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ORIGINAL ARTICLE

A new generalization of the Pareto–geometric distribution

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Abstract In this paper we introduce a new distribution called the beta Pareto–geometric. We provide a comprehensive treatment of the mathematical properties of the proposed distribution and derive expressions for its moment generating function and the rth generalized moment. We discuss estimation of the parameters by maximum likelihood and obtain the information matrix that is easily numerically determined. We also demonstrate its usefulness on a real data set.

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1. Introduction

The family of the Pareto distribution is well known in the literature for its capability in modeling the heavy-tailed distributions that are mostly common in data on income distribution [1], city population size [2,3], and size of firms [4]. Newman [5] also provided many other quantities measured in the physical, biological, technological and social systems of various kinds, where the Pareto law has been found to be an appropriate fit. Different methods may be used to introduce a shape parameter to an exponential model and they may result in a variety of weighted exponential (WE) distributions. For example, the gamma distribution and the generalized exponential distribution are different weighted versions of the exponential distribution.

Starting from a parent cumulative distribution function (Cdf) $F(x)$, Eugene et al. [6] defined a class of generalized distributions by

$$G(x) = \frac{1}{B(a,b)} \int_0^{F(x)} w^{a-1}(1-w)^{b-1}\,dw \tag{1}$$

with extra parameters $a, b > 0$ and

$$B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

is the beta function. This class of generalized distributions has been receiving considerable attention over the last years, in particular after the works of Eugene et al. [6] and Jones [7].

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The probability density function (pdf) corresponding to (1) can be written as

$$g(x) = \frac{1}{B(a,b)}[F(x)]^{a-1}[1-F(x)]^{b-1}f(x),$$

where $f(x) = \frac{dF(x)}{dx}$ is the parent density function.

Recently, attempts – by Nadarajah and Gupta [8], Nadarajah and Kotz [9,10], Kong et al. [11], Akinsete et al. [12], Pescim et al. [13], Souza et al. [14], Nassar and Nada [15] and others – have been made to extend families of probability distributions following the work proposed by Eugene et al. [6] and Jones [7]. Using the same idea of Eugene et al. [6], we generalize the Pareto–Geometric (PG) distribution – introduced by De Morais [16] – by considering the following probability density function (pdf).

$$g(x) = \frac{\beta(1-\theta)^b}{\alpha B(a,b)}\left[1-\left(\frac{\alpha}{x}\right)^{\beta}\right]^{(a-1)}\frac{\left(\frac{\alpha}{x}\right)^{\beta b+1}}{\left[1-\theta\left(\frac{\alpha}{x}\right)^{\beta}\right]^{a+b}},$$

$$x > \alpha,\quad \theta \in (0,1),\quad a,b,\beta > 0. \tag{2}$$

As distributions in this class have interesting applications to lifetime data due to the variety of the shapes of the hazard function-the Pareto distribution being a limiting special case of the PG distribution – in this paper, based on the importance of the Beta family in many areas in statistics, we introduce the five – parameter Beta Pareto Geometric (BPG) class of distributions which is obtained by compounding Pareto and geometric distributions and generalizing this compounded distribution using the logit of the beta random variable. A random variable X with the pdf given by (2) is said to follow the Beta Pareto–Geometric (BPG) distribution. This five-parameter beta-Pareto–geometric distribution is mostly common in data on income distribution, city population size and many other topics in physics, biology, hydrology and engineering, such as earthquakes, forest fire areas, fault lengths on Earth and Venus, oil and gas field sizes property.

In Section 2, we give the cumulative distribution function (Cdf) of the BPG distribution. Approximate forms for the median, mode and the failure rate function are derived in Section 3. Expressions for the moment generating function, and hence the mean and variance, are discussed in Section 4. The mean deviation about the mean and the median are calculated in Section 5. Moreover, we discuss, in Section 6, estimation by the maximum likelihood method. Section 7 is devoted to deriving Shannon's entropy. Explicit expressions for order statistics of the BPG distribution are obtained in Section 8. Finally, in Section 9, an application using a real data set is presented, followed by the concluding remarks.

2. Distribution function

The PG distribution is a special case of the BPG pdf given in (2) when $a = 1$, $b = 1$. Moreover, the GPG distribution is obtained from (2) when $b = 1$. Plots of the BPG pdf are given in Fig. 1.

For $|z| < 1$ and $b > 0$ real non-integer, we have

$$(1-z)^b = 1 - bz + \frac{b(b-1)}{2!}z^2 + \cdots + \frac{b(b-1)\cdots(b-k+1)}{k!}(-1)^k z^k + \cdots \tag{3}$$

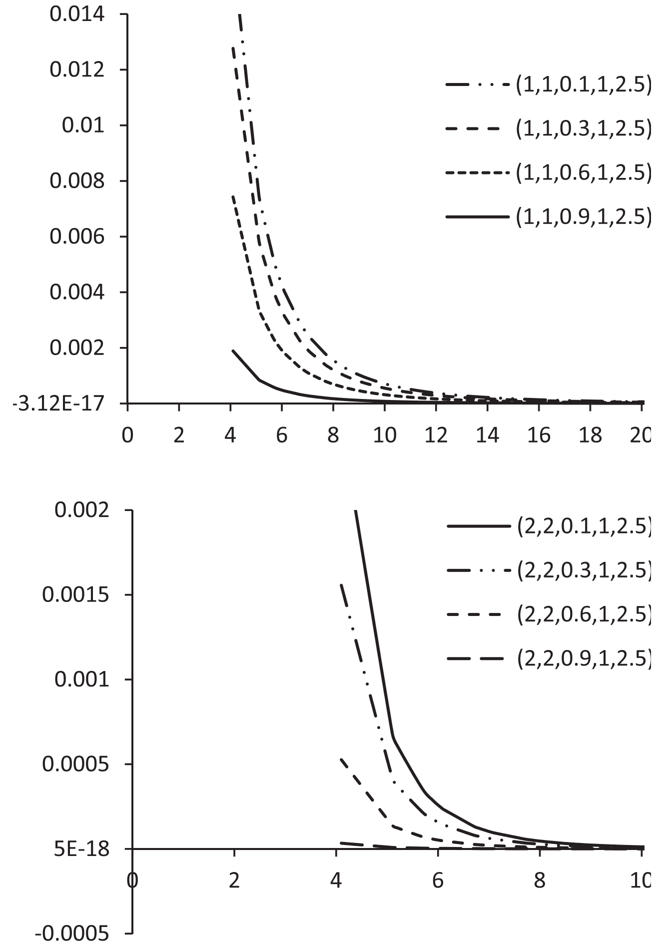


Figure 1 Plots of the BPG pdf for different values of parameters (a, b, θ, α, β).

If $b > 0$ is an integer, the sum in (2) stops at $b - 1$. Then the cumulative distribution function (Cdf) of the BPG distribution is given by

$$G(x) = \frac{1}{B(a,b)}\left\{\frac{1}{a} - \frac{b}{a+1}\left(\frac{1-\left(\frac{\alpha}{x}\right)^{\beta}}{1-\theta\left(\frac{\alpha}{x}\right)^{\beta}}\right)^{a} + \cdots + \frac{b(b-1)\cdots(b-k+1)}{k!(a+k)}(-1)^{a+k}\left(\frac{1-\left(\frac{\alpha}{x}\right)^{\beta}}{1-\theta\left(\frac{\alpha}{x}\right)^{\beta}}\right)^{a+k} + \cdots\right\},\qquad x > \alpha. \tag{4}$$

3. Median, mode and hazard rate function

Theorem 1. *The median of the BPG distribution is at the following approximate point*

$$m \approx \alpha\left[\frac{1-\theta\left(\frac{a}{2}B(a,b)\right)^{1/a}}{1-\left(\frac{a}{2}B(a,b)\right)^{1/a}}\right]^{1/\beta}. \tag{5}$$

Proof. Deriving the median m from the well-known fact $G(m) = \frac{1}{2}$ and the Cdf (4), we obtain

$$\frac{1}{B(a,b)}\left\{\frac{1}{a}-\frac{b}{a+1}\left(\frac{1-\left(\frac{\alpha}{m}\right)^{\beta}}{1-\theta\left(\frac{\alpha}{m}\right)^{\beta}}\right)^{a}+\cdots\right.$$
$$\left.+\frac{b(b-1)\ldots(b-k+1)}{k!(a+k)}(-1)^{a+k}\left(\frac{1-\left(\frac{\alpha}{m}\right)^{\beta}}{1-\theta\left(\frac{\alpha}{m}\right)^{\beta}}\right)^{a+k}+\cdots\right\}=\frac{1}{2}$$

Since the sum on the LHS is convergent for $\left(\frac{1-\left(\frac{\alpha}{m}\right)^{\beta}}{1-\theta\left(\frac{\alpha}{m}\right)^{\beta}}\right)^{\alpha}<1$, taking as a first approximation

$$\left(\frac{1-\left(\frac{\alpha}{m}\right)^{\beta}}{1-\theta\left(\frac{\alpha}{m}\right)^{\beta}}\right)^{a}\approx\frac{a}{2}B(a,b).$$

Thus an approximate value for the median will be the value given by (5). □

For the BPG distribution, we now investigate the unimodality property by the following theorem.

Theorem 2. *The BPG distribution is unimodal at the following approximate point*

$$x\approx\alpha\left(\frac{(a-1)\beta+\theta(1-a\beta)}{b\beta+1}+1\right)^{1/\beta}. \tag{6}$$

Proof. The derivative of Eq. (2) is calculated and setting

$$\eta(x)=-\frac{g'(x)}{g(x)}$$
$$=\frac{\beta}{\alpha}\left(\frac{(a+b)\theta\left(\frac{\alpha}{x}\right)^{\beta}}{1-\theta\left(\frac{\alpha}{x}\right)^{\beta}}-\frac{(a-1)\left(\frac{\alpha}{x}\right)^{\beta}}{1-\left(\frac{\alpha}{x}\right)^{\beta}}+\left(b+\frac{1}{\beta}\right)\left(\frac{\alpha}{x}\right)\right) \tag{7}$$

This leads to

$$\theta\left(1-\frac{1}{\beta}\right)\left(\frac{\alpha}{x}\right)^{2\beta}+\left(a+b-1+\frac{1}{\beta}+\theta\left(\frac{1}{\beta}-a\right)\right)\left(\frac{\alpha}{x}\right)^{\beta}-\left(b+\frac{1}{\beta}\right)=0.$$

Solving the second order equation for $\left(\frac{\alpha}{x}\right)^{\beta}$, we can then deduce the approximate mode in (6). □

The hazard (failure rate) function – an important quantity characterizing life phenomena – can, thus, be obtained from $h(x)=\frac{g(x)}{1-G(x)}$. Plots of the hazard rate function are given in Fig. 2 for different values of the parameters.

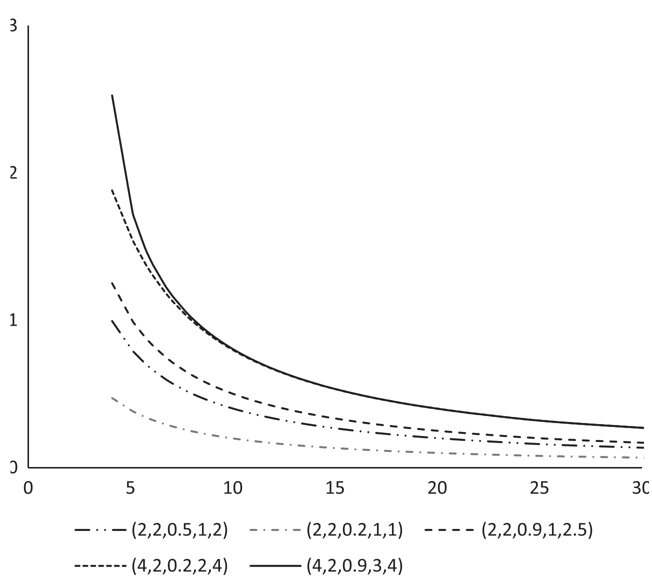


Figure 2 Plots of hazard rate function of BPG for different values of parameters (a, b, θ, α, β).

The derivative of Eq. (7) is

$$\eta'(x)=\frac{\beta}{\alpha}\left(\frac{(a-1)\frac{\beta}{\alpha}\left(\frac{\alpha}{x}\right)^{\beta+1}}{\left(1-\left(\frac{\alpha}{x}\right)^{\beta}\right)^{2}}-\frac{(a+b)\theta\frac{\beta}{\alpha}\left(\frac{\alpha}{x}\right)^{\beta+1}}{\left(1-\theta\left(\frac{\alpha}{x}\right)^{\beta}\right)^{2}}-\frac{1}{\alpha}\left(b+\frac{1}{\beta}\right)\left(\frac{\alpha}{x}\right)^{2}\right)$$

For $0<a<1$, we obtain $\eta'(x)<0,\forall x$. Hence, using Glaser's theorem [17], the hazard rate function $h(x)$ is decreasing. However, for $a>1$, for both cases $\theta<\frac{a-1}{a+b}$ and $\theta>\frac{a-1}{a+b}$, the hazard rate function $h(x)$ is decreasing.

4. Moment generating function

The rth moment of a random variable X following the BPG distribution is given by:

$$E(X^r)=\frac{\alpha^r(1-\theta)^b}{B(a,b)}B\left(b-\frac{r}{\beta},a\right){}_2F_1\left(a+b,b-\frac{r}{\beta},a+b-\frac{r}{\beta},\theta\right). \tag{8}$$

For the special case $a=b=1$, the rth moment derived in (8) reduces to the case given by the following

$$E(X^r)=\beta\alpha^r(1-\theta)\sum_{n=1}^{\infty}\frac{n\theta^{n-1}}{\beta n-r}.$$

Simply, the mean of a random variable X following the BPG distribution can be written as:

$$E(X)=\frac{\alpha(1-\theta)^b}{B(a,b)}B\left(b-\frac{1}{\beta},a\right){}_2F_1\left(a+b,b-\frac{1}{\beta},a+b-\frac{1}{\beta},\theta\right)$$

The moment generating function is then given as follows:

$$M_X(t)=\sum_{r=0}^{\infty}\frac{t^r}{r!}\frac{\alpha^r(1-\theta)^b}{B(a,b)}B\left(b-\frac{r}{\beta},a\right){}_2F_1\left(a+b,b-\frac{r}{\beta},a+b-\frac{r}{\beta},\theta\right). \tag{9}$$

where the hypergeometric function is defined as

$${}_2F_1(\alpha,\beta,\gamma,z)=\sum_{r=0}^{\infty}\frac{(\alpha)_r(\beta)_r}{(\gamma)_r}\frac{z^r}{r!},$$
$$(\alpha)_r=\alpha(\alpha+1)\ldots(\alpha+r-1),(\alpha)_0=1.$$

5. Mean deviation of the BPG

In case of symmetric distributions (the deviation from the mean) or in case of skewed distributions (deviation from the median), the deviations from the mean or the median can be used as a measure of spread in a population. Let X be a beta Pareto geometric random variable with mean $\mu=\mathrm{E(X)}$ and median M.

The mean deviation from the mean can be defined as

$$\mathrm{D}(\mu)=\mathrm{E}(|\mathrm{X}-\mu|)$$
$$=\int_{\alpha}^{\infty}|x-\mu|g(x)dx=2\mu G(\mu)-2\int_{\alpha}^{\mu}xg(x)\,dx$$
$$=2\int_{\alpha}^{\mu}G(x)\,dx=\frac{2\alpha}{B(a,b)}\sum_{k=0}^{\infty}\binom{b}{k}\frac{(-1)^k}{a+k}$$
$$\times\sum_{j,l=0}^{\infty}\binom{a+k}{j}\binom{a+k}{l}\theta^{l}\left(\frac{1-\left(\frac{\alpha}{\mu}\right)^{\beta(j+l)-1}}{\beta(j+l)-1}\right) \tag{10}$$

The mean deviation from the median is, also, defined by:

$$\mathsf{D}(m)=\mathsf{E}(|\mathsf{X}-m|)=\int_{\alpha}^{\infty}|x-m|g(x)\,dx=\mu-m+2\int_{\alpha}^{m}G(x)\,dx=\mu-m+\frac{2\alpha}{B(a,b)}\times\sum_{k=0}^{\infty}\binom{b}{k}\frac{(-1)^k}{a+k}\sum_{j,l=0}^{\infty}\binom{a+k}{j}\binom{a+k}{l}\theta^l\left(\frac{1-\left(\frac{\alpha}{m}\right)^{\beta(j+l)-1}}{\beta(j+l)-1}\right) \quad (11)$$

6. Maximum likelihood estimation

We consider estimation by the method of maximum likelihood. The log-likelihood for a random sample $\mathsf{X}_1,\mathsf{X}_2,\ldots,\mathsf{X}_n$ is

$$\ln L=n\ln\beta-n\ln\alpha-n\ln B(a,b)+nb\ln(1-\theta)+(\beta b+1)\sum_{i=1}^{n}\ln\left(\frac{\alpha}{x_i}\right)+(a-1)\sum_{i=1}^{n}\ln\left(1-\left(\frac{\alpha}{x_i}\right)^{\beta}\right)-(a+b)\sum_{i=1}^{n}\ln\left(1-\theta\left(\frac{\alpha}{x_i}\right)^{\beta}\right) \quad (12)$$

Differentiating (12) with respect to the parameters $(a, b, \alpha, \beta, \theta)$, we obtain

1. $$\frac{\partial\ln L(x)}{\partial\alpha}=\beta b\frac{n}{\alpha}-(a-1)\frac{\beta}{\alpha}\sum_{i=1}^{n}\frac{\left(\frac{\alpha}{x_i}\right)^{\beta}}{1-\left(\frac{\alpha}{x_i}\right)^{\beta}}+(a+b)\frac{\beta\theta}{\alpha}\sum_{i=1}^{n}\frac{\left(\frac{\alpha}{x_i}\right)^{\beta}}{1-\theta\left(\frac{\alpha}{x_i}\right)^{\beta}}$$

2. $$\frac{\partial\ln L(x)}{\partial\beta}=\frac{n}{\beta}+b\sum_{i=1}^{n}\ln\left(\frac{\alpha}{x_i}\right)-(a-1)\sum_{i=1}^{n}\left(\frac{\left(\frac{\alpha}{x_i}\right)^{\beta}\ln\left(\frac{\alpha}{x_i}\right)}{1-\left(\frac{\alpha}{x_i}\right)^{\beta}}\right)+(a+b)\theta\sum_{i=1}^{n}\left(\frac{\left(\frac{\alpha}{x_i}\right)^{\beta}\ln\left(\frac{\alpha}{x_i}\right)}{1-\theta\left(\frac{\alpha}{x_i}\right)^{\beta}}\right)$$

3. $$\frac{\partial\ln L(x)}{\partial\theta}=-\frac{nb}{1-\theta}+(a+b)\sum_{i=1}^{n}\left(\frac{\left(\frac{\alpha}{x_i}\right)^{\beta}}{1-\theta\left(\frac{\alpha}{x_i}\right)^{\beta}}\right) \quad (13)$$

4. $$\frac{\partial\ln L(x)}{\partial a}=n[\Psi(a+b)-\Psi(a)]+\sum_{i=1}^{n}\ln\left[1-\left(\frac{\alpha}{x_i}\right)^{\beta}\right]-\sum_{i=1}^{n}\ln\left[1-\theta\left(\frac{\alpha}{x_i}\right)^{\beta}\right]$$

5. $$\frac{\partial\ln L(x)}{\partial b}=n[\Psi(a+b)-\Psi(b)]+n\ln(1-\theta)+\beta\sum_{i=1}^{n}\ln\left(\frac{\alpha}{x_i}\right)-\sum_{i=1}^{n}\ln\left(1-\theta\left(\frac{\alpha}{x_i}\right)^{\beta}\right),$$

where $\Psi(p)=\frac{\partial\ln(\Gamma(p))}{\partial p}$ is the digamma function.

The maximum likelihood estimates of the parameters $(a, b, \alpha, \beta, \theta)$ are the solution of Eq. (13). For interval estimation and hypothesis tests on the model parameters, we require the information matrix. The Fisher information matrix $K = K(a, b, \alpha, \beta, \theta)^T$, is

$$K=\begin{pmatrix} K_{a,a} & K_{a,b} & K_{a,\theta} & K_{a,\beta} & K_{a,\alpha}\\ \cdot & K_{b,b} & K_{b,\theta} & K_{b,\beta} & K_{b,\alpha}\\ \cdot & \cdot & K_{\theta,\theta} & K_{\theta,\beta} & K_{\theta,\alpha}\\ \cdot & \cdot & \cdot & K_{\beta,\beta} & K_{\beta,\alpha}\\ \cdot & \cdot & \cdot & \cdot & K_{\alpha,\alpha}\end{pmatrix},$$

whose elements are

$$K_{\alpha,\alpha}=\frac{nb}{(1-\theta)^2}\left(\frac{\beta}{\alpha}\right)^2\left(\frac{a+b-1}{a-2}-2\theta+\frac{a}{a+b+1}\theta^2\right)$$

$$K_{\alpha,\beta}=-\frac{n}{\alpha}(a-1)\frac{(1-\theta)^b}{B(a,b)}\sum_{k=1}^{\infty}\frac{1}{k}B(a+k-2,b+1)\times{}_2F_1(b+1,a+b,a+b+k-1,\theta)+\frac{n}{\alpha}(a+b)\theta\frac{(1-\theta)^b}{B(a,b)}\sum_{k=1}^{\infty}\frac{1}{k}B(a+k,b+1)\times{}_2F_1(b+1,a+b+2,a+b+k+1,\theta)$$

$$K_{\alpha,\theta}=n\frac{\beta}{\alpha}\frac{1}{(1-\theta)^2}\left(1-\frac{a\theta}{a+b+1}\right)$$

$$K_{\alpha,a}=\frac{\beta}{\alpha}\frac{nb}{1-\theta}\left(\frac{1}{a-1}-\frac{\theta}{a+b}\right)$$

$$K_{\alpha,b}=-\beta\frac{n}{\alpha}\left(1+\frac{\theta}{1-\theta}\frac{b}{a+b}\right)$$

$$K_{\beta,\beta}=\frac{n}{\beta^2}+\frac{n(a-1)}{\beta^2}\frac{(1-\theta)^b}{B(a,b)}\sum_{k,j=1}^{\infty}\frac{1}{kj}B(a+k+j-2,b+1)\times{}_2F_1(b+1,a+b,a+b+k+j-1,\theta)-\frac{n(a+b)}{\beta^2}\theta^2\frac{(1-\theta)^b}{B(a,b)}\sum_{k,j=1}^{\infty}\frac{1}{kj}B(a+k+j,b+1)\times{}_2F_1(b+1,a+b+2,a+b+k+j+1,\theta)$$

$$K_{\beta,\theta}=(a+b)\frac{n(1-\theta)^b}{\beta B(a,b)}\sum_{k=1}^{\infty}\frac{1}{k}B(a+k,b+1)\times{}_2F_1(b+1,a+b+2,a+b+k+1,\theta)$$

$$K_{\beta,a}=\frac{n(1-\theta)^b}{\beta B(a,b)}\left(\theta\sum_{k=1}^{\infty}\frac{1}{k}B(a+k,b+1)\times{}_2F_1(b+1,a+b+1,a+b+k,\theta)-\sum_{k=1}^{\infty}\frac{1}{k}B(a+k-1,b+1)\times{}_2F_1(b+1,a+b,a+b+k-1,\theta)\right)$$

$$K_{\beta,b} = \frac{n(1-\theta)^b}{\beta B(a,b)}\left(\sum_{k=1}^{\infty}\frac{1}{k}B(a+k,b)\,{}_2F_1(b,a+b+1,a+b+k,\theta)\right)$$

$$K_{\theta,\theta} = \frac{nb}{(1-\theta)^2}\left(\frac{a}{a+b+1}\right)$$

$$K_{\theta,a} = -\frac{n}{1-\theta}\left(\frac{b}{a+b}\right)$$

$$K_{\theta,b} = -\frac{n}{1-\theta}\left(\frac{a}{a+b}\right)$$

$$K_{a,a} = n[\Psi'(a+b) - \Psi'(a)]$$

$$K_{a,b} = n\Psi'(a+b)$$

$$K_{b,b} = n[\Psi'(a+b) - \Psi'(b)]$$

7. The entropy

The entropy of a random variable X is a measure of uncertainty variation. Shannon's entropy is defined by

$$\hbar_{sh}(g) = -\int_{-\infty}^{\infty} g(x)\log g(x)\,dx = E(-\log g).$$

Substituting the BPG distribution given in Eq. (2), the entropy is given by

$$\begin{aligned}\hbar_{sh}(g) = &\log B(a,b) - \log\frac{\beta}{\alpha} - b\log(1-\theta)\\ &-(a-1)E\left[\log\left(1-\left(\frac{\alpha}{x}\right)^{\beta}\right)\right]\\ &+(a+b)E\left[\log\left(1-\theta\left(\frac{\alpha}{x}\right)^{\beta}\right)\right]\\ &-(\beta b+1)E\left[\log\left(\frac{\alpha}{x}\right)\right] \qquad (14)\end{aligned}$$

where

$$E\left[\log\left(\frac{\alpha}{x}\right)\right] = -\frac{(1-\theta)^b}{\beta B(a,b)}\sum_{k=1}^{\infty}\frac{1}{k}B(a+k,b)\,{}_2F_1(b,a+b,a+b+k,\theta) \qquad (15)$$

$$E\left[\log\left(1-\left(\frac{\alpha}{x}\right)^{\beta}\right)\right] = -\frac{(1-\theta)^b}{B(a,b)}\sum_{k=1}^{\infty}\frac{1}{k}B(a,b+k) \times {}_2F_1(b+k,a+b,a+b+k,\theta) \qquad (16)$$

$$E\left[\log\left(1-\theta\left(\frac{\alpha}{x}\right)^{\beta}\right)\right] = -\frac{(1-\theta)^b}{B(a,b)}\sum_{k=1}^{\infty}\frac{\theta^k}{k}B(a,b+k) \times {}_2F_1(b+k,a+b,a+b+k,\theta) \qquad (17)$$

Thus, we can finally write Shannon's entropy as follows:

$$\begin{aligned}\hbar_{sh}(g) = &\log B(a,b) - \log\frac{\beta}{\alpha}(1-\theta)^b\\ &-\frac{(\beta b+1)}{\beta}\frac{(1-\theta)^b}{B(a,b)}\sum_{k=1}^{\infty}\frac{1}{k}B(a+k,b)\\ &\times {}_2F_1(b,a+b,a+b+k,\theta)\\ &+\frac{(1-\theta)^b}{B(a,b)}\sum_{k=1}^{\infty}\frac{((a-1)-(a+b)\theta^k)}{k}B(a,b+k)\\ &\times {}_2F_1(b+k,a+b,a+b+k,\theta)\end{aligned}$$

8. Order statistics

Order statistics play an important role in life testing, reliability and replacement policy situations, where a practitioner needs to predict the failure of future items based on items of a few early failures. These predictors are often based on moments of order statistics. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed (i.i.d.) random variables with common Cdf $F(x)$ and pdf $f(x)$, $f(x) > 0$ for all x such that $0 < \alpha < x$. Let $X_{1:n} < X_{2:n} < \cdots < X_{n:n}$ be the corresponding order statistics.

We now derive an explicit expression for the density function of the ith order statistic $X_{i:n}$, say $f_{i:n}(x)$, in a random sample of size n from the BPG distribution. It can be easily written

$$f_{i:n}(x) = \frac{1}{B(i,n-i+1)}\sum_{l=0}^{n-i}\binom{n-i}{l}(-1)^l[G(x)]^{i+l-1}g(x).$$

Substituting with the Cdf given in Eq. (4), taking into account that

$$[G(x)]^{i+l-1} = \sum_{k_1,\ldots,k_{i+l-1}=0}^{\infty} C(k_1,\ldots,k_{i+l-1}) \times\left(\frac{1-\left(\frac{\alpha}{x}\right)^{\beta}}{1-\theta\left(\frac{\alpha}{x}\right)^{\beta}}\right)^{a(i+l-1)+k_*} \quad x > \alpha$$

where $C(k_1,\ldots,k_{i+l-1}) = c_{k_1}\times\ldots\times c_{k_{i+l-1}}$, $k_* = k_1+\cdots+k_{i+l-1}$, $c_{k_r} = \frac{\Gamma(b)(-1)^{k_r}}{B(a,b)\Gamma(b-k_r)k_r!(a+k_r)}$,

We obtain the following density function

$$f_{i:n}(x) = \sum_{l=0}^{n-i}\frac{\binom{n-i}{l}(-1)^l}{B(i,n-i+1)}\sum_{k_1,\ldots,k_{i+l-1}=0}^{\infty}C(k_1,\ldots,k_{i+l-1}) \times\left(\frac{1-\left(\frac{\alpha}{x}\right)^{\beta}}{1-\theta\left(\frac{\alpha}{x}\right)^{\beta}}\right)^{a(i+l-1)+k_*} g_{a,b}(x).$$

This can be rewritten as a linear combination of the form

$$f_{i:n}(x) = \sum_{l=0}^{n-i}\sum_{k_1,\ldots,k_{i+l-1}}^{\infty} w_{i,l}\, g_{a(i+l)+k_*,b}(x), \qquad (18)$$

where $g_{a,b}(x)$ is the pdf of the BPG distribution given in (1) and

$$w_{i,l} = \frac{\binom{n-i}{l}(-1)^l C(k_1,\ldots,k_{i+l-1})}{B(i,n-i+1)}\frac{B(a(i+l)+k_*,b)}{B(a,b)}.$$

Several mathematical properties of the BPG order statistics such as the ordinary moments, moment generating function, etc. can be then calculated. The first moment of the ith order statistic is determined as follows:

$$E(X_{i:n}) = \sum_{l=0}^{n-i}\sum_{k_1,\ldots,k_{i+l-1}}^{\infty} w_{i,l}\alpha(1-\theta)^b\frac{B\left(a(i+l)+k_*, b-\frac{1}{\beta}\right)}{B(a(i+l)+k_*,b)} \times {}_2F_1\left(b-\frac{1}{\beta},a+b,a+b-\frac{1}{\beta},\theta\right).$$

Table 1 Maximum likelihood estimates for the parameters.

	$\hat{\beta}$	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{p}$
P	0.361774			17.88	
GP	1.69531			17.88	67.8204
BP	10.1926	36.7657	0.0412693	17.88	
BGW	1.01962	8.85534	0.0208868	2	0.676527
BGP	1.10409	1.939	2.76615	17.88	17.931
BPG	1.87561	4.82583	0.267425	15.7246	0.385339

Table 2 The log–likelihood functions for the data sets.

Distribution	P	GP	BP	BGP	BPG	BGW
	−129.80	−416.84	−156.99	−330.10	−89.7034	−255.3088

Table 3 AIC and BIC values for the given data set.

	P	GP	BP	BGP	BPG	BGW
AIC	267.6117	841.673	317.9774	666.199	187.40425	520.61751
BIC	265.0587	839.1199	316.7008	664.284	184.85116	517.42615

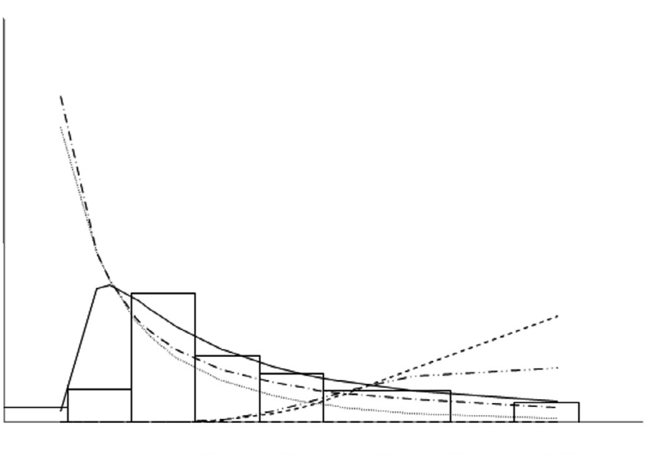
9. Application to real data

In this section we fit the BPG distribution to an example of an uncensored data set from Caroni [18] consisting of 23 observations of ball bearing data:

17.88, 28.92, 33.00, 41.52, 42.12, 45.60, 48.48, 51.84, 51.96, 54.12, 55.56, 67.80, 68.64, 68.64, 68.88, 84.12, 93.12, 98.64, 105.12, 105.84, 127.92, 128.04, 173.40.

The maximum likelihood parameters using the Pareto (P), Generalized Pareto (GP), Beta Pareto (BP). Beta Generalized Weibull (BGW), Beta Generalized Pareto (BGP), and Beta Pareto Geometric (BPG) distributions are given in Table 1.

The following table (Table 2) lists the values of the log-likelihood to all six models for this data set.



In addition to the MLEs of the parameters and the log-likelihood functions for the data set, the values of the Akaike information criterion (AIC) and the Bayesian information criterion (BIC) for the six distributions are deduced in Table 3.

It is clear that the BPG yields the highest value of the log-likelihood function for the data set, and hence we can conclude that the BPG model is better than the other distributions to fit this data set. Also, we can see from the numerical results in Table 3, the AIC value for the BPG model is the smallest value among those values, and hence our new model can be chosen as the best model.

10. Concluding remarks

In this work we introduce the Beta Pareto Geometric (BPG) distribution because of the wide usage of the Pareto geometric distribution and the fact that the current generalization provides means of its continuous extension to still more complex situations. We have derived various properties of the beta Pareto geometric distributions, including the moment generating function and the rth generalized moment. Discussion of the estimation procedure by maximum likelihood has been introduced followed by the Fisher information matrix. Finally, we demonstrate an application to real data. In conclusion, the beta Pareto geometric class of distributions provides a rather general and flexible framework for statistical analysis. It unifies several previously proposed families of distributions, therefore yielding a general overview of these families for theoretical studies, and it also provides a rather flexible mechanism for fitting a wide spectrum of real world data sets. We hope that this generalization may attract wider application in reliability and biology.

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Appendix A. The elements of the information matrix are calculated as follows:

$$K_{\alpha,\alpha}=E\left(-\frac{\partial^2\ln L}{\partial\alpha^2}\right)$$

$$=E\left(\beta b\frac{n}{\alpha^2}+(a-1)\left(\frac{\beta}{\alpha}\right)^2\sum_{i=1}^{n}\left(\frac{\left(\frac{\alpha}{x_i}\right)^\beta}{\left(1-\left(\frac{\alpha}{x_i}\right)^\beta\right)^2}\right)\right.$$

$$+(a+b)\theta\frac{\beta}{\alpha^2}\sum_{i=1}^{n}\left(\frac{\left(\frac{\alpha}{x_i}\right)^\beta}{\left(1-\left(\frac{\alpha}{x_i}\right)^\beta\right)}\right)$$

$$-(a-1)\frac{\beta}{\alpha^2}\sum_{i=1}^{n}\left(\frac{\left(\frac{\alpha}{x_i}\right)^\beta}{\left(1-\left(\frac{\alpha}{x_i}\right)^\beta\right)}\right)$$

$$\left.-(a+b)\theta\left(\frac{\beta}{\alpha}\right)^2\sum_{i=1}^{n}\left(\frac{\left(\frac{\alpha}{x_i}\right)^\beta}{\left(1-\left(\frac{\alpha}{x_i}\right)^\beta\right)^2}\right)\right)$$

$$K_{\alpha,\beta}=E\left(-\frac{\partial^2\ln L}{\partial\alpha\partial\beta}\right)$$

$$=E\left(-\frac{nb}{\alpha}+(a-1)\frac{\beta}{\alpha}\sum_{i=1}^{n}\left(\frac{\left(\frac{\alpha}{x_i}\right)^\beta\ln\left(\frac{\alpha}{x_i}\right)}{\left(1-\left(\frac{\alpha}{x_i}\right)^\beta\right)^2}\right)\right.$$

$$+\frac{(a-1)}{\alpha}\sum_{i=1}^{n}\left(\frac{\left(\frac{\alpha}{x_i}\right)^\beta}{\left(1-\left(\frac{\alpha}{x_i}\right)^\beta\right)}\right)$$

$$-(a+b)\theta\frac{\beta}{\alpha}\sum_{i=1}^{n}\left(\frac{\left(\frac{\alpha}{x_i}\right)^\beta\ln\left(\frac{\alpha}{x_i}\right)}{\left(1-\theta\left(\frac{\alpha}{x_i}\right)^\beta\right)^2}\right)$$

$$\left.+\frac{(a+b)}{\alpha}\theta\sum_{i=1}^{n}\left(\frac{\left(\frac{\alpha}{x_i}\right)^\beta}{\left(1-\theta\left(\frac{\alpha}{x_i}\right)^\beta\right)}\right)\right)$$

$$K_{\alpha,\theta}=E\left(-\frac{\partial^2\ln L}{\partial\alpha\partial\theta}\right)=-(a+b)\frac{\beta}{\alpha}E\left(\sum_{i=1}^{n}\left[\frac{(\alpha/x_i)^\beta}{\left(1-\theta(\alpha/x_i)^\beta\right)^2}\right]\right)$$

$$K_{\alpha,a}=E\left(-\frac{\partial^2\ln L}{\partial\alpha\partial a}\right)$$

$$=E\left(\frac{\beta}{\alpha}\sum_{i=1}^{n}\left(\frac{\left(\frac{\alpha}{x_i}\right)^\beta}{1-\left(\frac{\alpha}{x_i}\right)^\beta}\right)-\theta\frac{\beta}{\alpha}\sum_{i=1}^{n}\left(\frac{\left(\frac{\alpha}{x_i}\right)^\beta}{1-\theta\left(\frac{\alpha}{x_i}\right)^\beta}\right)\right)$$

$$K_{\alpha,b}=E\left(-\frac{\partial^2\ln L}{\partial\alpha\partial b}\right)=E\left(\beta\frac{n}{\alpha}\sum_{i=1}^{n}\frac{1}{1-\theta\left(\frac{\alpha}{x_i}\right)^\beta}\right)$$

$$K_{\beta,\beta}=E\left(-\frac{\partial^2\ln L}{\partial\beta^2}\right)$$

$$=E\left(\frac{n}{\beta^2}+(a-1)\sum_{i=1}^{n}\left[\frac{\left(\frac{\alpha}{x_i}\right)^\beta\left(\ln\left(\frac{\alpha}{x_i}\right)\right)^2}{\left(1-\left(\frac{\alpha}{x_i}\right)^\beta\right)^2}\right]\right.$$

$$\left.-(a+b)\theta^2\sum_{i=1}^{n}\left[\frac{\left(\frac{\alpha}{x_i}\right)^\beta\left(\ln\left(\frac{\alpha}{x_i}\right)\right)^2}{\left(1-\theta\left(\frac{\alpha}{x_i}\right)^\beta\right)^2}\right]\right)$$

$$K_{\beta,\theta}=E\left(-\frac{\partial^2\ln L}{\partial\beta\partial\theta}\right)=E\left(-(a+b)\sum_{i=1}^{n}\left[\frac{-\left(\frac{\alpha}{x_i}\right)^\beta\ln\left(\frac{\alpha}{x_i}\right)}{\left(1-\theta\left(\frac{\alpha}{x_i}\right)^\beta\right)^2}\right]\right)$$

$$K_{\beta,a}=E\left(-\frac{\partial^2\ln L}{\partial\beta\partial a}\right)$$

$$=E\left(\sum_{i=1}^{n}\left[\frac{\left(\frac{\alpha}{x_i}\right)^\beta\ln\left(\frac{\alpha}{x_i}\right)}{1-\left(\frac{\alpha}{x_i}\right)^\beta}\right]-\theta\sum_{i=1}^{n}\left[\frac{\left(\frac{\alpha}{x_i}\right)^\beta\ln\left(\frac{\alpha}{x_i}\right)}{1-\left(\frac{\alpha}{x_i}\right)^\beta}\right]\right)$$

$$K_{\beta,b}=E\left(-\frac{\partial^2\ln L}{\partial\beta\partial b}\right)=E\left(-\sum_{i=1}^{n}\left(\frac{\ln\left(\frac{\alpha}{x_i}\right)}{1-\theta\left(\frac{\alpha}{x_i}\right)^\beta}\right)\right)$$

$$K_{\theta,\theta}=E\left(-\frac{\partial^2\ln L}{\partial\theta^2}\right)$$

$$=E\left(\frac{nb}{(1-\theta)^2}+(a+b)\sum_{i=1}^{n}\frac{\left(\frac{\alpha}{x_i}\right)^{2\beta}}{\left(1-\theta\left(\frac{\alpha}{x_i}\right)^\beta\right)^2}\right)$$

$$K_{\theta,a}=E\left(-\frac{\partial^2\ln L}{\partial\theta\partial a}\right)=E\left(-\sum_{i=1}^{n}\frac{\left(\frac{\alpha}{x_i}\right)^\beta}{1-\theta\left(\frac{\alpha}{x_i}\right)^\beta}\right)$$

$$K_{\theta,b}=E\left(-\frac{\partial^2\ln L}{\partial\theta\partial b}\right)=E\left(-\frac{n}{1-\theta}+\sum_{i=1}^{n}\left(\frac{\left(\frac{\alpha}{x_i}\right)^\beta}{1-\theta\left(\frac{\alpha}{x_i}\right)^\beta}\right)\right)$$

$$K_{a,b}=E\left(-\frac{\partial^2\ln L}{\partial a\partial b}\right)$$

$$K_{a,a} = E\left(-\frac{\partial^2 \ln L}{\partial a^2}\right)$$

$$K_{b,b} = E\left(-\frac{\partial^2 \ln L}{\partial b^2}\right)$$

References

[1] N. Krishnaji, Characterization of the Pareto distribution through a model of underreported incomes, Econometrica 38 (2) (1970) 251–255.

[2] G. Alperovich, An explanatory model of city-size distribution: evidence from cross-country data, Urban Studies 30 (9) (1993) 1591–1601.

[3] W.J. Reed, On the rank-size distribution for human settlements, Regional Scientific Journal 42 (1) (2002) 1–17.

[4] J.L. Devore, Probability and Statistics for Engineering and the Sciences, sixth ed., Duxbury, Ontario, 2004.

[5] M.E.J. Newman, Power laws, Pareto distributions and Zipf's law, Contemporary Physics 46 (5) (2005) 323–351.

[6] N. Eugene, C. Lee, F. Famoye, Beta normal distribution and its applications, Communications in Statistics – Theory Methods 31 (2002) 497–512.

[7] M.C. Jones, Families of distributions arising from distributions of order statistics, Test 13 (2004) 1–43.

[8] S. Nadarajah, A.K. Gupta, The beta Fréchet distribution, Far East Journal of Theoretical Statistics 14 (2004) 15–24.

[9] S. Nadarajah, S. Kotz, The beta Gumbel distribution, Mathematics and Probability for Engineers 10 (2004) 323–332.

[10] S. Nadarajah, S. Kotz, The beta exponential distribution, Reliability Engineering and System Safety 91 (2005) 689–697.

[11] L. Kong, C. Lee, J.H. Sepanski, On the properties of beta Gamma distributions, Journal of Modern Applied Statistical Methods 6 (2007) 173–187.

[12] A. Akinsete, F. Famoye, C. Lee, The beta-Pareto distribution, Statistics 42 (2008) 547–563.

[13] R.R. Pescim, C.G.B. Demetrio, G.M. Cordeiro, E.M.M. Ortega, M.R. Urbano, The beta generalized half-normal distribution, Computational Statistics and Data Analysis 54 (2010) 945–957.

[14] W.B. Souza, A.H.S. Santos, G.M. Cordeiro, The beta generalized exponential distribution, Journal of Statistical Computation and Simulation 80 (2010) 159–172.

[15] M.M. Nassar, N.K. Nada, The beta generalized Pareto distribution, Journal of Statistics – Advances in Theory and Applications 6 (2011) 1–17.

[16] A.L. De Morais, A Class of Generalized Beta Distributions, Pareto Power Series and Weibull Power Series, Dissertação (mestrado) – Universidade Federal de Pernambuco. CCEN, 2009.

[17] R.E. Glaser, Bathtub and related failure rate characterizations, Journal of the American Statistical Association 75 (1980) 667–672.

[18] C. Caroni, The correct ball bearings'' data'', Lifetime Data Analysis 8 (2002) 395–399.