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On involute-evolute curve couple in the hyperbolic and de Sitter spaces

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Abstract
This paper aims at showing that *Frenet* apparatus of an evolute curve can be formed in terms of *Frenet* apparatus of its involute curve in the hyperbolic (*de Sitter*) space. Also, we establish relationships among *Frenet* frame of the considered curve couple. Finally, we defray some examples to confirm our main results.

Keywords: Frenet frames, Involute-evolute curves, Hyperbolic space

MSC: 53A25, 53C50

Introduction

The idea of a string involute is due to Christian Huygens (1658), who is also known for his work in optics. He discovered involutes while trying to develop a more accurate clock [1]. The involute of a given curve is a well-known concept in *Euclidean* 3-space R^3. It is well-known that if a curve is differentiable at each point of an open interval, a set of mutually orthogonal unit vectors can be constructed and called *Frenet* frame or moving frame vectors. The rates of these frame vectors along the curve define curvatures of the curves. The set, whose elements are frame vector and curvatures of a curve, is called *Frenet* apparatus of the curve.

It is safe to report that the many important results in the theory of the curves in R^3 were initiated by G. Monge, and G. Darboux pioneered the moving frame idea (for more details see [2]). Thereafter, *Frenet* defined his moving frame and his special equations which play important role in mechanics and kinematics as well as in differential geometry. At the beginning of the twentieth century, A. Einstein's theory opened a door to the use of new geometries. One of them, *Minkowski* space-time, which is simultaneously the geometry of special relativity and the geometry induced on each fixed tangent space of an arbitrary *Lorentzian* manifold, was introduced, and some of the classical differential geometry topics have been treated by the researchers. In the recent years, the theory of degenerate submanifolds has been treated by researchers and some classical differential geometry topics have been extended to *Lorentz* manifolds. For instance, in [3–5], the authors extended and studied spacelike involute-evolute curves in *Euclidean* 4-space and *Minkowski* space-time.

An evolute and its involute are defined in mutual pairs. The evolute and the involute of the curve pair are well known by the mathematicians especially the differential geometry scientists. The evolute of any curve is defined as the locus of the centers of curvature of the curve. The original curves are then defined as the involute of the evolute. The simplest case is that of a circle, which has only one center of curvature (its center), which is a degenerate evolute and the circle itself is the involute of this point.

Izumiya et al. defined the evolute curve in hyperbolic 2-space and found its equation. Following the works of them, we defined the evolute curve in hyperbolic 3-space and *de Sitter* 3-space and found its equations (see Definitions (2) and (3), and for more details, see [1, 6–10]).

In this paper, we calculate the *Frenet* apparatus of the evolute curve in terms of the apparatus of its involute curve in hyperbolic 2-space, hyperbolic 3-space, and *de Sitter* 3-space. We hope that our results can be seen as refinement and generalization of many corresponding results existing in the literature and useful in mathematical modeling and some other applications.

Preliminaries

In this section, we give the basic notions and familiar results in Lorentzian geometry which we need in this paper (for more details, see [7–12]).

Hyperbolic 2-space

Let $R^3 = \{(x_1, x_2, x_3) \mid x_1, x_2, x_3 \in R\}$ be a three-dimensional vector space and $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ be two vectors in R^3. The pseudo-scalar product of x and y is defined by $\langle x, y\rangle = -x_1y_1 + x_2y_2 + x_3y_3$. $\left(R^3, \langle,\rangle\right)$ is called a three-dimensional pseudo-*Euclidean* space or *Minkowski* 3-space. We write E_1^3 instead of $\left(R^3, \langle,\rangle\right)$. We say that a vector x in E_1^3 is *spacelike*, *lightlike*, or *timelike* if $\langle x, x\rangle > 0$, $\langle x, x\rangle = 0$ or $\langle x, x\rangle < 0$, respectively. We now define spheres in E_1^3 as follows:

$$\begin{cases} H_+^2 = \{x \in E_1^3 \mid -x_1^2 + x_2^2 + x_3^2 = -1, x_1 \geq 1\} \\ H_-^2 = \{x \in E_1^3 \mid -x_1^2 + x_2^2 + x_3^2 = -1, x_1 \leq -1\} \\ S_1^2 = \{x \in E_1^3 \mid -x_1^2 + x_2^2 + x_3^2 = 1\}. \end{cases}$$

We call $H_\pm^2$ a hyperbola and S_1^2 a pseudo-sphere. Now, we discuss some basic facts of curves in hyperbolic 2-space, which are needed in the sequel.

Let $\alpha : I \longrightarrow H_+^2 \subset E_1^3; \quad \alpha(t) = (x_1(t), x_2(t), x_3(t))$ be a smooth regular curve in H_+^2 (*i.e.*, $\alpha'(t) \neq 0$) for any $t \in I$, where I is an open interval. It is easy to show that $\langle \alpha'(t), \alpha'(t)\rangle > 0$, for any $t \in I$. We call such a curve *a spacelike* curve. The *norm* of the vector $x \in E_1^3$ is defined by $\|x\| = \sqrt{|\langle x, x\rangle|}$. The *arc-length* of a spacelike curve α, measured from $\alpha(t_0), t_0 \in I$ is $s(t) = \int_{t_\circ}^{t} \|\alpha'(t)\| dt$. Then, the parameter s is determined such that $\|\dot\alpha(s)\| = 1$, where $\dot\alpha(s) = \frac{d\alpha(s)}{ds}$. So, we say that a spacelike curve α is parameterized by *arc-length*, if it satisfies $\|\dot\alpha(s)\| = 1$. Throughout the remainder in this paper, we denote the parameter s of α as the *arc-length* parameter. Let us denote $\mathbf{T}(s) = \dot\alpha(s)$, and we call $\mathbf{T}(s)$ a unit tangent vector of α at s.

For any $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in E_1^3$, the pseudo-vector product of x and y is defined as follows:

$$x \wedge y = \begin{vmatrix} -e_1 & e_2 & e_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = (-(x_2y_3 - x_3y_2), x_3y_1 - x_1y_3, x_1y_2 - x_2y_1).$$

We remark that $\langle x \wedge y, z\rangle = det(x\, y\, z)$. Hence, $x \wedge y$ is pseudo-orthogonal to x, y. We now set a vector $\mathbf{E}(s) = \alpha(s) \wedge \mathbf{T}(s)$. By definition, we can calculate that $\langle \mathbf{E}(s), \mathbf{E}(s)\rangle = 1$ and $\langle \alpha(s), \alpha(s)\rangle = -1$. We can also show that $\mathbf{T}(s) \wedge \mathbf{E}(s) = -\alpha(s)$ and $\alpha(s) \wedge \mathbf{E}(s) = -\mathbf{T}(s)$. Therefore, we have a pseudo-orthonormal frame $\{\alpha(s), \mathbf{T}(s), \mathbf{E}(s)\}$ along $\alpha(s)$. We have the following hyperbolic *Frenet-Serret* formula of plane curves:

$$\begin{cases} \dot{\alpha}(s) = \mathbf{T}(s) \\ \dot{\mathbf{T}}(s) = \alpha(s) + \kappa_g(s)\mathbf{E}(s) \\ \dot{\mathbf{E}}(s) = -\kappa_g(s)\mathbf{T}(s), \end{cases} \tag{1}$$

or in the matrix form:

$$\begin{bmatrix} \dot{\alpha}(s) \\ \dot{\mathbf{T}}(s) \\ \dot{\mathbf{E}}(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & \kappa_g \\ 0 & -\kappa_g & 0 \end{bmatrix} \begin{bmatrix} \alpha(s) \\ \mathbf{T}(s) \\ \mathbf{E}(s) \end{bmatrix} \tag{2}$$

where κ_g is the geodesic curvature of the curve α in H^2_+, which is given by

$$\kappa_g(s) = det(\alpha(s)\ \mathbf{T}(s)\ \dot{\mathbf{T}}(s)).$$

Hyperbolic 3-space

Let R^4 be a four-dimensional vector space. For any $x = (x_1, x_2, x_3, x_4), y = (y_1, y_2, y_3, y_4) \in R^4$, the pseudo-scalar product of x and y is defined by $\langle x, y\rangle = -x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4$. $(R^4, \langle,\rangle)$ is called a *Minkowski* 4-space and denoted by E^4_1. We say that a vector $x \in E^4_1$ is *spacelike, lightlike*, or *timelike* if $\langle x_1, x_2\rangle > 0$, $\langle x_1, x_2\rangle = 0$ or $\langle x_1, x_2\rangle < 0$, respectively. The *norm* of the vector $x \in E^4_1$ is defiend by $\|x\| = \sqrt{|\langle x, x\rangle|}$. For a non-zero vector $\upsilon \in E^4_1$ and a real number c, we define a space with pseudo-normal υ by

$$S(\upsilon, c) = \{x \in E^4_1 \mid \langle x, \upsilon\rangle = c\}.$$

The space $S(\upsilon, c)$ is called a *spacelike* space, a *timelike* space, or a *lightlike* space if υ is *timelike, spacelike,* or *lightlike*, respectively.

Now, we define a hyperbolic 3-space by

$$H^3_+(-1) = \left\{x \in E^4_1 \mid \langle x, x\rangle = -1, x_1 > 0\right\}.$$

For any $x = (x_1, x_2, x_3, x_4)$, $y = (y_1, y_2, y_3, y_4)$ and $z = (z_1, z_2, z_3, z_4) \in E^4_1$, the pseudo-vector product of x, y, and z is defined as follows:

$$x \wedge y \wedge z = \begin{vmatrix} -i & j & k & l \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix}$$

$$= \left(-\begin{vmatrix} x_2 & x_3 & x_4 \\ y_2 & y_3 & y_4 \\ z_2 & z_3 & z_4 \end{vmatrix}, -\begin{vmatrix} x_1 & x_3 & x_4 \\ y_1 & y_3 & y_4 \\ z_1 & z_3 & z_4 \end{vmatrix}, \begin{vmatrix} x_1 & x_2 & x_4 \\ y_1 & y_2 & y_4 \\ z_1 & z_2 & z_4 \end{vmatrix}, -\begin{vmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{vmatrix} \right).$$

We now prepare some basic facts of curves in hyperbolic 3-space.

Let $\beta : I \longrightarrow H^3_+ \subset E^4_1$; $\beta(t) = (x_1(t), x_2(t), x_3(t), x_4(t))$ be a smooth regular curve in H^3_+ (*i.e.*, $\beta'(t) \neq 0$) for any $t \in I$ where I is an open interval. So that $\langle \beta'(t), \beta'(t)\rangle > 0$ for any $t \in I$. The *arc-length* of β measured from $\beta(t_\circ), t_\circ \in I$ is $s(t) = \int_{t_\circ}^{t} \|\beta'(t)\| dt$. Then, the parameter s is determined such that $\|\dot{\beta}(s)\| = 1$, where $\dot{\beta}(s) = \frac{d\beta(s)}{ds}$. So, we say that a spacelike curve β is parameterized by *arc-length* if it satisfies that $\|\dot{\beta}(s)\| = 1$. Let us denote $\mathbf{T}(s) = \dot{\beta}(s)$, and we call $\mathbf{T}(s)$ a unit tangent vector of β at s.

Here, we construct the explicit differential geometry on curves in $H^3_+(-1)$. Let $\beta : I \longrightarrow H^3_+(-1)$ be a regular curve. Since $H^3_+(-1)$ is a *Riemannian* manifold, we can reparameterize β by the *arc-length*. Hence, we may assume that $\beta(s)$ is a unit speed curve. So, we have the tangent vector $\mathbf{T}(s) = \dot{\beta}(s)$ with $\|\mathbf{T}\| = 1$. In case when $\langle \dot{\mathbf{T}}(s), \dot{\mathbf{T}}(s) \rangle \neq -1$, then we have a unit vector

$$\mathbf{N}(s) = \frac{\dot{\mathbf{T}}(s) - \beta(s)}{\|\dot{\mathbf{T}}(s) - \beta(s)\|}.$$

Moreover, define $\mathbf{E}(s) = \beta(s) \wedge \mathbf{T}(s) \wedge \mathbf{N}(s)$, then we have a pseudo-orthonormal frame $\{\beta(s), \mathbf{T}(s), \mathbf{N}(s), \mathbf{E}(s)\}$ of E^4_1 along β. By standard arguments, under the assumption that $\langle \dot{\mathbf{T}}(s), \dot{\mathbf{T}}(s) \rangle \neq -1$, we have the following *Frenet* formula:

$$\begin{cases} \dot{\beta}(s) = \mathbf{T}(s), \\ \dot{\mathbf{T}}(s) = \beta(s) + \kappa_g \mathbf{N}(s), \\ \dot{\mathbf{N}}(s) = -\kappa_g \mathbf{T}(s) + \tau_g \mathbf{E}(s), \\ \dot{\mathbf{E}}(s) = -\tau_g \mathbf{N}(s). \end{cases} \tag{3}$$

In another form:

$$\begin{bmatrix} \dot{\beta}(s) \\ \dot{\mathbf{T}}(s) \\ \dot{\mathbf{N}}(s) \\ \dot{\mathbf{E}}(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & \kappa_g & 0 \\ 0 & -\kappa_g & 0 & \tau_g \\ 0 & 0 & -\tau_g & 0 \end{bmatrix} \begin{bmatrix} \beta(s) \\ \mathbf{T}(s) \\ \mathbf{N}(s) \\ \mathbf{E}(s) \end{bmatrix}$$

where

$$\begin{aligned} \kappa_g &= \|\dot{\mathbf{T}}(s) - \beta(s)\|, \\ \tau_g &= -\frac{det\left(\beta(s), \dot{\beta}(s), \ddot{\beta}(s), \dddot{\beta}(s)\right)}{(\kappa_g(s))^2} \end{aligned} \tag{4}$$

are the geodesic curvature and geodesic torsion of the curve β in $H^3_+(-1)$, respectively.

Since $\langle \dot{\mathbf{T}}(s) - \beta(s), \dot{\mathbf{T}}(s) - \beta(s) \rangle = \langle \dot{\mathbf{T}}(s), \dot{\mathbf{T}}(s) \rangle + 1$, the condition

$$\langle \dot{\mathbf{T}}(s), \dot{\mathbf{T}}(s) \rangle \neq -1,$$

is equivalent to the condition $\kappa_g(s) \neq 0$. Moreover, we can show that the curve $\beta(s)$ satisfies the condition $\kappa_g(s) \equiv 0$ if and only if there exists a *lightlike* vector c such that $\beta(s) - c$ is a geodesic. Such a curve is called an equidistant curve (see [9, 12]).

De Sitter 3-space

Let $\gamma : I \longrightarrow S^3_1$ be a smooth and regular spacelike curve in S^3_1. We can parameterize it by *arc-length* s. Hence, we may assume that $\gamma(s)$ is a unit speed curve and we have the tangent vector $\mathbf{T}(s) = \dot{\gamma}(s)$ with $\|\mathbf{T}\| = 1$. In this case, we call γ a unit speed spacelike curve. If $\langle \dot{\mathbf{T}}(s), \dot{\mathbf{T}}(s) \rangle \neq 1$, then $\|\dot{\mathbf{T}}(s) + \gamma(s)\| \neq 0$, and we define the unit vector $\mathbf{N}(s) = \dfrac{\dot{\mathbf{T}}(s) + \gamma(s)}{\|\dot{\mathbf{T}}(s) + \gamma(s)\|}$. Moreover, define $\mathbf{E}(s) = \gamma(s) \wedge \mathbf{T}(s) \wedge \mathbf{N}(s)$, then we have a pseudoorthonormal frame $\{\gamma(s), \mathbf{T}(s), \mathbf{N}(s), \mathbf{E}(s)\}$ of E^4_1 along γ. By standard arguments, under the assumption that $\langle \dot{\mathbf{T}}(s), \dot{\mathbf{T}}(s) \rangle \neq 1$, we have the following *Frenet-Serret* type formula:

$$\begin{cases} \dot{\gamma}(s) = \mathbf{T}(s) \\ \dot{\mathbf{T}}(s) = -\gamma(s) + \kappa_g \mathbf{N}(s) \\ \dot{\mathbf{N}}(s) = -\delta(\gamma)\kappa_g \mathbf{T}(s) + \tau_g \mathbf{E}(s) \\ \dot{\mathbf{E}}(s) = \tau_g \mathbf{N}(s). \end{cases} \tag{5}$$

It can be written as:

$$\begin{bmatrix} \dot{\gamma}(s) \\ \dot{\mathbf{T}}(s) \\ \dot{\mathbf{N}}(s) \\ \dot{\mathbf{E}}(s) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & \kappa_g & 0 \\ 0 & -\delta(\gamma)\kappa_g & 0 & \tau_g \\ 0 & 0 & \tau_g & 0 \end{bmatrix} \begin{bmatrix} \gamma(s) \\ \mathbf{T}(s) \\ \mathbf{N}(s) \\ \mathbf{E}(s) \end{bmatrix}$$

where $\delta(\gamma) = sign\ (\mathbf{N}(s))$ (which we shall write as simply δ) and

$$\begin{cases} \kappa_g = \|\dot{\mathbf{T}}(s) + \gamma(s)\|, \\ \\ \tau_g = \dfrac{\delta \det\left(\gamma(s), \dot{\gamma}(s), \ddot{\gamma}(s), \dddot{\gamma}(s)\right)}{(\kappa_g(s))^2}, \end{cases} \tag{6}$$

are the geodesic curvature and geodesic torsion of the curve γ in S_1^3, respectively.

Since $\langle \dot{\mathbf{T}}(s) + \gamma(s), \dot{\mathbf{T}}(s) + \gamma(s) \rangle = \langle \dot{\mathbf{T}}(s), \dot{\mathbf{T}}(s) \rangle - 1$, the condition $\langle \dot{\mathbf{T}}(s), \dot{\mathbf{T}}(s) \rangle \neq 1$ is equivalent to the condition $\kappa_g(s) \neq 0$ (see [4]).

The Frenet apparatus of an evolute curve in hyperbolic 2-space

In this section, we introduce the *Frenet* apparatus of an evolute curve in terms of *Frenet* apparatus of its involute curve in H_+^2.

Definition 1 *Let $\alpha : I \longrightarrow H_+^2$ be a smooth and regular spacelike curve in hyperbolic 2-space. We define the hyperbolic evolute curve of $\alpha(s)$ under the assumption that $\kappa_g^2(s) \neq \pm 1$ in H_+^2 as;*

$$h_\alpha(s) = \frac{1}{\sqrt{\left|\kappa_g^2(s) - 1\right|}} \left(\kappa_g(s)\alpha(s) + \mathbf{E}(s)\right).$$

We remark that h_α is located in $H_+^2 \cup H_-^2$ if and only if $\kappa_g^2 > 1$. If h_α is located in H_-^2, we may consider $-h_\alpha(s)$ instead of $h_\alpha(s)$ and we call $\alpha(s)$ an involute curve of $h_\alpha(s)$ (for more details, see [10]).

We denote by the *Frenet* apparatus of an evolute curve $h_\alpha(s)$ by $\left\{h_\alpha(s), \mathbf{T}_{h_\alpha}(s), \mathbf{E}_{h_\alpha}(s), \mathcal{K}_{h_\alpha}(s)\right\}$.

Theorem 1 *Let $\alpha(s) : I \longrightarrow H_+^2 \subset E_1^3$ be a unit speed spacelike involute curve in H_+^2, then the Frenet apparatus of the hyperbolic evolute curve $h_\alpha(s)$ are given as follows:*

$$\begin{cases} h_\alpha(s) = \dfrac{1}{\sqrt{\left|\kappa_g^2 - 1\right|}} \left(\kappa_g \alpha(s) + \mathbf{E}(s)\right), \\ \mathbf{T}_{h_\alpha}(s) = \dfrac{-\dot{\kappa_g}}{\left(\left|\kappa_g^2 - 1\right|\right)^{\frac{3}{2}}} \left(\alpha(s) + \kappa_g \mathbf{E}(s)\right), \end{cases} \quad \begin{cases} \mathbf{E}_{h_\alpha}(s) = \left(\dfrac{\dot{\kappa_g}}{\kappa_g^2 - 1}\right) \mathbf{T}(s), \\ \mathcal{K}_{h_\alpha}(s) = \dfrac{\dot{\kappa_g}^2}{\left(\left|\kappa_g^2 - 1\right|\right)^{\frac{3}{2}}}. \end{cases} \tag{7}$$

Proof From the definition of the evolute curve in hyperbolic 2-space, we can write

$$h_\alpha(s) = \frac{1}{\sqrt{\left|\kappa_g^2 - 1\right|}} \left(\kappa_g \alpha(s) + \mathbf{E}(s)\right). \tag{8}$$

Differentiating both sides of the previous equation with respect to s and substitute from Eq. (1), we obtain

$$\mathbf{T}_{h_\alpha}(s) = \frac{-\kappa_g \dot{\kappa_g}}{\left(\left|\kappa_g^2 - 1\right|\right)^{\frac{3}{2}}} \left(\kappa_g \alpha + \mathbf{E}\right) + \frac{1}{\sqrt{\left|\kappa_g^2 - 1\right|}} \left(\dot{\kappa_g} \alpha + \kappa_g \mathbf{T} - \kappa_g \mathbf{T}\right),$$

which can be written as

$$\begin{aligned} \mathbf{T}_{h_\alpha}(s) &= \frac{-\kappa_g \dot{\kappa_g}}{\left(\left|\kappa_g^2 - 1\right|\right)^{\frac{3}{2}}} \left(\kappa_g \alpha + \mathbf{E}\right) + \frac{\kappa_g^2 - 1}{\left(\left|\kappa_g^2 - 1\right|\right)^{\frac{3}{2}}} \left(\dot{\kappa_g} \alpha + \kappa_g \mathbf{T} - \kappa_g \mathbf{T}\right) \\ &= \frac{-\kappa_g^2 \dot{\kappa_g}}{\left(\left|\kappa_g^2 - 1\right|\right)^{\frac{3}{2}}} \alpha - \frac{\kappa_g \dot{\kappa_g}}{\left(\left|\kappa_g^2 - 1\right|\right)^{\frac{3}{2}}} \mathbf{E} + \frac{\kappa_g^2 \dot{\kappa_g}}{\left(\left|\kappa_g^2 - 1\right|\right)^{\frac{3}{2}}} \alpha - \frac{\dot{\kappa_g}}{\left(\left|\kappa_g^2 - 1\right|\right)^{\frac{3}{2}}} \alpha, \end{aligned}$$

then, we get

$$\mathbf{T}_{h_\alpha}(s) = \frac{-\dot{\kappa_g}}{\left(\left|\kappa_g^2 - 1\right|\right)^{\frac{3}{2}}} \alpha(s) - \frac{\kappa_g \dot{\kappa_g}}{\left(\left|\kappa_g^2 - 1\right|\right)^{\frac{3}{2}}} \mathbf{E}(s). \tag{9}$$

Since $\mathbf{E}_{h_\alpha}(s) = h_\alpha(s) \wedge \mathbf{T}_{h_\alpha}(s)$, we have

$$\mathbf{E}_{h_\alpha}(s) = \begin{vmatrix} -\alpha(s) & \mathbf{T}(s) & \mathbf{E}(s) \\ \dfrac{\kappa_g}{\sqrt{\left|\kappa_g^2 - 1\right|}} & 0 & \dfrac{1}{\sqrt{\left|\kappa_g^2 - 1\right|}} \\ \dfrac{-\dot{\kappa_g}}{\left(\left|\kappa_g^2 - 1\right|\right)^{\frac{3}{2}}} & 0 & \dfrac{-\kappa_g \dot{\kappa_g}}{\left(\left|\kappa_g^2 - 1\right|\right)^{\frac{3}{2}}} \end{vmatrix},$$

then, we get

$$\mathbf{E}_{h_\alpha}(s) = \left(\frac{\dot{\kappa_g}}{\kappa_g^2 - 1}\right) \mathbf{T}(s).$$

Also, by differentiating the Eq. (9) and substitute from Eq. (1), we obtain

$$\dot{\mathbf{T}}_{h_\alpha}(s) = \begin{cases} \left(\dfrac{\ddot{\kappa_g} - \kappa_g^2\ddot{\kappa_g} + 3\kappa_g\dot{\kappa_g}}{\left(\left|\kappa_g^2 - 1\right|\right)^{\frac{5}{2}}}\right)\alpha(s) + \dfrac{\dot{\kappa_g}}{\sqrt{\left|\kappa_g^2 - 1\right|}}\mathbf{T}(s) \\ + \left(\dfrac{\kappa_g\ddot{\kappa_g} - \kappa_g^3\ddot{\kappa_g} + 3\kappa_g^2\dot{\kappa_g} - \kappa_g^2\dot{\kappa_g}^2 + \dot{\kappa_g}^2}{\left(\left|\kappa_g^2 - 1\right|\right)^{\frac{5}{2}}}\right)\mathbf{E}(s). \end{cases} \tag{10}$$

Because $\mathcal{K}_{h_\alpha} = \det\left(h_\alpha(s)\ \mathbf{T}_{h_\alpha}\ \dot{\mathbf{T}}_{h_\alpha}\right)$, then from Eqs. (8–10), we get

$$\mathcal{K}_{h_\alpha}(s) = \begin{vmatrix} \dfrac{\kappa_g}{\sqrt{\left|\kappa_g^2-1\right|}} & 0 & \dfrac{1}{\sqrt{\left|\kappa_g^2-1\right|}} \\ \dfrac{-\dot{\kappa_g}}{\left(\left|\kappa_g^2-1\right|\right)^{\frac{3}{2}}} & 0 & \dfrac{-\kappa_g\dot{\kappa_g}}{\left(\left|\kappa_g^2-1\right|\right)^{\frac{3}{2}}} \\ \dfrac{\ddot{\kappa_g} - \kappa_g^2\ddot{\kappa_g} + 3\kappa_g\dot{\kappa_g}}{\left(\left|\kappa_g^2-1\right|\right)^{\frac{5}{2}}} & \dfrac{\dot{\kappa_g}}{\sqrt{\left|\kappa_g^2-1\right|}} & \dfrac{\kappa_g\ddot{\kappa_g} - \kappa_g^3\ddot{\kappa_g} + 3\kappa_g^2\dot{\kappa_g} - \kappa_g^2\dot{\kappa_g}^2 + \dot{\kappa_g}^2}{\left(\left|\kappa_g^2-1\right|\right)^{\frac{5}{2}}} \end{vmatrix},$$

which can be written as:

$$\begin{aligned} \mathcal{K}_{h_\alpha} &= \left(\frac{\kappa_g}{\sqrt{\left|\kappa_g^2-1\right|}}\right)\left(\frac{\dot{\kappa_g}}{\sqrt{\left|\kappa_g^2-1\right|}}\right)\left(\frac{\kappa_g\dot{\kappa_g}}{\left(\left|\kappa_g^2-1\right|\right)^{\frac{3}{2}}}\right) \\ &\quad - \left(\frac{1}{\sqrt{\left|\kappa_g^2-1\right|}}\right)\left(\frac{\dot{\kappa_g}}{\left(\left|\kappa_g^2-1\right|\right)^{\frac{3}{2}}}\right)\left(\frac{\dot{\kappa_g}}{\sqrt{\left|\kappa_g^2-1\right|}}\right) \\ &= \left(\frac{\kappa_g^2\dot{\kappa_g}^2}{\left(\left|\kappa_g^2-1\right|\right)^{\frac{5}{2}}}\right) - \left(\frac{\dot{\kappa_g}^2}{\left(\left|\kappa_g^2-1\right|\right)^{\frac{5}{2}}}\right) \\ &= \left(\frac{\dot{\kappa_g}^2\left(\kappa_g^2-1\right)}{\left(\left|\kappa_g^2-1\right|\right)^{\frac{5}{2}}}\right). \end{aligned}$$

In a simple form

$$\mathcal{K}_{h_\alpha} = \frac{\dot{\kappa_g}^2}{\left(\left|\kappa_g^2-1\right|\right)^{\frac{3}{2}}}.$$

In the light of the above calculations, the proof is completed. □

The Frenet apparatus of an evolute curve in hyperbolic 3-space

Here, as in the case of hyperbolic 2-space, we construct the *Frenet* apparatus of an evolute curve using *Frenet* apparatus of its involute curve in the hyperbolic 3-space $H_+^3(-1)$ and define its equation. We start as follows:

Definition 2 *For a given involute curve β in $H_+^3(-1)$, the hyperbolic evolute curve $h_\beta : I \longrightarrow H_+^3$ of $\beta(s)$ is defined by*

$$h_\beta(s) = \frac{1}{\sqrt{\left|\kappa_g^2 - \left(\frac{\dot{\kappa_g}}{\kappa_g\tau_g}\right)^2 - 1\right|}}\left(\kappa_g\beta(s) + \mathbf{N}(s) - \frac{\dot{\kappa_g}}{\kappa_g\tau_g}\mathbf{E}(s)\right),$$

under the assumption that $\kappa_g^2 - \left(\frac{\dot{\kappa_g}}{\kappa_g\tau_g}\right)^2 > 1$.

We remark that $h_\beta(s)$ is located in $H_+^3(-1)$ if and only if $\kappa_g^2 - \left(\frac{\dot{\kappa_g}}{\kappa_g\tau_g}\right)^2 > 1$, where $\beta(s)$ is an involute curve of $h_\beta(s)$ (see [6]).

Now, we denote by $\{h_\beta(s), \mathbf{T}_{h_\beta}(s), \mathbf{N}_{h_\beta}(s), \mathbf{E}_{h_\beta}(s), \mathcal{K}_{h_\beta}(s), \mathcal{T}_{h_\beta}(s)\}$ the *Frenet* apparatus of $h_\beta(s)$ and we formulate the following theorem.

Theorem 2 *Let $h_\beta(s)$ be a unit speed spacelike evolute curve of $\beta(s)$, then the Frenet apparatus of $h_\beta(s)$ can be expressed as:*

$$\mathbf{T}_{h_\beta}(s) = \dot{\mu}_1\beta(s) + (\mu_1 - \kappa_g\mu_2)\mathbf{T}(s) + (\dot{\mu}_2 - \tau_g\mu_3)\mathbf{N}(s) + (\dot{\mu}_3 + \tau_g\mu_2)\mathbf{E}(s),$$

$$\begin{aligned}\mathbf{N}_{h_\beta}(s) = &\left(\left|-(\eta_1-\mu_1)^2 + \eta_2^2 + (\eta_3-\mu_2)^2 + (\eta_4-\mu_3)^2\right|\right)^{\frac{-1}{2}} (((\eta_1-\mu_1)\beta(s)\\ &+ \eta_2\mathbf{T}(s) + (\eta_3-\mu_2)\mathbf{N}(s) + (\eta_4-\mu_3)\mathbf{E}(s))),\end{aligned}$$

$$\mathcal{K}_{h_\beta} = \sqrt{\left|-(\eta_1-\mu_1)^2 + \eta_2^2 + (\eta_3-\mu_2)^2 + (\eta_4-\mu_3)^2\right|},$$

$$\begin{aligned}\mathbf{E}_{h_\beta}(s) = &-\frac{1}{\mathcal{K}_\beta}((-\mu_2((\mu_1-\kappa_g\mu_2)(\eta_4-\mu_3) - \eta_2(\dot{\mu}_2-\tau_g\mu_3)) + \mu_3((\mu_1-\kappa_g\mu_2)(\eta_3-\mu_2)\\ &- \eta_2(\dot{\mu}_2-\tau_g\mu_3)))\beta(s) + (\mu_1((\dot{\mu}_2-\tau_g\mu_3)(\eta_4-\mu_3) - (\eta_3-\mu_2)(\dot{\mu}_3+\tau_g\mu_2))\\ &- \mu_2(\dot{\mu}_1(\eta_4-\mu_3) - (\eta_1-\mu_1)(\dot{\mu}_3+\tau_g\mu_2)) + \mu_3(\dot{\mu}_1(\eta_3-\mu_2) - (\eta_1-\mu_1)(\dot{\mu}_2-\tau_g\mu_3)))\mathbf{T}(s)\\ &- (\mu_1((\mu_1-\kappa_g\mu_2)(\eta_4-\mu_3) - \eta_2(\dot{\mu}_3+\tau_g\mu_2)) + \mu_3(\dot{\mu}_1\eta_2 - (\eta_1-\mu_1)(\mu_1-\kappa_g\mu_2)))\mathbf{N}(s)\\ &+ (\mu_1((\mu_1-\kappa_g\mu_2)(\eta_3-\mu_2) - \eta_2(\dot{\mu}_2+\tau_g\mu_3)) + \mu_2(\dot{\mu}_1\eta_2 - (\eta_1-\mu_1)(\mu_1-\kappa_g\mu_2)))\mathbf{E}(s)),\end{aligned}$$

$$\begin{aligned}\mathcal{T}_{h_\beta}(s) = &\frac{1}{\mathcal{K}_\beta^2}(\mu_1((\mu_1-\kappa_g\mu_2)(\eta_3\zeta_4-\eta_4\zeta_3) - (\dot{\mu}_2-\tau_g\mu_3)(\eta_2\zeta_4-\eta_4\zeta_2) + (\dot{\mu}_3-\tau_g\mu_2)(\eta_2\zeta_3-\eta_3\zeta_2))\\ &- \mu_2(\dot{\mu}_1(\eta_2\zeta_4-\eta_4\zeta_2) - (\mu_1-\kappa_g\mu_2)(\eta_1\zeta_4-\eta_4\zeta_1) + (\dot{\mu}_3-\tau_g\mu_2)(\eta_1\zeta_2-\eta_2\zeta_1))\\ &+ \mu_3(\dot{\mu}_1(\eta_2\zeta_3-\eta_3\zeta_2) - (\mu_1-\kappa_g\mu_2)(\eta_1\zeta_3-\eta_3\zeta_1) + (\dot{\mu}_2-\tau_g\mu_3)(\eta_1\zeta_2-\eta_2\zeta_1))),\end{aligned}$$

where $\mu_1, \mu_2, \mu_3, \eta_1, \eta_2, \eta_3, \eta_4, \zeta_1, \zeta_2, \zeta_3,$ and ζ_4 are smooth functions.

Proof According to the Definition (2), we have

$$h_\beta(s) = \frac{\kappa_g}{\sqrt{\left|\kappa_g^2 - \left(\frac{\dot{\kappa_g}}{\kappa_g\tau_g}\right)^2 - 1\right|}}\beta + \frac{1}{\sqrt{\left|\kappa_g^2 - \left(\frac{\dot{\kappa_g}}{\kappa_g\tau_g}\right)^2 - 1\right|}}\mathbf{N} - \frac{\dot{\kappa_g}}{\kappa_g\tau_g\sqrt{\left|\kappa_g^2 - \left(\frac{\dot{\kappa_g}}{\kappa_g\tau_g}\right)^2 - 1\right|}}\mathbf{E}.$$

If we denote

$$\mu_1 = \frac{\kappa_g}{\sqrt{\left|\kappa_g^2 - \left(\frac{\dot{\kappa}_g}{\kappa_g \tau_g}\right)^2 - 1\right|}}, \quad \mu_2 = \frac{1}{\sqrt{\left|\kappa_g^2 - \left(\frac{\dot{\kappa}_g}{\kappa_g \tau_g}\right)^2 - 1\right|}},$$
$$\mu_3 = -\frac{\dot{\kappa}_g}{\kappa_g \tau_g \sqrt{\left|\kappa_g^2 - \left(\frac{\dot{\kappa}_g}{\kappa_g \tau_g}\right)^2 - 1\right|}}, \tag{11}$$

then, the evolute curve $h_\beta(s)$ can take the form

$$h_\beta(s) = \mu_1 \beta(s) + \mu_2 \mathbf{N}(s) + \mu_3 \mathbf{E}(s). \tag{12}$$

Differentiating both sides of the previous equation with respect to s and substitute from Eq. (3), we obtain

$$\begin{aligned} \mathbf{T}_{h_\beta}(s) =& \dot{\mu}_1 \beta(s) + \mu_1 \dot{\beta}(s) + \dot{\mu}_2 \mathbf{N}(s) + \mu_2 \dot{\mathbf{N}}(s) + \dot{\mu}_3 \mathbf{E}(s) + \mu_3 \dot{\mathbf{E}}(s) \\ =& \dot{\mu}_1 \beta(s) + \mu_1 \mathbf{T}(s) + \dot{\mu}_2 \mathbf{N}(s) + \mu_2(-\kappa_g \mathbf{T}(s) + \tau_g \mathbf{E}(s)) \\ & + \dot{\mu}_3 \mathbf{E}(s) - \mu_3(\tau_g \mathbf{N}(s)) \\ =& \dot{\mu}_1 \beta(s) + (\mu_1 - \kappa_g \mu_2)\mathbf{T}(s) + (\dot{\mu}_2 - \tau_g \mu_3)\mathbf{N}(s) \\ & + (\dot{\mu}_3 + \tau_g \mu_2)\mathbf{E}(s). \end{aligned} \tag{13}$$

Following the differentiating of Eq. (13), we get

$$\dot{\mathbf{T}}_{h_\beta}(s) = \begin{cases} (\ddot{\mu}_1 + \mu_1 - \kappa_g \mu_2)\beta(s) + (2\dot{\mu}_1 - \dot{\kappa}_g \mu_2 - 2\kappa_g \dot{\mu}_2 + \kappa_g \tau_g \mu_3)\mathbf{T}(s) \\ +(\kappa_g \mu_1 - \kappa_g^2 \mu_2 + \ddot{\mu}_2 - \dot{\tau}_g \mu_3 - 2\tau_g \dot{\mu}_3 - \tau_g^2 \mu_2)\mathbf{N}(s) \\ +(\tau_g \dot{\mu}_2 - \tau_g^2 \mu_3 + \ddot{\mu}_3 + \dot{\tau}_g \mu_2 + \tau_g \dot{\mu}_2)\mathbf{E}(s), \end{cases} \tag{14}$$

which can be written in a simple form

$$\dot{\mathbf{T}}_{h_\beta}(s) = \eta_1 \beta(s) + \eta_2 \mathbf{T}(s) + \eta_3 \mathbf{N}(s) + \eta_4 \mathbf{E}(s), \tag{15}$$

where

$$\begin{cases} \eta_1 = (\ddot{\mu}_1 + \mu_1 - \kappa_g \mu_2) \\ \eta_2 = (2\dot{\mu}_1 - \dot{\kappa}_g \mu_2 - 2\kappa_g \dot{\mu}_2 + \kappa_g \tau_g \mu_3) \\ \eta_3 = (\kappa_g \mu_1 - \kappa_g^2 \mu_2 + \ddot{\mu}_2 - \dot{\tau}_g \mu_3 - 2\tau_g \dot{\mu}_3 - \tau_g^2 \mu_2) \\ \eta_4 = (\tau_g \dot{\mu}_2 - \tau_g^2 \mu_3 + \ddot{\mu}_3 + \dot{\tau}_g \mu_2 + \tau_g \dot{\mu}_2). \end{cases} \tag{16}$$

Thus, from Eqs. (12) and (14), we obtain

$$\mathbf{N}_{h_\beta}(s) = \frac{\dot{\mathbf{T}}_\beta(s) - h_\beta(s)}{\|\dot{\mathbf{T}}_\beta(s) - h_\beta(s)\|},$$

Following the above, we can write

$$\begin{aligned} \mathbf{N}_{h_\beta}(s) =& \left(\left|-(\eta_1 - \mu_1)^2 + \eta_2^2 + (\eta_3 - \mu_2)^2 + (\eta_4 - \mu_3)^2\right|\right)^{\frac{-1}{2}} (((\eta_1 - \mu_1)\beta(s) \\ & + \eta_2 \mathbf{T}(s) + (\eta_3 - \mu_2)\mathbf{N}(s) + (\eta_4 - \mu_3)\mathbf{E}(s))). \end{aligned} \tag{17}$$

Also, from Eqs. (4) and (17), we obtain

$$\mathcal{K}_{h_\beta} = \sqrt{\left|-(\eta_1 - \mu_1)^2 + \eta_2^2 + (\eta_3 - \mu_2)^2 + (\eta_4 - \mu_3)^2\right|}.$$

Therefore, by differentiating (14) and using (3), we can obtain

$$\dddot{h}_\beta(s) = (\ddot{\mu}_1 + 3\dot{\mu}_1 - 2\dot{\kappa}_g\mu_2 - 3\kappa_g\dot{\mu}_2 + \kappa_g\tau_g\mu_3)\beta(s) + (3\ddot{\mu}_1 + \mu_1 - \kappa_g\mu_2 - \ddot{\kappa}_g\mu_2 - 3\dot{\kappa}_g\dot{\mu}_2 - 3\kappa_g\ddot{\mu}_2 + \dot{\kappa}_g\tau_g\mu_3 + 2\kappa_g\dot{\tau}_g\mu_3 + 3\kappa_g\tau_g\dot{\mu}_3)\mathbf{T}(s) + (3\kappa_g\dot{\mu}_1 - 3\kappa_g\dot{\kappa}_g\mu_2 - 3\kappa_g^2\dot{\mu}_2 + \dot{\kappa}_g\mu_1 + \kappa_g^2\tau_g\mu_3 - 3\tau_g\dot{\tau}_g\mu_2 - 2\tau_g^2\dot{\mu}_2 + \ddot{\mu}_2 - \ddot{\tau}_g\mu_3 - 3\dot{\tau}_g\dot{\mu}_3 - 3\tau_g\ddot{\mu}_3 - \tau_g^2\dot{\mu}_2 - \tau_g^2\mu_3)\mathbf{N}(s) + (\kappa_g\tau_g\mu_1 - \kappa_g^2\tau_g\mu_2 - \tau_g^3\mu_2 + 3\tau_g\ddot{\mu}_2 - 3\tau_g\dot{\tau}_g\mu_3 + \ddot{\mu}_3 - 3\tau_g^2\dot{\mu}_3 + 3\dot{\tau}_g\dot{\mu}_2 + \ddot{\tau}_g\mu_2)\mathbf{E}(s), \tag{18}$$

which can be written as:

$$\dddot{h}_\beta(s) = \zeta_1\beta(s) + \zeta_2\mathbf{T}(s) + \zeta_3\mathbf{N}(s) + \zeta_4\mathbf{E}(s), \tag{19}$$

Noting that

$$\begin{cases} \zeta_1 = (\ddot{\mu}_1 + 3\dot{\mu}_1 - 2\dot{\kappa}_g\mu_2 - 3\kappa_g\dot{\mu}_2 + \kappa_g\tau_g\mu_3) \\ \zeta_2 = (3\ddot{\mu}_1 + \mu_1 - \kappa_g\mu_2 - \ddot{\kappa}_g\mu_2 - 3\dot{\kappa}_g\dot{\mu}_2 - 3\kappa_g\ddot{\mu}_2 \\ \quad + \dot{\kappa}_g\tau_g\mu_3 + 2\kappa_g\dot{\tau}_g\mu_3 + 3\kappa_g\tau_g\dot{\mu}_3) \\ \zeta_3 = (3\kappa_g\dot{\mu}_1 - 3\kappa_g\dot{\kappa}_g\mu_2 - 3\kappa_g^2\dot{\mu}_2 + \dot{\kappa}_g\mu_1 + \kappa_g^2\tau_g\mu_3 \\ \quad - 3\tau_g\dot{\tau}_g\mu_2 - 2\tau_g^2\dot{\mu}_2 + \ddot{\mu}_2 - \ddot{\tau}_g\mu_3 - 3\dot{\tau}_g\dot{\mu}_3 - 3\tau_g\ddot{\mu}_3 \\ \quad - \tau_g^2\dot{\mu}_2 - \tau_g^2\mu_3) \\ \zeta_4 = (\kappa_g\tau_g\mu_1 - \kappa_g^2\tau_g\mu_2 - \tau_g^3\mu_2 + 3\tau_g\ddot{\mu}_2 - 3\tau_g\dot{\tau}_g\mu_3 \\ \quad + \ddot{\mu}_3 - 3\tau_g^2\dot{\mu}_3 + 3\dot{\tau}_g\dot{\mu}_2 + \ddot{\tau}_g\mu_2). \end{cases} \tag{20}$$

From Eqs. (4), (12), (13), (15), and (19), the torsion $\mathcal{T}_\beta$ is given by

$$\mathcal{T}_{h_\beta}(s) = -\frac{1}{\mathcal{K}_{h_\beta}^2}\begin{vmatrix} \mu_1 & 0 & \mu_2 & \mu_3 \\ \dot{\mu}_1 & (\mu_1 - \kappa_g\mu_2) & (\dot{\mu}_2 - \tau_g\mu_3) & (\dot{\mu}_3 + \tau_g\mu_2) \\ \eta_1 & \eta_2 & \eta_3 & \eta_4 \\ \zeta_1 & \zeta_2 & \zeta_3 & \zeta_4 \end{vmatrix},$$

or

$$\mathcal{T}_{h_\beta}(s) = \frac{1}{\mathcal{K}_{h_\beta}^2}(\mu_1((\mu_1 - \kappa_g\mu_2)(\eta_3\zeta_4 - \eta_4\zeta_3) - (\dot{\mu}_2 - \tau_g\mu_3)(\eta_2\zeta_4 - \eta_4\zeta_2) + (\dot{\mu}_3 - \tau_g\mu_2)(\eta_2\zeta_3 - \eta_3\zeta_2)) - \mu_2(\dot{\mu}_1(\eta_2\zeta_4 - \eta_4\zeta_2) - (\mu_1 - \kappa_g\mu_2)(\eta_1\zeta_4 - \eta_4\zeta_1) + (\dot{\mu}_3 - \tau_g\mu_2)(\eta_1\zeta_2 - \eta_2\zeta_1)) + \mu_3(\dot{\mu}_1(\eta_2\zeta_3 - \eta_3\zeta_2) - (\mu_1 - \kappa_g\mu_2)(\eta_1\zeta_3 - \eta_3\zeta_1) + (\dot{\mu}_2 - \tau_g\mu_3)(\eta_1\zeta_2 - \eta_2\zeta_1))). \tag{21}$$

In addition, from (12), (13), and (17), the vector $\mathbf{E}_{h_\beta}(s) = h_\beta(s) \wedge \mathbf{T}_{h_\beta}(s) \wedge \mathbf{N}_{h_\beta}(s)$ is computed as:

$$\mathbf{E}_{h_\beta}(s) = \frac{1}{\mathcal{K}_{h_\beta}}\begin{vmatrix} -\beta(s) & \mathbf{T}(s) & \mathbf{N}(s) & \mathbf{E}(s) \\ \mu_1 & 0 & \mu_2 & \mu_3 \\ \dot{\mu}_1 & (\mu_1 - \kappa_g\mu_2) & (\dot{\mu}_2 - \tau_g\mu_3) & (\dot{\mu}_3 + \tau_g\mu_2) \\ (\eta_1 - \mu_1) & \eta_2 & (\eta_3 - \mu_2) & (\eta_4 - \mu_3) \end{vmatrix},$$

which can be taken in the form

$$\mathbf{E}_{h_\beta}(s) = -\frac{1}{\mathcal{K}_{h_\beta}}\begin{vmatrix} 0 & \mu_2 & \mu_3 \\ (\mu_1 - \kappa_g\mu_2) & (\dot{\mu}_2 - \tau_g\mu_3) & (\dot{\mu}_3 + \tau_g\mu_2) \\ \eta_2 & (\eta_3 - \mu_2) & (\eta_4 - \mu_3) \end{vmatrix}\beta(s) - \frac{1}{\mathcal{K}_{h_\beta}}\begin{vmatrix} \mu_1 & \mu_2 & \mu_3 \\ \dot{\mu}_1 & (\dot{\mu}_2 - \tau_g\mu_3) & (\dot{\mu}_3 + \tau_g\mu_2) \\ (\eta_1 - \mu_1) & (\eta_3 - \mu_2) & (\eta_4 - \mu_3) \end{vmatrix}\mathbf{T}(s)$$

$$+\frac{1}{\mathcal{K}_{h_\beta}}\begin{vmatrix}\mu_1 & 0 & \mu_3\\ \dot{\mu}_1 & (\mu_1-\kappa_g\mu_2) & (\dot{\mu}_3+\tau_g\mu_2)\\ (\eta_1-\mu_1) & \eta_2 & (\eta_4-\mu_3)\end{vmatrix}\mathbf{N}(s)$$

$$-\frac{1}{\mathcal{K}_{h_\beta}}\begin{vmatrix}\mu_1 & 0 & \mu_2\\ \dot{\mu}_1 & (\mu_1-\kappa_g\mu_2) & (\dot{\mu}_2-\tau_g\mu_3)\\ (\eta_1-\mu_1) & \eta_2 & (\eta_3-\mu_2)\end{vmatrix}\mathbf{E}(s),$$

In the light of the previous computations, the proof is completed. □

The Frenet apparatus of an evolute curve in de Sitter 3-space

Similar to the case of the hyperbolic evolute curve, we introduce the definition *Frenet* apparatus of an evolute curve in the three-dimensional *de Sitter* space.

Definition 3 *Let γ be an involute curve in S^3_1, then the de Sitter evolute curve $d_\gamma : I \longrightarrow S^3_1$ of $\gamma(s)$ is expressed as*

$$d_\gamma(s) = \frac{1}{\sqrt{\left|\left(\frac{\dot{\kappa_g}}{\kappa_g\tau_g}\right)^2-\kappa_g^2+1\right|}}\left(\kappa_g\gamma(s)+\mathbf{N}(s)+\frac{\dot{\kappa_g}}{\kappa_g\tau_g}\mathbf{E}(s)\right)$$

under the assumption that $\kappa_g^2-\left(\frac{\dot{\kappa_g}}{\kappa_g\tau_g}\right)^2<1$.

We remark that $d_\gamma(s)$ is located in S^3_1 if and only if $\kappa_g^2-\left(\frac{\dot{\kappa_g}}{\kappa_g\tau_g}\right)^2<1$, where $\gamma(s)$ is an involute curve of d_γ (see [1]).

We refer to $\{d_\gamma(s),\ \mathbf{T}_{d_\gamma}(s),\ \mathbf{N}_{d_\gamma}(s),\ \mathbf{E}_{d_\gamma}(s),\ \mathcal{K}_\gamma(s),\ \mathcal{T}_{d_\gamma}(s)\}$ as *Frenet* apparatus of the evolute curve in S^3_1.

Theorem 3 *For a given spacelike evolute curve $d_\gamma(s)$ of $\gamma(s)$, the Frenet apparatus are introduced as:*

$$\mathbf{T}_{d_\gamma}(s)=\dot{\lambda}_1\gamma(s)+(\lambda_1-\delta\kappa_g\lambda_2)\mathbf{T}(s)+(\dot{\lambda}_2+\tau_g\lambda_3)\mathbf{N}(s)+(\dot{\lambda}_3+\tau_g\lambda_2)\mathbf{E}(s),$$

$$\begin{aligned}\mathbf{N}_{d_\gamma}(s)=&\left(\left|-(\xi_1+\lambda_1)^2+\xi_2^2+(\xi_3+\lambda_2)^2+(\xi_4+\lambda_3)^2\right|\right)^{\frac{-1}{2}}(((\xi_1+\lambda_1)\gamma(s)\\&+\xi_2\mathbf{T}(s)+(\xi_3+\lambda_2)\mathbf{N}(s)+(\xi_4+\lambda_3)\mathbf{E}(s))),\end{aligned}$$

$$\mathcal{K}_{d_\gamma}=\sqrt{\left|-(\xi_1+\lambda_1)^2+\xi_2^2+(\xi_3+\lambda_2)^2+(\xi_4+\lambda_3)^2\right|},$$

$$\begin{aligned}\mathbf{E}_{d_\gamma}(s)=&-\frac{1}{\mathcal{K}_{d_\gamma}}((-\lambda_2((\lambda_1-\kappa_g\lambda_2)(\xi_4+\lambda_3)-\xi_2(\dot{\lambda}_2+\tau_g\lambda_3))+\lambda_3((\lambda_1-\kappa_g\lambda_2)(\xi_3+\lambda_2)\\&-\xi_2(\dot{\lambda}_2+\tau_g\lambda_3)))\gamma(s)+(\lambda_1((\dot{\lambda}_2+\tau_g\lambda_3)(\xi_4+\lambda_3)-(\xi_3+\lambda_2)(\dot{\lambda}_3+\tau_g\lambda_2))\\&-\lambda_2(\dot{\lambda}_1(\xi_4+\lambda_3)-(\xi_1+\lambda_1)(\dot{\lambda}_3+\tau_g\lambda_2))+\lambda_3(\dot{\lambda}_1(\xi_3+\lambda_2)-(\xi_1+\lambda_1)(\dot{\lambda}_2+\tau_g\lambda_3)))\mathbf{T}(s)\\&-(\lambda_1((\lambda_1-\kappa_g\lambda_2)(\xi_4+\lambda_3)-\xi_2(\dot{\lambda}_3+\tau_g\lambda_2))+\lambda_3(\dot{\lambda}_1\xi_2-(\xi_1+\lambda_1)(\lambda_1-\kappa_g\lambda_2)))\mathbf{N}(s)\\&+(\lambda_1((\lambda_1-\kappa_g\lambda_2)(\xi_3+\lambda_2)-\xi_2(\dot{\lambda}_2+\tau_g\lambda_3))+\lambda_2(\dot{\lambda}_1\xi_2-(\xi_1+\lambda_1)(\lambda_1-\kappa_g\lambda_2)))\mathbf{E}(s)),\end{aligned}$$

$$\mathcal{T}_{d_\gamma}(s) = \frac{1}{\mathcal{K}^2_{d_\gamma}}(\lambda_1((\lambda_1-\kappa_g\lambda_2)(\xi_3B_4-\xi_4B_3)-(\dot{\lambda}_2+\tau_g\lambda_3)(\xi_2B_4-\xi_4B_2)+(\dot{\lambda}_3+\tau_g\lambda_2)(\xi_2B_3-\xi_3B_2))$$
$$-\lambda_2(\dot{\lambda}_1(\xi_2B_4-\xi_4B_2)-(\lambda_1-\kappa_g\lambda_2)(\xi_1B_4-\xi_4B_1)+(\dot{\lambda}_3+\tau_g\lambda_2)(\xi_1B_2-\xi_2B_1))$$
$$+\lambda_3(\dot{\lambda}_1(\xi_2B_3-\xi_3B_2)-(\lambda_1-\kappa_g\lambda_2)(\xi_1B_3-\xi_3B_1)+(\dot{\lambda}_2+\tau_g\lambda_3)(\xi_1B_2-\xi_2B_1))),$$

where the functions $\lambda_1, \lambda_2, \lambda_3, \xi_1, \xi_2, \xi_3, \xi_4, B_1, B_2, B_3,$ *and* B_4 *being smooth functions.*

Proof From the definition of an evolute curve in *de Sitter* 3-space, we have

$$d_\gamma(s) = \frac{\kappa_g}{\sqrt{\left|\left(\frac{\dot{\kappa_g}}{\kappa_g\tau_g}\right)^2-\kappa_g^2+1\right|}}\gamma + \frac{1}{\sqrt{\left|\left(\frac{\dot{\kappa_g}}{\kappa_g\tau_g}\right)^2-\kappa_g^2+1\right|}}\mathbf{N} - \frac{\dot{\kappa_g}}{\kappa_g\tau_g\sqrt{\left|\left(\frac{\dot{\kappa_g}}{\kappa_g\tau_g}\right)^2-\kappa_g^2+1\right|}}\mathbf{E}. \tag{22}$$

If we denote by

$$\lambda_1 = \frac{\kappa_g}{\sqrt{\left|\left(\frac{\dot{\kappa_g}}{\kappa_g\tau_g}\right)^2-\kappa_g^2+1\right|}}, \quad \lambda_2 = \frac{1}{\sqrt{\left|\left(\frac{\dot{\kappa_g}}{\kappa_g\tau_g}\right)^2-\kappa_g^2+1\right|}},$$
$$\lambda_3 = -\frac{\dot{\kappa_g}}{\kappa_g\tau_g\sqrt{\left|\left(\frac{\dot{\kappa_g}}{\kappa_g\tau_g}\right)^2-\kappa_g^2+1\right|}}, \tag{23}$$

then (22) can be written as

$$d_\gamma(s) = \lambda_1\gamma(s) + \lambda_2\mathbf{N}(s) + \lambda_3\mathbf{E}(s). \tag{24}$$

Differentiating both sides of (24), and using (5), we get

$$\begin{aligned}\mathbf{T}_{d_\gamma}(s) &= \dot{\lambda}_1\gamma(s) + \lambda_1\dot{\gamma}(s) + \dot{\lambda}_2\mathbf{N}(s) + \lambda_2\dot{\mathbf{N}}(s) + \dot{\lambda}_3\mathbf{E}(s) + \lambda_3\dot{\mathbf{E}}(s)\\ &= \dot{\lambda}_1\gamma(s) + \lambda_1\mathbf{T}(s) + \dot{\lambda}_2\mathbf{N}(s) + \lambda_2(-\delta\kappa_g\mathbf{T}(s) + \tau_g\mathbf{E}(s))\\ &\quad + \dot{\lambda}_3\mathbf{E}(s) + \lambda_3(\tau_g\mathbf{N}(s))\\ &= \dot{\lambda}_1\gamma(s) + (\lambda_1 - \delta\kappa_g\lambda_2)\mathbf{T}(s) + (\dot{\lambda}_2+\tau_g\lambda_3)\mathbf{N}(s)\\ &\quad + (\dot{\lambda}_3+\tau_g\lambda_2)\mathbf{E}(s),\end{aligned} \tag{25}$$

which gives by differentiating

$$\dot{\mathbf{T}}_{d_\gamma}(s) = \begin{cases}(\ddot{\lambda}_1 - \lambda_1 + \delta\kappa_g\lambda_2)\gamma(s) + (2\dot{\lambda}_1 - \delta\dot{\kappa_g}\lambda_2 - 2\delta\kappa_g\dot{\lambda}_2 - \delta\kappa_g\tau_g\lambda_3)\mathbf{T}(s)\\ +(\kappa_g\lambda_1 - \delta\kappa_g^2\lambda_2 + \ddot{\lambda}_2 + \dot{\tau}_g\lambda_3 + 2\tau_g\dot{\lambda}_3 + \tau_g^2\lambda_2)\mathbf{N}(s)\\ +(\tau_g\dot{\lambda}_2 + \tau_g^2\lambda_3 + \ddot{\lambda}_3 + \dot{\tau}_g\lambda_2 + \tau_g\dot{\lambda}_2)\mathbf{E}(s).\end{cases} \tag{26}$$

In another form, (26) is written

$$\dot{\mathbf{T}}_{d_\gamma}(s) = \xi_1\gamma(s) + \xi_2\mathbf{T}(s) + \xi_3\mathbf{N}(s) + \xi_4\mathbf{E}(s), \tag{27}$$

where

$$\begin{cases}\xi_1 = (\ddot{\lambda}_1 - \lambda_1 + \delta\kappa_g\lambda_2)\\ \xi_2 = (2\dot{\lambda}_1 - \delta\dot{\kappa_g}\lambda_2 - 2\delta\kappa_g\dot{\lambda}_2 - \delta\kappa_g\tau_g\lambda_3)\\ \xi_3 = (\kappa_g\lambda_1 - \delta\kappa_g^2\lambda_2 + \ddot{\lambda}_2 + \dot{\tau}_g\lambda_3 + 2\tau_g\dot{\lambda}_3 + \tau_g^2\lambda_2)\\ \xi_4 = (\tau_g\dot{\lambda}_2 + \tau_g^2\lambda_3 + \ddot{\lambda}_3 + \dot{\tau}_g\lambda_2 + \tau_g\dot{\lambda}_2).\end{cases}$$

Thus, from Eqs. (6), (24), and (27), we obtain

$$\mathbf{N}_{d_\gamma}(s) = \left(\left|-(\xi_1+\lambda_1)^2+\xi_2^2+(\xi_3+\lambda_2)^2+(\xi_4+\lambda_3)^2\right|\right)^{\frac{-1}{2}} (((\xi_1+\lambda_1)\gamma(s) + \xi_2\mathbf{T}(s) + (\xi_3+\lambda_2)\mathbf{N}(s) + (\xi_4+\lambda_3)\mathbf{E}(s))), \tag{28}$$

and

$$\mathcal{K}_{d_\gamma} = \sqrt{\left|-(\xi_1+\lambda_1)^2+\xi_2^2+(\xi_3+\lambda_2)^2+(\xi_4+\lambda_3)^2\right|}.$$

Moreover, by differentiating Eq. (26) with respect to s, one can obtain

$$\begin{aligned}\dddot{d}_\gamma(s) = {} & (\dddot{\lambda}_1 - 3\dot{\lambda}_1 + 2\delta\dot{\kappa}_g\lambda_2 + 3\delta\kappa_g\dot{\lambda}_2 + \delta\kappa_g\tau_g\lambda_3)\gamma(s) + (3\ddot{\lambda}_1 - \lambda_1 + \delta\kappa_g\lambda_2 \\ & - \delta\ddot{\kappa}_g\lambda_2 - 3\delta\dot{\kappa}_g\dot{\lambda}_2 - 3\delta\kappa_g\ddot{\lambda}_2 - \delta\dot{\kappa}_g\tau_g\lambda_3 - \delta\kappa_g\dot{\tau}_g\lambda_3 - 3\kappa_g\tau_g\dot{\lambda}_3 - \delta\kappa_g^2\lambda_1 \\ & + \delta\kappa_g^3\lambda_2 - \delta\kappa_g\tau_g^2\lambda_2)\mathbf{T}(s) + (3\kappa_g\dot{\lambda}_1 - 3\delta\kappa_g\dot{\kappa}_g\lambda_2 - 3\delta\kappa_g^2\dot{\lambda}_2 + \dot{\kappa}_g\lambda_1 \\ & - \delta\kappa_g^2\tau_g\lambda_3 + 3\tau_g\dot{\tau}_g\lambda_2 + 3\tau_g^2\dot{\lambda}_2 + \dddot{\lambda}_2 + \ddot{\tau}_g\lambda_3 + 3\dot{\tau}_g\dot{\lambda}_3 + \tau_g\ddot{\lambda}_3 + \tau_g^3\lambda_3 \\ & + 2\dot{\tau}_g\ddot{\lambda}_3)\mathbf{N}(s) + (\kappa_g\tau_g\lambda_1 - \delta\kappa_g^2\tau_g\lambda_2 + \tau_g^3\lambda_2 + 3\tau_g\ddot{\lambda}_2 + 3\tau_g\dot{\tau}_g\lambda_3 + \dddot{\lambda}_3 \\ & + 3\tau_g^2\dot{\lambda}_3 + 3\dot{\tau}_g\dot{\lambda}_2 + \ddot{\tau}_g\lambda_2)\mathbf{E}(s),\end{aligned} \tag{29}$$

which in abbreviated form

$$\dddot{d}_\gamma(s) = B_1\gamma(s) + B_2\mathbf{T}(s) + B_3\mathbf{N}(s) + B_4\mathbf{E}(s), \tag{30}$$

where

$$\begin{cases} B_1 & = (\dddot{\lambda}_1 - 3\dot{\lambda}_1 + 2\delta\dot{\kappa}_g\lambda_2 + 3\delta\kappa_g\dot{\lambda}_2 + \delta\kappa_g\tau_g\lambda_3) \\ B_2 & = (3\ddot{\lambda}_1 - \lambda_1 + \delta\kappa_g\lambda_2 - \delta\ddot{\kappa}_g\lambda_2 - 3\delta\dot{\kappa}_g\dot{\lambda}_2 - 3\delta\kappa_g\ddot{\lambda}_2 - \delta\dot{\kappa}_g\tau_g\lambda_3 \\ & \quad -\delta\kappa_g\dot{\tau}_g\lambda_3 - 3\kappa_g\tau_g\dot{\lambda}_3 - \delta\kappa_g^2\lambda_1 + \delta\kappa_g^3\lambda_2 - \delta\kappa_g\tau_g^2\lambda_2) \\ B_3 & = (3\kappa_g\dot{\lambda}_1 - 3\delta\kappa_g\dot{\kappa}_g\lambda_2 - 3\delta\kappa_g^2\dot{\lambda}_2 + \dot{\kappa}_g\lambda_1 - \delta\kappa_g^2\tau_g\lambda_3 + 3\tau_g\dot{\tau}_g\lambda_2 \\ & \quad +3\tau_g^2\dot{\lambda}_2 + \dddot{\lambda}_2 + \ddot{\tau}_g\lambda_3 + 3\dot{\tau}_g\dot{\lambda}_3 + \tau_g\ddot{\lambda}_3 + \tau_g^3\lambda_3 + 2\dot{\tau}_g\ddot{\lambda}_3) \\ B_4 & = (\kappa_g\tau_g\lambda_1 - \delta\kappa_g^2\tau_g\lambda_2 + \tau_g^3\lambda_2 + 3\tau_g\ddot{\lambda}_2 + 3\tau_g\dot{\tau}_g\lambda_3 + \dddot{\lambda}_3 + 3\tau_g^2\dot{\lambda}_3 \\ & \quad +3\dot{\tau}_g\dot{\lambda}_2 + \ddot{\tau}_g\lambda_2). \end{cases} \tag{31}$$

Now, after using (24), (25), (27), and (30), the torsion $\mathcal{T}_{d_\gamma}$ of the evolute curve $d_\gamma(s)$ is obtained

$$\begin{aligned}\mathcal{T}_{d_\gamma}(s) = {} & \frac{1}{\mathcal{K}^2_{d_\gamma}}(\lambda_1((\lambda_1 - \kappa_g\lambda_2)(\xi_3B_4 - \xi_4B_3) - (\dot{\lambda}_2 + \tau_g\lambda_3)(\xi_2B_4 - \xi_4B_2) + (\dot{\lambda}_3 + \tau_g\lambda_2)(\xi_2B_3 - \xi_3B_2)) \\ & - \lambda_2(\dot{\lambda}_1(\xi_2B_4 - \xi_4B_2) - (\lambda_1 - \kappa_g\lambda_2)(\xi_1B_4 - \xi_4B_1) + (\dot{\lambda}_3 + \tau_g\lambda_2)(\xi_1B_2 - \xi_2B_1)) \\ & + \lambda_3(\dot{\lambda}_1(\xi_2B_3 - \xi_3B_2) - (\lambda_1 - \kappa_g\lambda_2)(\xi_1B_3 - \xi_3B_1) + (\dot{\lambda}_2 + \tau_g\lambda_3)(\xi_1B_2 - \xi_2B_1))).\end{aligned} \tag{32}$$

The only vector remaining from *Frenet* apparatus is the binormal vector $\mathbf{E}_\gamma(s)$, which is given directly by

$$\mathbf{E}_{d_\gamma}(s) = -\frac{1}{\mathcal{K}_{d_\gamma}}\begin{vmatrix} 0 & \lambda_2 & \lambda_3 \\ (\lambda_1 - \kappa_g\lambda_2) & (\dot{\lambda}_2 + \tau_g\lambda_3) & (\dot{\lambda}_3 + \tau_g\lambda_2) \\ \xi_2 & (\xi_3 + \lambda_2) & (\xi_4 + \lambda_3) \end{vmatrix}\gamma(s)$$

$$-\frac{1}{\mathcal{K}_{d_\gamma}}\begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \dot{\lambda}_1 & (\dot{\lambda}_2 + \tau_g\lambda_3) & (\dot{\lambda}_3 + \tau_g\lambda_2) \\ (\xi_1 + \lambda_1) & (\xi_3 + \lambda_2) & (\xi_4 + \lambda_3) \end{vmatrix}\mathbf{T}(s)$$

$$+\frac{1}{\mathcal{K}_{d_\gamma}}\begin{vmatrix}\lambda_1 & 0 & \lambda_3\\ \dot{\lambda}_1 & (\lambda_1-\kappa_g\lambda_2) & (\dot{\lambda}_3+\tau_g\lambda_2)\\ (\xi_1+\lambda_1) & \xi_2 & (\xi_4+\lambda_3)\end{vmatrix}\mathbf{N}(s)$$

$$-\frac{1}{\mathcal{K}_{d_\gamma}}\begin{vmatrix}\lambda_1 & 0 & \lambda_2\\ \dot{\lambda}_1 & (\lambda_1-\kappa_g\lambda_2) & (\dot{\lambda}_2+\tau_g\lambda_3)\\ (\xi_1+\lambda_1) & \xi_2 & (\xi_3+\lambda_2)\end{vmatrix}\mathbf{E}(s),$$

Thus, the theorem has been proven. □

Examples

In this section, to support the theoretical results of this paper, we construct two examples of an evolute curve in the two- and three-dimensional hyperbolic spaces, then we calculate its *Frenet* apparatus using *Frenet* apparatus of its involute curve.

Example 1 *Consider the general helix curve* α *in* $H^2_+(-1)$*, where*

$$\alpha(s)=\left(\frac{s}{\sqrt{2}},\sin\left(\sqrt{\frac{3}{2}}s\right),\cos\left(\sqrt{\frac{3}{2}}s\right)\right). \tag{33}$$

Firstly, we compute the Frenet frame, the curvature and the torsion of the curve α*. For this, from Eq. (33), the tangent vector of the curve* α *is given by*

$$\mathbf{T}(s)=\dot{\alpha}(s)=\left(\frac{1}{\sqrt{2}},\sqrt{\frac{3}{2}}\cos\left(\sqrt{\frac{3}{2}}s\right),-\sqrt{\frac{3}{2}}\sin\left(\sqrt{\frac{3}{2}}s\right)\right). \tag{34}$$

And from (33) and (34), we get

$$\alpha(s)\wedge\mathbf{T}(s)=\begin{vmatrix}-i & j & k\\ \frac{s}{\sqrt{2}} & \sin\left(\sqrt{\frac{3}{2}}s\right) & \cos\left(\sqrt{\frac{3}{2}}s\right)\\ \frac{1}{\sqrt{2}} & \sqrt{\frac{3}{2}}\cos\left(\sqrt{\frac{3}{2}}s\right) & -\sqrt{\frac{3}{2}}\sin\left(\sqrt{\frac{3}{2}}s\right)\end{vmatrix}, \tag{35}$$

which enable us to obtain

$$\begin{aligned}\mathbf{E}(s)=&\left(\sqrt{\frac{3}{2}}\right)i+\left(\frac{\sqrt{3}}{2}s\sin\left(\sqrt{\frac{3}{2}}s\right)+\frac{1}{\sqrt{2}}\cos\left(\sqrt{\frac{3}{2}}s\right)\right)j\\ &+\left(\frac{\sqrt{3}}{2}s\cos\left(\sqrt{\frac{3}{2}}s\right)-\frac{1}{\sqrt{2}}\sin\left(\sqrt{\frac{3}{2}}s\right)\right)k;\end{aligned} \tag{36}$$

the curvature of the curve $\alpha(s)$ *is given as follows:*

$$\begin{aligned}\kappa_g(s)=&-\frac{s}{\sqrt{2}}\left(\frac{3}{2}\sqrt{\frac{3}{2}}\cos^2\left(\sqrt{\frac{3}{2}}s\right)+\frac{3}{2}\sqrt{\frac{3}{2}}\sin^2\left(\sqrt{\frac{3}{2}}s\right)\right)\\ &+\frac{3}{2\sqrt{2}}\left(\sin\left(\sqrt{\frac{3}{2}}s\right)\cos\left(\sqrt{\frac{3}{2}}s\right)-\sin\left(\sqrt{\frac{3}{2}}s\right)\cos\left(\sqrt{\frac{3}{2}}s\right)\right),\end{aligned}$$

which takes the simple form

$$\kappa_g(s)=-\frac{\sqrt{27}}{4}s.$$

Thus, from Theorem (1), the Frenet apparatus of the evolute curve h_α of $\alpha(s)$ (see Figs. 1 and 2) are expressed respectively, by

$$h_\alpha(s) = \frac{1}{\sqrt{|27s^2-16|}} \left\{ \begin{array}{l} \left(\sqrt{\frac{3}{2}}(4-3s^2), \frac{4}{\sqrt{2}}\cos\left(\sqrt{\frac{3}{2}}s\right) - \sqrt{3}s\sin\left(\sqrt{\frac{3}{2}}s\right)\right. \\ \left., -\frac{4}{\sqrt{2}}\sin\left(\sqrt{\frac{3}{2}}s\right) - \sqrt{3}s\cos\left(\sqrt{\frac{3}{2}}s\right)\right), \end{array}\right.$$

$$\mathbf{T}_{h_\alpha}(s) = \frac{6\sqrt{3}}{|27s^2-16|^{\frac{3}{2}}} \left\{ \begin{array}{l} \left(-5\sqrt{2}s, (9s^2-2)\sin\left(\sqrt{\frac{3}{2}}s\right) - 3\sqrt{6}s\cos\left(\sqrt{\frac{3}{2}}s\right)\right. \\ \left., (9s^2-2)\cos\left(\sqrt{\frac{3}{2}}s\right) + 3\sqrt{6}s\sin\left(\sqrt{\frac{3}{2}}s\right)\right), \end{array}\right.$$

$$\mathbf{E}_{h_\alpha}(s) = \frac{-12\sqrt{3}}{|27s^2-16|}\left(\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}\cos\left(\sqrt{\frac{3}{2}}s\right), -\sqrt{\frac{3}{2}}\sin\left(\sqrt{\frac{3}{2}}s\right)\right),$$

and

$$\mathcal{K}_{h_\alpha}(s) = \frac{27}{|27s^2-16|^{\frac{3}{2}}}.$$

Example 2 *Let β be a general helix in $H_+^3(-1)$, where*

$$\beta(s) = \left(\sqrt{2}\cosh(s), \sqrt{2}\sinh(s), \sin(s), \cos(s)\right). \tag{37}$$

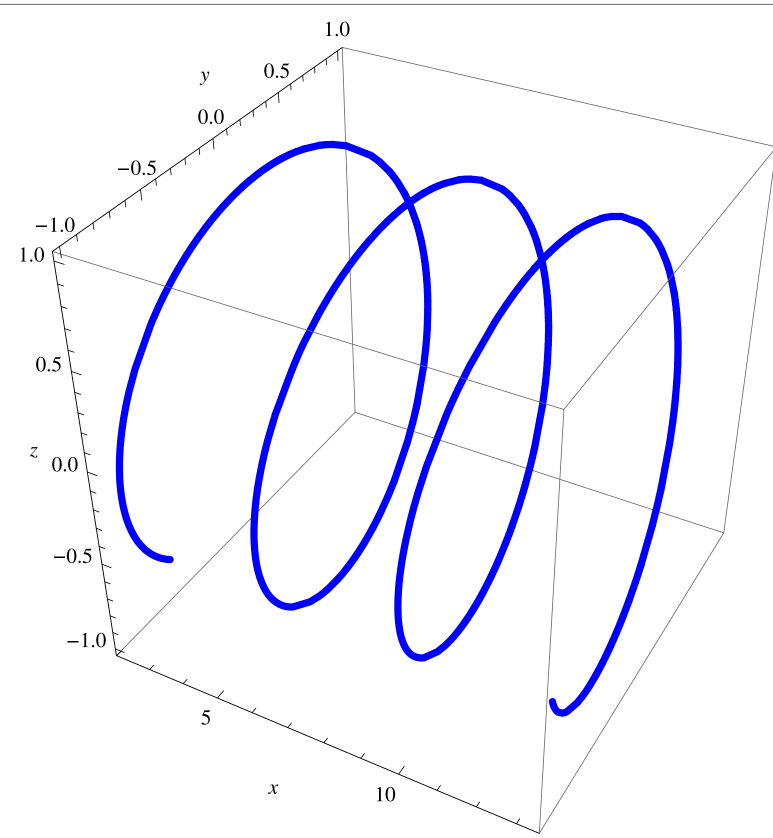

Fig. 1 Evolute curve $h_\alpha(s)$

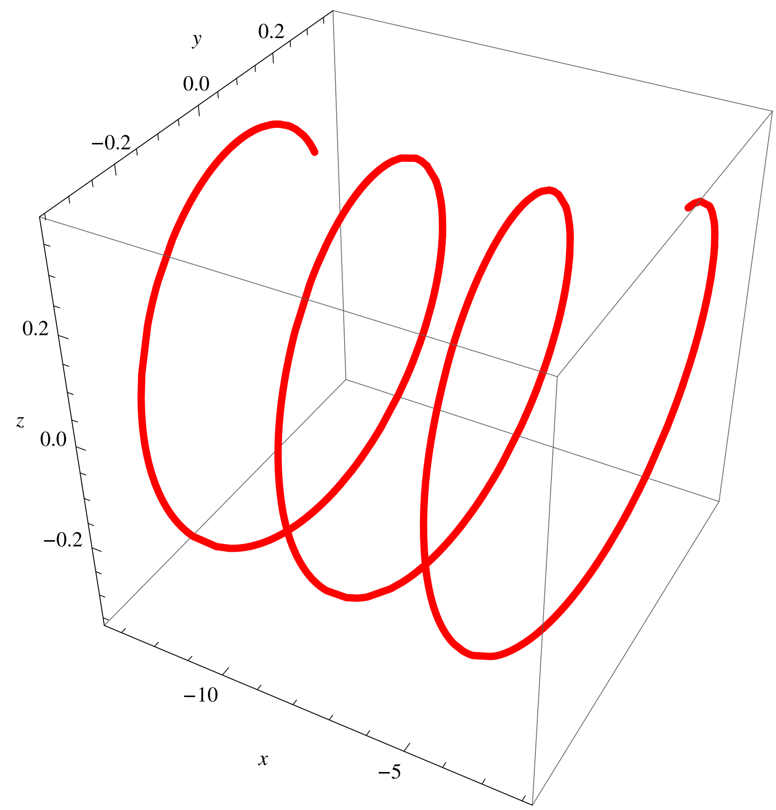

Fig. 2 Involute curve $\alpha(s)$

From (37), the tangent vector of the curve β is given by

$$\mathbf{T}(s) = \left(\sqrt{2}\sinh(s), \sqrt{2}\cosh(s), \cos(s), -\sin(s)\right), \tag{38}$$

which gives by differentiating

$$\dot{\mathbf{T}}(s) = \left(\sqrt{2}\cosh(s), \sqrt{2}\sinh(s), -\sin(s), -\cos(s)\right). \tag{39}$$

From Eqs. (37) and (39), we get

$$\mathbf{N}(s) = \frac{\dot{\mathbf{T}}(s) - \beta(s)}{\|\dot{\mathbf{T}}(s) - \beta(s)\|} = (0, 0, -\sin(s), -\cos(s)), \tag{40}$$

and

$$\kappa_g(s) = \|\dot{\mathbf{T}}(s) - \beta(s)\| = 2.$$

Also, we get

$$\mathbf{E}(s) = \left(\sqrt{2}\sinh(s), \sqrt{2}\cosh(s), -2\cos(s), 2\sin(s)\right).$$

Using the differentiation of Eq. (37) three times with respect to s to obtain

$$\begin{aligned}\det(\beta;\dot{\beta};\ddot{\beta};\dddot{\beta}) =& \sqrt{2}\cosh(s)[-\sqrt{2}\cosh(s)-\cos(s)(\sqrt{2}\sin(s)\sinh(s)\\ &+\sqrt{2}\cos(s)\cosh(s))-\sin(s)(-\sqrt{2}\cos(s)\sinh(s)\\ &+\sqrt{2}\sin(s)\cosh(s))]-\sqrt{2}\sinh(s)[-\sqrt{2}\sinh(s)\\ &-\cos(s)(\sqrt{2}\sin(s)\cosh(s)+\sqrt{2}\cos(s)\sinh(s))\\ &-\sin(s)(-\sqrt{2}\cos(s)\cosh(s)+\sqrt{2}\sin(s)\sinh(s)]\\ &+\sin(s)[\sqrt{2}\sinh(s)(\sqrt{2}\sin(s)\sinh(s)+\sqrt{2}\cos(s)\cosh(s))\\ &-\sqrt{2}\cosh(s)(\sqrt{2}\sin(s)\cosh(s)+\sqrt{2}\cos(s)\sinh(s))\\ &-2\sin(s))]-\cos(s)[\sqrt{2}\sinh(s)(-\sqrt{2}\cos(s)\sinh(s)\\ &+\sqrt{2}\sin(s)\cosh(s))-\sqrt{2}\cosh(s)(-\sqrt{2}\cos(s)\cosh(s)\\ &+\sqrt{2}\sin(s)\sinh(s))+2\cos(s))],\end{aligned}$$

then, we get

$$\tau_g(s) = -\frac{\det(\beta,\dot{\beta},\ddot{\beta},\dddot{\beta})}{\kappa_g^2} = 2.$$

From Theorem (2) and Eqs. (11), (16) and (20), the Frenet apparatus of the evolute curve h_β of $\beta(s)$ are respectively, expressed by

$$h_\beta(s) = \frac{1}{\sqrt{3}}\left(2\sqrt{2}\cosh(s), 2\sqrt{2}\sinh(s), \sin(s), \cos(s)\right),$$

$$\begin{cases}\mu_1 = \frac{2}{\sqrt{3}}, \mu_2 = \frac{1}{\sqrt{3}}, \mu_3 = 0,\\ \eta_1 = \eta_2 = \eta_4 = 0, \quad \eta_3 = \frac{-4}{3},\\ \zeta_1 = \zeta_2 = \zeta_3 = 0, \quad \zeta_4 = \frac{-8}{\sqrt{3}},\end{cases}$$

$$\begin{aligned}\mathbf{T}_{h_\beta}(s) &= \frac{2}{\sqrt{3}}\left(\sqrt{2}\sinh(s), \sqrt{2}\cosh(s), -2\cos(s), 2\sin(s)\right),\\ \mathbf{N}_{h_\beta}(s) &= \frac{-1}{\sqrt{5}}\left(-2\sinh(s), -2\cosh(s), (-2\sqrt{2}+5)\sin(s), (-2\sqrt{2}+5)\cos(s)\right),\\ \mathbf{E}_{h_\beta}(s) &= \frac{-16}{3\sqrt{5}}\left(\sqrt{2}\sinh(s), \sqrt{2}\cosh(s), \cos(s), -\sin(s)\right),\end{aligned}$$

and

$$\mathcal{K}_{h_\beta}(s) = \sqrt{\frac{5}{3}}, \quad \mathcal{T}_{h_\beta}(s) = 0.$$

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