced the notion of a pointwise asymptotically non-expansive mapping as follows:

Definition 1.5 13. Let K be a non-empty subset of Banach space X. A mapping T: $K \to K$ is said to be a pointwise asymptotically non-expansive, if for each integer $n \geqslant 1$,

$$\|T^n(x) - T^n(y)\| \leqslant \alpha_n(x)\|x - y\| \quad \text{for all } x, y \in K,$$

where $\alpha_n \to 1$ pointwise on K.

In 2009, Anuradha and Veeramani in [14] introduced a new class of mappings; they called each mapping of this class a proximal pointwise contraction:

Definition 1.6 14. Let A and B be non-empty subsets of a metric space (X, d). Let T: $A \cup B \to A \cup B$ be a cyclic mapping. The mapping T is said to be a proximal pointwise contraction if for each $(x, y) \in A \times B$ there exist $0 \leqslant \alpha(x) < 1$, $0 \leqslant \alpha(y) < 1$ such that

$$d(T(x), T(y)) \leqslant \max\{\alpha(x)d(x, y), dist(A, B)\} \quad \text{for all } y \in B,$$
$$d(T(x), T(y)) \leqslant \max\{\alpha(y)d(x, y), dist(A, B)\} \quad \text{for all } x \in A.$$

Recently, Abkar and Gabeleh [15] introduced a new notion of an asymptotic proximal pointwise contraction mapping as follows:

Definition 1.7 15. Let (A, B) be a non-empty pair in a Banach space X. A mapping T: $A \cup B \to A \cup B$ is said to be an asymptotic proximal pointwise contraction if T is cyclic and there exists a function α: $A \cup B \to [0,1)$ such that for any integer $n \geqslant 1$ and $(x, y) \in A \times B$,

$$\|T^{2n}x - T^{2n}y\| \leqslant \max\{\alpha_n(x)\|x - y\|, dist(A, B)\} \quad \text{for all } y \in B,$$
$$\|T^{2n}x - T^{2n}y\| \leqslant \max\{\alpha_n(y)\|x - y\|, dist(A, B)\} \quad \text{for all } x \in A,$$

where $\alpha_n \to \alpha$ pointwise on $A \cup B$.

Just recently, Chen [16] defined the following new notion of the weaker Meir–Keeler-type function and an asymptotic pointwise weaker Meir–Keeler-type contraction, $\mathbb{R}_+$ denoted the set of all non-negative numbers.

Definition 1.8 16. The function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is called a weaker Meir–Keeler-type function, if for each $\eta > 0$, there exists $\delta > \eta$ such that for $t \in \mathbb{R}_+$ with $\eta \leqslant t < \delta$, there exists $n_0 \in \mathbb{N}$ such that $\psi^{n_0}(t) < \eta$.

Definition 1.9 16. Let X be a Banach space, and $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ be a weaker Meir–Keeler-type function. A mapping T: $X \to X$ is said to be an asymptotic pointwise weaker Meir–Keeler-type ψ-contraction, if for each $n \in \mathbb{N}$,

$$\|T^n x - T^n y\| \leqslant \psi^n(\|x\|)\|x - y\| \quad \text{for all } x, y \in X.$$

For example of a weaker Meir–Keeler-type mapping and a weaker Meir–Keeler-type mapping which is not a Meir–Keeler-type mapping, we can see in [17]. Best proximity point theorems for several types of contractions, for examples see in [18–23].

In this paper, we give the notion of new class of an asymptotic proximal pointwise weaker Meir–Keeler-type ψ-contraction and prove the existence of a best proximity point theorem for this mapping. Also, we give some an example for support our main Theorem.

2. Asymptotic proximal pointwise weaker Meir–Keeler-type ψ-contraction

In this section, we prove the existence of a best proximity point for an asymptotic proximal pointwise weaker Meir–Keeler-type ψ-contraction in a uniformly convex Banach space. First, we introduce below notion of an asymptotic proximal pointwise weaker Meir–Keeler-type ψ-contraction mapping.

Definition 2.1. Let (A, B) be a non-empty pair in Banach space X, and let $\psi : \mathbb{R}_+ \to \mathbb{R}_+$be a weaker Meir–Keeler-type function. A mapping T: $A \cup B \to A \cup B$ is said to be an asymptotic proximal pointwise weaker Meir–Keeler-type ψ-contraction, if for each $n \in \mathbb{N}$ and $(x, y) \in A \times B$,

$$\|T^{2n}x - T^{2n}y\| \leqslant \max\{\psi^n(\|x\|)\|x - y\|, dist(A, B)\}$$
for all $y \in B$,

$$\|T^{2n}x - T^{2n}y\| \leqslant \max\{\psi^n(\|y\|)\|x - y\|, dist(A, B)\}$$
for all $x \in A$.

Before stating the main result, we recall definition and fact of asymptotic centers. Let X be a Banach space, C subset of X and $\{x_n\}$ is a bounded sequence in X. The asymptotic centers of $\{x_n\}$ relative to C denoted by $A_C(x_n)$ is the set of minimizers in A (if any) of the function f given by

$$f(x) = \limsup_{n\to\infty}\|x_n - x\|.$$

That is,

$$A_C(x_n) = \{x \in C : f(x) = inf_{u\in C}f(u)\},$$

and we can see that, if X is uniformly convex and C is closed and convex, then $A_C(x_n)$ consists of exactly one point.

Theorem 2.2. *Let (A, B) be a non-empty bounded closed convex pair in a uniformly convex Banach space X and T: $A \cup B \to A \cup B$ be an asymptotic proximal pointwise weaker Meir–Keeler-type ψ-contraction. If T is a relatively non-expansive mapping, then there exists a unique pair $(v_0, u_0) \in A \times B$ such that*

$$\|u_0 - Tu_0\| = \|v_0 - Tv_0\| = dist(A, B).$$

Moreover, if $x_0 \in A$ and $x_{n+1} = Tx_n$, then $\{x_{2n}\}$ converges in norm to v_0 and $\{x_{2n+1}\}$ converges in norm to u_0.

Proof. Fix an $x_0 \in A$ and define a function f: $B \to [0,\infty)$ by

$$f(u) = \limsup_{n\to\infty}\|T^{2n}(x_0) - u\| \text{ for } u \in B.$$

Since X is uniformly convex and B is bounded closed and convex, it follow that f has unique minimizer over B; that is, we have a unique point $u_0 \in B$ satisfying

$$f(u_0) = \inf_{u\in B}f(u).$$

Indeed, for all $m \geqslant 1$ and $u \in B$, we have

$$\begin{aligned} f(T^{2m}(u)) &= \limsup_{n\to\infty}\|T^{2n}(x_0) - T^{2m}u\| \\ &= \limsup_{n\to\infty}\|T^{2n+2m}(x_0) - T^{2m}u\| \\ &= \limsup_{n\to\infty}\|T^{2m}(T^{2n}(x_0)) - T^{2m}u\| \\ &\leqslant \limsup_{n\to\infty} \max\{\psi^m(\|u\|)\|T^{2n}(x_0) \\ &\quad - u\|, \mathrm{dist}(A, B)\} \\ &= \max\{\psi^m(\|u\|)f(u), \mathrm{dist}(A, B)\}. \end{aligned} \tag{2.1}$$

Since $u_0 \in B$ is the minimum of f, for all $m \geqslant 1$, we have

$$f(u_0) \leqslant f(T^{2m}u_0) \leqslant \max\{\psi^m(\|u_0\|)f(u_0), \mathrm{dist}(A, B)\}. \tag{2.2}$$

We now claim that $f(u_0) = \mathrm{dist}(A, B)$. Since for each $u \in B$, $\{\psi^m(\|u\|)\}$ is non-increasing, it must converges to some $\eta \geqslant 0$. Suppose that $\eta > 0$, by definition of weaker Meir–Keeler-type function, there exists $\delta > \eta$ such that for $u \in B$ with $\eta \leqslant \|u\| < \delta$, there exists $n_0 \in \mathbb{N}$ such that $\psi^{n_0}(\|u\|) < \eta$. Since $\lim_{m\to\infty}\psi^m(\|u\|) = \eta$ there exists $m_0 \in \mathbb{N}$ such that $\eta \leqslant \psi^m(\|u\|) < \delta$, for all $m \geqslant m_0$. Thus we conclude that $\psi^{m_0+n_0}(\|u\|) < \eta$, thus we get the contradiction. So

$$\lim_{m\to\infty}\psi^m(\|u\|) = 0. \tag{2.3}$$

Taking $m \to \infty$ in the inequality (2.2),we get

$$f(u_0) = \mathrm{dist}(A, B).$$

On the other hand, by the relatively non-expansive of T, we have

$$\begin{aligned} f(T^2u_0) &= \limsup_{n\to\infty}\|(T^{2n}(x_0)) - T^2u_0\| \\ &\leqslant \limsup_{n\to\infty}\|(T^{2n-2}(x_0)) - u_0\| = f(u_0), \end{aligned}$$

which implies that $T^2u_0 = u_0$, by the uniqueness of minimum of f, then u_0 is a fixed point of T^2 in B. Hence,

$$\begin{aligned} \lim_{m\to\infty}\sup_{n\geqslant m}\|(T^{2m}(x_0)) - T^{2n}u_0\| &= \lim_{m\to\infty}\|(T^{2m}(x_0)) - u_0\| \\ &= f(u_0) = \mathrm{dist}(A, B). \end{aligned}$$

By the property UC of (A, B), it follows from Lemma 1.2 that $\{T^{2n}(x_0)\}$ is a Cauchy sequence, so there exists $x' \in A$ such that $T^{2n}x_0 \to x'$ as $n \to \infty$. By the similar argument as above, if $y_0 \in B$ and g: $A \to [0,\infty)$ is given by $g(v) = \limsup_{n\to\infty}\|T^{2n}(y_0) - v\|$ for $v \in A$, we get v_0 is a fixed point of T^2, where v_0 is a minimum in exactly one point in A, and also $T^{2n}y_0 \to y' \in B$. Hence, we obtain

$$u_0 = T^{2n}u_0 \to y' \text{ and } v_0 = T^{2n}v_0 \to x'.$$

This show that $(v_0, u_0) = (x', y')$, and $T^{2n}x_0 \to v_0$, $T^{2n}y_0 \to u_0$. Moreover,

$$\begin{aligned} \|u_0 - v_0\| &= \|T^{2n}(u_0) - T^{2n}v_0\| \\ &\leqslant \max\{\psi^n(\|u_0\|)\|u_0 - v_0\|, \mathrm{dist}(A, B)\}. \end{aligned} \tag{2.4}$$

Taking $n \to \infty$ in the inequality (2.4), by (2.3) and definition of $\mathrm{dist}(A, B)$, we get

$$\|u_0 - v_0\| = \mathrm{dist}(A, B).$$

Since T is relatively non-expansive mapping, we have

$$\mathrm{dist}(A, B) \leqslant \|Tu_0 - Tv_0\| \leqslant \|u_0 - v_0\| = \mathrm{dist}(A, B).$$

Therefore $Tu_0 = v_0$ and $Tv_0 = u_0$. This implies that

$$\|Tu_0 - u_0\| = \|v_0 - Tv_0\| = \mathrm{dist}(A, B). \quad \square$$

Now, we shall give a validate example of Theorem 2.2.

Example 2.3. Consider $X = \mathbb{R}^2$ with the metric $d((x_1, y_1), (x_2, y_2)) = \max\{|x_1 - x_2|, |y_1 - y_2|\}$ for all $(x_1, y_1), (x_2, y_2) \in \mathbb{R}_+$. Let

$$A = \{(1, a) : a \geqslant 0\} \text{ and } B = \{(-1, b) : b \geqslant 0\},$$

then A and B be a non-empty closed and convex subset of X and $dist(A, B) = 2$. Define T:$A \cup B \to A \cup B$, by

$$T(1, a) = \left(-1, \frac{a}{2}\right) \text{ and } T(-1, b) = \left(1, \frac{b}{2}\right) \quad \text{for all } a, b \geqslant 0.$$

Then T is a cyclic mapping, relatively non-expansive and for each $(1, a) \in A$ and $(-1, b) \in B$, we have

$T^{2n}(1,a) = \left(1, \frac{a}{2^{2n}}\right)$ and $T^{2n}(-1,b) = \left(-1, \frac{b}{2^{2n}}\right)$. Next, we will show that T is an asymptotic proximal pointwise weaker Meir–Keeler-type ψ-contraction with weaker Meir–Keeler-type function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$defined by

$$\psi(t) = \frac{t}{2} \text{ for all } t \geq 0.$$

Since,

$$\begin{aligned} d(T^{2n}(1,a), T^{2n}(-1,b)) &= d((1, \frac{a}{2^{2n}}), (-1, \frac{b}{2^{2n}})) \\ &= \max\{2, |\frac{a-b}{2^{2n}}|\} \\ &\leq \max\{2, |\frac{a-b}{2^{n}}|\} \\ &\leq \max\{2, \psi^n(d((0,0),(1,a))|a-b|\} \\ &\leq \max\{\psi^n(d((0,0),(1,a))d((1,a),(-1,b)),\ dist(A,B)\}. \end{aligned}$$

Similarly, we can conclude that

$$d(T^{2n}(1,a), T^{2n}(-1,b)) \leqslant \max\{\psi^n(d((0,0), (-1,b))d((1,a),(-1,b))), dist(A,B)\},$$

and hence T is an asymptotic proximal pointwise weaker Meir–Keeler-type ψ-contraction. Moreover $((1,0),(-1,0)) \in A \times B$ is a pair of best proximity point of T, because

$$d((1,0), T(1,0)) = d((-1,0), T(-1,0)) = 2 = dist(A,B).$$

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