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# Separation Problem for Bi-Harmonic Differential Operators in $L^p-$ spaces on Manifolds

H. A. Atia

Correspondence:
h_a_atia@hotmail.com
Faculty of Science, Department of Mathematics, Zagazig University, Zagazig, Egypt

**Abstract**

Consider the bi-harmonic differential expression of the form

$$A = \Delta_M^2 + q$$

on a manifold of bounded geometry $(M, g)$ with metric $g$, where $\Delta_M$ is the scalar Laplacian on $M$ and $q \geq 0$ is a locally integrable function on $M$.

In the terminology of Everitt and Giertz, the differential expression $A$ is said to be separated in $L^p(M)$, if for all $u \in L^p(M)$ such that $Au \in L^p(M)$, we have $qu \in L^p(M)$. In this paper, we give sufficient conditions for $A$ to be separated in $L^p(M)$ ,where $1 < p < \infty$.

**Keywords:** Separation problem, Bi-harmonic differential operator, Manifold

**AMS Subject Classification :** 47F05, 58J99

## Introduction

In the terminology of Everitt and Giertz, the concept of separation of differential operators was first introduced in [1]. Several results of the separation problem are given in a series of pioneering papers [2–5]. For more backgrounds concerning to our problem, see [6–8]. Atia et al. [9] have studied the separation property of the bi-harmonic differential expression $A = \Delta_M^2 + q$ , on a Riemannian manifold $(M, g)$ without boundary in $L^2(M)$ , where $\Delta_M$ is the Laplacian on $M$ and $0 \leq q \in L^2_{loc}(M)$ is a real-valued function.

Recently, Atia [10] has studied the sufficient conditions for the magnetic bi-harmonic differential operator $B$ of the form $B = \Delta_E^2 + q$ to be separated in $L^2(M)$, on a complete Riemannian manifold $(M, g)$ with metric $g$ , where $\Delta_E$ is the magnetic Laplacian on $M$ and $q \geq 0$ is a locally square integrable function on $M$. In [11], Milatovic has studied the separation property for the Schrodinger-type expression of the form $L = \Delta_M + q$ , on non-compact manifolds in $L^p(M)$ . Let $(M, g)$ be a Riemannian manifold without boundary, with metric $g$ (i.e., $M$ is a $C^\infty-manifold$ without boundary and $g = (g_{jk})$ is a Riemannian metric on $M$) and $dimM = n$. We will assume that $M$ is connected. We will also assume that we are given a positive smooth measure $d\mu$, i.e., in any local coordinates $x^1, x^2, \ldots, x^n$ , there exists a strictly positive $C^\infty-$density $\rho(x)$ such that $d\mu = \rho(x)\, dx^1 dx^2 \ldots dx^n$. In the sequel, $L^2(M)$ is the space of complex-valued square integrable functions on $M$ with the inner product:

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$$(u, v) = \int_M \left(uv^-\right) d\mu, \tag{1}$$

and $\|.\|$ is the norm in $L^2(M)$ corresponding to the inner product (1). We use the notation $L^2\left(\Lambda^1 T^* M\right)$ for the space of complex-valued square integrable 1-forms on $M$ with the inner product:

$$(W, \Psi)_{L^2(\Lambda^1 T^* M)} = \int_M \left\langle W, \overline{\Psi}\right\rangle d\mu, \tag{2}$$

where for 1-forms $W = W_j dx^j$ and $\Psi = \Psi_k dx^k$, we define $\langle W, \Psi\rangle = g^{jk} W_j \Psi_k$ , where $(g^{jk})$ is the inverse matrix to $(g_{jk})$ , and $\overline{\Psi} = \overline{\Psi_k} dx^k$ (above, we use the standard Einstein summation convention).

The notation $\|.\|_{L^2(\Lambda^1 T^* M)}$ stands for the norm in $L^2\left(\Lambda^1 T^* M\right)$ corresponding to the inner product (2). To simplify notations, we will denote the inner products (1) and (2) by $(.,.)$. In the sequel, for $1 \le p < \infty$, $L^p(M)$ is the space of complex-valued $p$-integrable functions on $M$ with the norm:

$$\|u\|_p = \left(\int_M |u|^p \; d\mu\right)^{\frac{1}{p}}, \tag{3}$$

In what follows, by $C^1(M)$ , we denote the space of continuously differentiable complex-valued functions on $M$, and by $C^\infty(M)$ , we denote the space of smooth complex-valued functions on $M$, by $C_c^\infty(M)$ −the space of smooth compactly supported complex-valued functions on $M$, by $\Omega^1(M)$− the space of smooth 1-forms on $M$ , and by $\Omega_c^1(M)$ −the space of smooth compactly supported 1-forms on $M$. In the sequel, the operator $d : C^\infty(M) \to \Omega^1(M)$ is the standard differential and $d^* : \Omega^1(M) \to C^\infty(M)$ is the formal adjoint of $d$ defined by the identity: $(du, v)_{L^2(\Lambda^1 T^* M)} = (u, d^* v)$ , $\quad u \in C_c^\infty(M)$ , $v \in \Omega^1(M)$ .By $\Delta_M = d^* d$ , we will denote the scalar Laplacian on $M$. We will use the product rule for $d^*$ as follows:

$$d^*(uv) = ud^* v - \langle du, v\rangle, \; u \in C^1(M), \; v \in \Omega^1(M). \tag{4}$$

We consider the bi-harmonic differential expression:

$$A = \triangle_M^2 + q, \tag{5}$$

where $q \ge 0$ is a locally integrable function on $M$.

**Definition 1** *The set* $D_p$ :

Let $A$ be as in (5), we will use the notation

$$D_p = \{u \in L^p(M) : Au \in L^p(M)\}. \tag{6}$$

**Remark 1** *In general, it is not true that for all* $u \in D_p$*, we have* $\triangle_M^2 u \in L^p(M)$ *and* $qu \in L^p(M)$ *separately. Using the terminology of Everitt and Giertz, we will say that the differential expression* $A = \triangle_M^2 + q$ *is separated in* $L^p(M)$ *when the following statement holds true: for all* $u \in D_p$ *, we have* $qu \in L^p(M)$ *.*

We will give sufficient conditions for $A$ to be separated in $L^p(M)$ . Assume that the manifold $(M, g)$ has bounded geometry, that is

(a) $\inf_{x\in M} r_{inj}(x) > 0$, where $r_{inj}(x)$ is the injectivity radius of $(M, g)$,

(b) all covariant derivatives $\nabla^j R$ of the Riemann curvature tensor $R$ are bounded: $\left|\nabla^j R\right| \leq K_j,\ \ j = 0, 1, 2, ...,$where $K_j$ are constants.

Let $(M, g)$ be a manifold of bounded geometry. Then, there exists a sequence of functions (called cut-off functions) $\{\phi_j\}$ in $C_c^\infty(M)$ such that for all $j = 1, 2, 3...,$

(i) $0 \leq \phi_j \leq 1$;

(ii) $\phi_j \leq \phi_{j+1}$;

(iii) for every compact set $S \subset M$ , there exists $j$ such that $\phi_j|_S = 1$;

(iv) $\sup_{x\in M} \left|d\phi_j\right| \leq C_1,\ \ \sup_{x\in M} \left|\triangle_M \phi_j\right| \leq C_1$ , and $\sup_{x\in M} \left|\triangle_M^2 \phi_j\right| \leq C_1$, where $C_1 > 0$ is a constant independent of $j$. For the construction of $\phi_j$ satisfying the above properties, see [12].

## Preliminary lemma

In the following, we introduce a preliminary lemma which will be used in the sequel.

**Lemma 1** *Assume that $(M, g)$ is a connected $C^\infty-$Riemannian manifold without boundary, with metric g and has bounded geometry. Assume that there exist a constant $\gamma$ such that $0 < \gamma \leq q \in C^1(M)$, and*

$$\left|\triangle_M q(x)\right| \leq \sigma q^{\frac{3}{2}}(x), \text{ for all } x \in M, \tag{7}$$

*where $0 < \sigma < \frac{2}{\sqrt{p-1}}$, $1 < p < \infty$ , and $\left|\triangle_M q(x)\right|$ denotes the norm of $\triangle_M q(x) \in T_x^* M$ with respect to the inner product in $T_x^* M$ induced by the metric g. Assume that $f \in L^p(M)$ and that $u \in L^p(M) \cap C^1(M)$ is a solution of the equation*

$$\triangle_M^2 u + qu = f. \tag{8}$$

*Additionally assume that for all $k \in \left[-\frac{1}{2}, p-1\right]$,*

$$|u|^p q^{k+\frac{1}{2}} \in L^1(M) \text{ and } \lim_{j\to\infty} \left(\triangle_M u q^k du, u\,|u|^{p-4} \phi_j du\right) = 0. \tag{9}$$

*Then, the following properties hold:*

$$\lim_{j\to\infty} \left(\triangle_M u, q^k u\,|u|^{p-2} \triangle_M \phi_j\right) = 0, \tag{10}$$

*and*

$$q^{k+1} |u|^p \in L^1(M), \text{ and } \int_M q^{k+1} |u|^p \ \ d\mu \leq C_1 \|f\|_p^p, \tag{11}$$

*for all $k \in \left[-\frac{1}{2}, p-1\right]$, where $\{\phi_j\}$ is as in (i-iv) and $C_1 \geq 0$ is a constant independent of u.*

*Proof* We first prove (10): Since $u \in L^p(M) \cap C^1(M)$, using integration by parts, product rule of $d$, the definition of $\Delta_M = d^*d$, and the formula $d(u_\epsilon) = \frac{udu}{u_\epsilon}$, we have

$$\begin{aligned}
\left(\triangle_M u, q^k u\,|u|^{p-2}\,\triangle_M \phi_j\right) &= \lim_{\epsilon\to 0^+}\left(\triangle_M u, q^k u(u_\epsilon)^{p-2}\triangle_M\phi_j\right)\\
&= \lim_{\epsilon\to 0^+}\left(du, d\left(q^k u(u_\epsilon)^{p-2}\triangle_M\phi_j\right)\right)\\
&= \lim_{\epsilon\to 0^+}\left(dudq^k, u(u_\epsilon)^{p-2}\triangle_M\phi_j\right)\\
&\quad+\lim_{\epsilon\to 0^+}\left(duq^k du, (u_\epsilon)^{p-2}\triangle_M\phi_j\right)\\
&\quad+(p-2)\lim_{\epsilon\to 0^+}\left(duq^k du, u^2(u_\epsilon)^{p-4}\triangle_M\phi_j\right)\\
&\quad+\lim_{\epsilon\to 0^+}\left(du, q^k u(u_\epsilon)^{p-2}d(\triangle_M\phi_j)\right)\\
&= \lim_{\epsilon\to 0^+}\left(du, dq^k u(u_\epsilon)^{p-2}\triangle_M\phi_j\right)\\
&\qquad+\lim_{\epsilon\to 0^+}\left(du, q^k u(u_\epsilon)^{p-2}d(\triangle_M\phi_j)\right)\\
&\quad+(p-1)\left(du, q^k du\,|u|^{p-2}\,\triangle_M\phi_j\right)\\
&= \lim_{\epsilon\to 0^+}\left(u, d^*\left(dq^k u(u_\epsilon)^{p-2}\triangle_M\phi_j\right)\right)\\
&\quad+\lim_{\epsilon\to 0^+}\left(u, d^*\left(q^k u(u_\epsilon)^{p-2}d(\triangle_M\phi_j)\right)\right)\\
&\quad+(p-1)\lim_{\epsilon\to 0^+}\left(u, d^*\left(q^k du(u_\epsilon)^{p-2}\triangle_M\phi_j\right)\right),
\end{aligned}$$

using the product rule (4) of $d^*$, we get

$$\begin{aligned}
\left(\triangle_M u, q^k u\,|u|^{p-2}\,\triangle_M \phi_j\right) &= -\lim_{\epsilon\to 0^+}\left(ud\left(u(u_\epsilon)^{p-2}\triangle_M\phi_j\right), dq^k\right)\\
&\quad+\lim_{\epsilon\to 0^+}\left(u, u(u_\epsilon)^{p-2}\triangle_M\phi_j\triangle_M q^k\right)\\
&\quad-(p-1)\lim_{\epsilon\to 0^+}\left(ud\left(q^k(u_\epsilon)^{p-2}\triangle_M\phi_j\right), du\right)\\
&\quad+(p-1)\lim_{\epsilon\to 0^+}\left(u, q^k(u_\epsilon)^{p-2}\triangle_M\phi_j\triangle_M u\right)\\
&\quad-\lim_{\epsilon\to 0^+}\left(ud\left(q^k u(u_\epsilon)^{p-2}\right), d(\triangle_M\phi_j)\right)\\
&\qquad+\lim_{\epsilon\to 0^+}\left(u, q^k u(u_\epsilon)^{p-2}\triangle_M^2\phi_j\right),
\end{aligned}$$

using the product rule of $d$ again, we get

$$\begin{aligned}
\left(\triangle_M u, q^k u\,|u|^{p-2}\,\triangle_M \phi_j\right) &= -\lim_{\epsilon\to 0^+}\left(ud\left(\triangle_M\phi_j\right), u(u_\epsilon)^{p-2}dq^k\right)\\
&\quad-\lim_{\epsilon\to 0^+}\left(u\triangle_M\phi_j du, (u_\epsilon)^{p-2}dq^k\right)\\
&\quad-(p-2)\lim_{\epsilon\to 0^+}\left(u\triangle_M\phi_j du, u^2(u_\epsilon)^{p-4}dq^k\right)\\
&\quad+\lim_{\epsilon\to 0^+}\left(u, u(u_\epsilon)^{p-2}\triangle_M\phi_j\triangle_M q^k\right)\\
&\qquad+\lim_{\epsilon\to 0^+}\left(u, q^k u(u_\epsilon)^{p-2}\triangle_M^2\phi_j\right)
\end{aligned}$$

$$+(p-1)\lim_{\epsilon\to 0^+}\left(udq^k,(u_\epsilon)^{p-2}\triangle_M\phi_j du\right)$$
$$+(p-1)\lim_{\epsilon\to 0^+}\left(uq^k d\left(\triangle_M\phi_j\right),(u_\epsilon)^{p-2}du\right)$$
$$-(p-1)(p-2)\lim_{\epsilon\to 0^+}\left(uq^k du,u(u_\epsilon)^{p-4}\triangle_M\phi_j du\right)$$
$$-(p-1)\lim_{\epsilon\to 0^+}\left(u,q^k(u_\epsilon)^{p-2}\triangle_M\phi_j\triangle_M u\right)$$
$$+\lim_{\epsilon\to 0^+}\left(udq^k,u(u_\epsilon)^{p-2}d(\triangle_M\phi_j)\right)$$
$$-\lim_{\epsilon\to 0^+}\left(udq^k du,(u_\epsilon)^{p-2}d(\triangle_M\phi_j)\right)$$
$$-(p-2)\lim_{\epsilon\to 0^+}\left(udq^k du,u^2(u_\epsilon)^{p-4}d(\triangle_M\phi_j)\right).$$

Hence, we obtain

$$p\left(\triangle_M u,q^k u\,|u|^{p-2}\,\triangle_M\phi_j\right)=\left(u\triangle_M q^k,u\,|u|^{p-2}\,\triangle_M\phi_j\right)+\left(u,q^k u\,|u|^{p-2}\,\triangle_M^2\phi_j\right)$$
$$-(p-1)(p-2)\left(uq^k du,u\,|u|^{p-4}\,\triangle_M\phi_j du\right).$$

Taking the limit as $j\to\infty$, we get

$$p\lim_{j\to\infty}\left(\triangle_M u,q^k u\,|u|^{p-2}\,\triangle_M\phi_j\right)=\lim_{j\to\infty}\left(u\triangle_M q^k,u\,|u|^{p-2}\,\triangle_M\phi_j\right)$$
$$+\lim_{j\to\infty}\left(u,q^k u\,|u|^{p-2}\,\triangle_M^2\phi_j\right)$$
$$-(p-1)(p-2)\lim_{j\to\infty}\left(uq^k du,u\,|u|^{p-4}\,\triangle_M\phi_j du\right).$$

By properties of $\left\{\phi_j\right\}$, it follows that for all $x\in M$, $\phi_j(x)\to 1$, $d\phi_j(x)\to 0$, $\triangle_M\phi_j(x)\to 0$ and $\triangle_M^2\phi_j(x)\to 0$ as $j\to\infty$, we apply dominated convergence theorem by using the assumption (7), the assumption $|u|^p\,q^{k+\frac{1}{2}}\in L^1(M)$ and the condition (iv), we obtain (10).

We now prove (11): Since $u\in L^p(M)\cap C^1(M)$, using (8), integration by parts, product rule of $d$, the definition of $\Delta_M=d^*d$, and the formula $d(u_\epsilon)=\frac{udu}{u_\epsilon}$, we have

$$\left(f,q^k u\,|u|^{p-2}\,\phi_j\right)=\left(\triangle_M^2 u,q^k u\,|u|^{p-2}\,\phi_j\right)+\left(qu,q^k u\,|u|^{p-2}\,\phi_j\right)$$
$$=\lim_{\epsilon\to 0^+}\left(\triangle_M^2 u,q^k u(u_\epsilon)^{p-2}\phi_j\right)+\left(qu,q^k u\,|u|^{p-2}\,\phi_j\right)$$
$$=\lim_{\epsilon\to 0^+}\left(d\left(\triangle_M u\right),d\left(q^k u(u_\epsilon)^{p-2}\phi_j\right)\right)+\left(qu,q^k u\,|u|^{p-2}\,\phi_j\right)$$

$$=\lim_{\epsilon\to 0^+}\left(d\left(\triangle_M u\right),q^k u(u_\epsilon)^{p-2}d\phi_j\right)+\lim_{\epsilon\to 0^+}\left(d\left(\triangle_M u\right),q^k(u_\epsilon)^{p-2}\phi_j du\right)$$
$$+(p-2)\lim_{\epsilon\to 0^+}\left(d\left(\triangle_M u\right),q^k u^2(u_\epsilon)^{p-4}\phi_j du\right)$$
$$+\lim_{\epsilon\to 0^+}\left(d\left(\triangle_M u\right),u(u_\epsilon)^{p-2}\phi_j dq^k\right)+\left(qu,q^k u\,|u|^{p-2}\,\phi_j\right)$$
$$=\left(d\left(\triangle_M u\right),q^k u\,|u|^{p-2}\,d\phi_j\right)+\left(d\left(\triangle_M u\right),u\,|u|^{p-2}\,\phi_j dq^k\right)$$
$$+(p-1)\left(d\left(\triangle_M u\right),q^k\,|u|^{p-2}\,\phi_j du\right)+\left(qu,q^k u\,|u|^{p-2}\,\phi_j\right)$$
$$=\lim_{\epsilon\to 0^+}\left(\triangle_M u,d^*\left(q^k u(u_\epsilon)^{p-2}d\phi_j\right)\right)+(p-1)\lim_{\epsilon\to 0^+}\left(\triangle_M u,d^*\left(q^k(u_\epsilon)^{p-2}\phi_j du\right)\right)$$
$$+\lim_{\epsilon\to 0^+}\left(\triangle_M u,d^*\left(u(u_\epsilon)^{p-2}\phi_j dq^k\right)\right)+\left(qu,q^k u\,|u|^{p-2}\,\phi_j\right),$$

using the product rule (4) of $d^*$, we get

$$\begin{aligned}\left(f, q^k u\,|u|^{p-2}\,\phi_j\right) = &-\lim_{\epsilon\to 0^+}\left(\triangle_M u d\left(q^k u(u_\epsilon)^{p-2}\right), d\phi_j\right) + \lim_{\epsilon\to 0^+}\left(\triangle_M u, q^k u(u_\epsilon)^{p-2}\triangle_M\phi_j\right)\\ &-\lim_{\epsilon\to 0^+}\left(\triangle_M u d\left(u(u_\epsilon)^{p-2}\phi_j\right), dq^k\right) + \lim_{\epsilon\to 0^+}\left((\triangle_M u)u(u_\epsilon)^{p-2}\phi_j, \triangle_M q^k\right)\\ &-(p-1)\lim_{\epsilon\to 0^+}\left(\triangle_M u d\left(q^k(u_\epsilon)^{p-2}\phi_j\right), du\right)\\ &+(p-1)\lim_{\epsilon\to 0^+}\left(\triangle_M u q^k(u_\epsilon)^{p-2}\phi_j, \triangle_M u\right) + \left(qu, q^k u\,|u|^{p-2}\,\phi_j\right),\end{aligned}$$

using the product rule of $d$ again, we get

$$\begin{aligned}\left(f, q^k u\,|u|^{p-2}\,\phi_j\right) = &-\lim_{\epsilon\to 0^+}\left(\triangle_M u dq^k, u(u_\epsilon)^{p-2}d\phi_j\right) - \lim_{\epsilon\to 0^+}\left(\triangle_M u q^k du, (u_\epsilon)^{p-2}d\phi_j\right)\\ &-(p-2)\lim_{\epsilon\to 0^+}\left(\triangle_M u q^k du, u^2(u_\epsilon)^{p-4}d\phi_j\right)\\ &+\left(\triangle_M u, q^k u\,|u|^{p-2}\,\triangle_M\phi_j\right) + \lim_{\epsilon\to 0^+}\left(\triangle_M u d\phi_j, u(u_\epsilon)^{p-2}dq^k\right)\\ &-\lim_{\epsilon\to 0^+}\left(\triangle_M u\phi_j du, (u_\epsilon)^{p-2}dq^k\right) + \left((\triangle_M u)u\,|u|^{p-2}\,\phi_j, \triangle_M q^k\right)\\ &-(p-2)\lim_{\epsilon\to 0^+}\left(\triangle_M u\phi_j du, u^2(u_\epsilon)^{p-4}dq^k\right)\\ &+(p-1)\lim_{\epsilon\to 0^+}\left(\triangle_M u dq^k, (u_\epsilon)^{p-2}\phi_j du\right)\\ &+(p-1)\lim_{\epsilon\to 0^+}\left(\triangle_M u q^k, (u_\epsilon)^{p-2}d\phi_j du\right)\\ &-(p-1)(p-2)\lim_{\epsilon\to 0^+}\left(\triangle_M u q^k du, u(u_\epsilon)^{p-4}\phi_j du\right)\\ &+(p-1)\left(\triangle_M u q^k\,|u|^{p-2}\,\phi_j, \triangle_M u\right) + \left(qu, q^k u\,|u|^{p-2}\,\phi_j\right).\end{aligned}$$

Hence, we obtain

$$\begin{aligned}\left(f, q^k u\,|u|^{p-2}\,\phi_j\right) = &-(p-1)(p-2)\left(\triangle_M u q^k du, u\,|u|^{p-4}\,\phi_j du\right)\\ &+\left(\triangle_M u, q^k u\,|u|^{p-2}\,\triangle_M\phi_j\right) + (p-1)\left(\triangle_M u, q^k\,|u|^{p-2}\,\phi_j\triangle_M u\right)\\ &+\left(\triangle_M u, u\,|u|^{p-2}\,\phi_j\triangle_M q^k\right) + \left(qu, q^k u\,|u|^{p-2}\,\phi_j\right). \qquad (12)\end{aligned}$$

We now estimate the term $\left(\triangle_M u, u\,|u|^{p-2}\,\phi_j\triangle_M q^k\right)$.

Using the assumption (7), we get

$$\left|\triangle_M q^k\right| \le \sigma\,|k|\,q^{k+\frac{1}{2}}. \qquad (13)$$

Using (13) and the inequality $ab \le (p-1)a^2 + \frac{b^2}{4(p-1)}$, for all $0 \le a, b \in R$, we have

$$\left|\left(\triangle_M u, u\,|u|^{p-2}\,\phi_j\triangle_M q^k\right)\right| \le \int_M |\triangle_M u|\left|\triangle_M q^k\right||u|^{p-1}\,\phi_j\,d\mu$$

$$\leq \int_M \sigma \left|\triangle_M u\right| |k| \, q^{k+\frac{1}{2}} \left|u\right|^{p-1} \phi_j \, d\mu$$
$$= \int_M \left(\left|\triangle_M u\right| |u|^{\frac{p-2}{2}} \phi_j^{\frac{1}{2}} q^{\frac{k}{2}}\right)\left(\sigma \, |k| \, q^{\frac{k+1}{2}} \phi_j^{\frac{1}{2}} |u|^{\frac{p}{2}}\right) d\mu$$
$$\leq (p-1) \int_M \left|\triangle_M u\right|^2 |u|^{p-2} \phi_j q^k \, d\mu + \frac{\sigma^2 k^2}{4(p-1)} \int_M q^{k+1} \phi_j \, |u|^p \, d\mu$$
$$= (p-1)\left(\triangle_M u, q^k |u|^{p-2} \phi_j \triangle_M u\right) + \frac{\sigma^2 k^2}{4(p-1)}\left(qu, q^k u \, |u|^{p-2} \phi_j\right)$$
$$= (p-1)\left(\triangle_M u, q^k |u|^{p-2} \phi_j \triangle_M u\right) + (1-\alpha)\left(qu, q^k u \, |u|^{p-2} \phi_j\right), \tag{14}$$

where $\alpha = 1 - \frac{\sigma^2 k^2}{4(p-1)}$, and $\alpha \in (0, 1]$.

From (14), we get

$$\left(\triangle_M u, u \, |u|^{p-2} \phi_j \triangle_M q^k\right) \geq -\left|\left(\triangle_M u, u \, |u|^{p-2} \phi_j \triangle_M q^k\right)\right|$$

$$\geq (1-p)\left(\triangle_M u, q^k |u|^{p-2} \phi_j \triangle_M u\right) + (\alpha - 1)\left(qu, q^k u \, |u|^{p-2} \phi_j\right). \tag{15}$$

From (15) into (12), we obtain

$$\begin{aligned}
\left(f, q^k u \, |u|^{p-2} \phi_j\right) \geq\; & -(p-1)(p-2)\left(\triangle_M u q^k du, u \, |u|^{p-4} \phi_j du\right) \\
& + \left(\triangle_M u, q^k u \, |u|^{p-2} \triangle_M \phi_j\right) + (p-1)\left(\triangle_M u, q^k |u|^{p-2} \phi_j \triangle_M u\right) \\
& + (1-p)\left(\triangle_M u, q^k |u|^{p-2} \phi_j \triangle_M u\right) \\
& + (\alpha - 1)\left(qu, q^k u \, |u|^{p-2} \phi_j\right) + \left(qu, q^k u \, |u|^{p-2} \phi_j\right) \\
=\; & -(p-1)(p-2)\left(\triangle_M u q^k du, u \, |u|^{p-4} \phi_j du\right) \\
& + \left(\triangle_M u, q^k u \, |u|^{p-2} \triangle_M \phi_j\right) + \alpha\left(u, q^{k+1} u \, |u|^{p-2} \phi_j\right).
\end{aligned} \tag{16}$$

Now, we use the inequality:

$$|ab| \leq \frac{|a|^p}{\lambda^p} + \lambda \, |b|^t, \tag{17}$$

where $\frac{1}{p} + \frac{1}{t} = 1$, $a, \; b \in R$, and $\lambda \in (0, 1)$. Since $\phi_j \leq 1$ and $t = \frac{p}{p-1} > 1$, this implies $(\phi_j)^t \leq \phi_j$.

Using this and (17), we have

$$\begin{aligned}
\left(f, q^k u \, |u|^{p-2} \phi_j\right) &\leq \left|\left(f, q^k u \, |u|^{p-2} \phi_j\right)\right| \\
&\leq \frac{1}{\lambda^p} \int_M |f|^p \, d\mu + \lambda \int_M (\phi_j)^t q^{kt} \, |u|^t \, |u|^{(p-2)t} \, d\mu \\
&\leq \lambda^{-p} \left\|f\right\|_p^p + \lambda \int_M \phi_j q^{kt} \, |u|^t \, |u|^{(p-2)t} \, d\mu \\
&= \lambda^{-p} \left\|f\right\|_p^p + \lambda\left(q^{\frac{kp}{p-1}} |u|, \phi_j \, |u|^{p-1}\right).
\end{aligned} \tag{18}$$

From (18) into (16), we get

$$\left(\triangle_M u, q^k u\,|u|^{p-2}\,\triangle_M \phi_j\right) + \alpha\left(u, q^{k+1} u\,|u|^{p-2}\,\phi_j\right)$$
$$-(p-1)(p-2)\left(\triangle_M u q^k du, u\,|u|^{p-4}\,\phi_j du\right) \le \lambda^{-p}\,\|f\|_p^p + \lambda\left(q^{\frac{kp}{p-1}}\,|u|\,, \phi_j\,|u|^{p-1}\right).$$

Since $k \le p-1$ and $\lambda \in (0,1)$ is arbitrary, we can choose a sufficiently small $\lambda > 0$ such that

$$-(p-1)(p-2)\left(\triangle_M u q^k du, u\,|u|^{p-4}\,\phi_j du\right)$$
$$+\left(\triangle_M u, q^k u\,|u|^{p-2}\,\triangle_M \phi_j\right) + \frac{\alpha}{2}\left(u, q^{k+1} u\,|u|^{p-2}\,\phi_j\right) \le \lambda^{-p}\,\|f\|_p^p. \tag{19}$$

By Fatou's lemma, we have

$$\int_M q^{k+1}\,|u|^p\ d\mu \le \lim_{j\to\infty}\inf\left(u, q^{k+1} u\,|u|^{p-2}\,\phi_j\right). \tag{20}$$

Combining (19) and (20) and using (9) and (10), we obtain $\int_M q^{k+1}\,|u|^p\ d\mu \le C_1\,\|f\|_p^p$, where $C_1 \ge 0$ is a constant independent of $u$, which is the proof of (11) and the lemma. □

## Preparatory result

The following proposition is the most important result of this section.

**Proposition 1** *Assume that $(M, g)$ is a connected $C^\infty$−Riemannian manifold without boundary, with metric g and has bounded geometry. Assume that the hypotheses (7), (8), and (9) of the Lemma 1 are satisfied. Then*

$$\|qu\|_p \le C\,\|f\|_p, \tag{21}$$

*where $C \ge 0$ is a constant independent of u.*

*Proof* Let m be an integer such that $\frac{m}{2} < p \le \frac{m+1}{2}$. By the result (11) in Lemma 1 with $k = -\frac{1}{2},\ 0,\ \frac{1}{2},\ 1,\ \frac{3}{2}, ..., \frac{m}{2}$, we get $q^{\frac{1}{2}}\,|u|^p \in L^1(M)\,,\ q\,|u|^p \in L^1(M)\,, ..., q^{\frac{m}{2}+1}\,|u|^p \in L^1(M)$. Since $q(x) \ge \gamma > 0$, thus $|u|^p\,q^{p-\frac{1}{2}} = |u|^p\,q^{\frac{m}{2}+1} q^\beta \le |u|^p\,q^{\frac{m}{2}+1}\gamma^\beta$,where $\beta = p - \frac{m+1}{2} \le 0$. This implies $|u|^p\,q^{(p-1)+\frac{1}{2}} \in L^1(M)$, so by (11) (for $k = p-1$), we obtain $q^p\,|u|^p \in L^1(M)$ and $\int_M q^p\,|u|^p\ d\mu \le C_1\,\|f\|_p^p$, which implies $\|qu\|_p^p \le C_1\,\|f\|_p^p$, that is $\|qu\|_p \le C\,\|f\|_p$ ,where $C \ge 0$ is a constant independent of $u$. Hence, the proof of the proposition. □

**Lemma 2** *Let $(M, g)$ be a Remannian manifold, and let $u \in L^1_{loc}(M)\,,\ \triangle_M u \in L^1_{loc}(M)$. Then, $\triangle_M^2\,|u| \le Re\left((\triangle_M^2 u) sign\overline{u}\right)$, where $signu(x) = \begin{cases} \frac{u(x)}{|u(x)|} & if\ u(x) \ne 0 \\ 0 & otherwise \end{cases}$. See [13].*

**Distributional inequality** For $1 < p < \infty$ and $\lambda > 0$, we consider the inequality, $\left(\triangle_M^2 + \lambda\right) u = \nu \ge 0,\ u \in L^p(M)$ ,where $\nu \ge 0$ means that $\nu$ is a positive distribution, i.e., $\langle \nu, \phi\rangle \ge 0$ for every $0 \le \phi \in C_c^\infty(M)$. See [14].

**Lemma 3** *Let $(M, g)$ be a manifold of bounded geometry and let $1 < p < \infty$. If $u \in L^p(M)$ satisfies the distributional inequality: $\left(\triangle_M^2 + \lambda\right) u \ge 0$,then $u \ge 0$ (almost every where or, equivalently, as a distribution). See [15].*

**Lemma 4** *If $u \in L^p(M)$ satisfies the equation $\triangle_M^2 u + qu = 0$, (which is understood in distributional sense), then $u = 0$.*

*Proof* Since $q \in C^1(M) \subset L^\infty_{loc}(M)$, it follows that $qu \in L^1_{loc}(M)$. Since we have $\triangle_M^2 u + qu = 0$, it follows that $\triangle_M^2 u = -qu \in L^1_{loc}(M)$. From Lemma 2 and the assumption $q \geq \gamma > 0$, we get

$$\triangle_M^2 |u| \leq \mathrm{Re}\left((\triangle_M^2 u) sign\overline{u}\right) = -\mathrm{Re}\left((qu) sign\overline{u}\right) = -qu\frac{\overline{u}}{|\overline{u}|} = -q\frac{|u|^2}{|u|} = -q|u| \leq -\gamma |u|,$$

which implies $\left(\triangle_M^2 + \gamma\right)|u| \leq 0$. From Lemma 3, we get $|u| \leq 0$. This implies $u = 0$, hence the proof. □

## The Main result

We now introduce our main result of this paper.

**Theorem 1** *Assume that $(M, g)$ is a connected $C^\infty-$Riemannian manifold without boundary, with metric g and has bounded geometry. Assume that the assumption (7) of the Lemma 1 is satisfied. Then*

$$\|qu\|_p \leq C\|Au\|_p, \text{ for all } u \in D_p, \tag{22}$$

*where $C \geq 0$ is a constant independent of u.*

*Proof* Let $u \in D_p$ and

$$\left(\triangle_M^2 + q\right)u = f, \tag{23}$$

so $f \in L^p(M)$. Thus, there exist a sequence $(f_j)$ in $C_c^\infty(M)$ such that $f_j \to f$ in $L^p(M)$ as $j \to \infty$. Let $T$ be the closure of $\left(\triangle_M^2 + q\right)|_{C_c^\infty(M)}$ in $L^p(M)$. By [15], it follows that:

(i) $Dom(T) = D_p$, and $Tu = \left(\triangle_M^2 + q\right)u$ for all $u \in D_p$.

(ii) The operator $T$ is invertible, and $T^{-1} : L^p(M) \to L^p(M)$ is a bounded linear operator.

Consider the sequence $T^{-1}f_j = w_j$, since $T^{-1} : L^p(M) \to L^p(M)$ is a bounded linear operator, so $w_j \to T^{-1}f$ in $L^p(M)$ as $j \to \infty$. Let $w = T^{-1}f$. Using the property (i) of $T$, we get

$$\left(\triangle_M^2 + q\right)w = f. \tag{24}$$

From (23) and (24), we get $\left(\triangle_M^2 + q\right)(u - w) = 0$. By Lemma 4, we obtain $u = w$. Since $T^{-1}f_j = w_j$, it follows that $w_j \in D_p$, and by the property (i) of $T$, we get

$$\left(\triangle_M^2 + q\right)w_j = f_j. \tag{25}$$

In (25), we have $q \in C^1(M)$ and $f_j \in C_c^\infty(M)$, so by elliptic regularity, we get $w_j \in W^{2,p}_{loc}(M)$. By Sobolev embedding theorem [16], we get $w_j \in W^{2,p}_{loc}(M) \subset L^t_{loc}(M)$, where $\frac{1}{t} = \frac{1}{p} - \frac{2}{m}$. Hence, $qw_j \in L^t_{loc}(M)$. Using elliptic regularity again, we get $w_j \in W^{2,t}_{loc}(M)$ with $t > p$. Applying the same procedure, we will obtain $w_j \in C^1(M)$. Thus, $w_j \in C^1(M) \cap L^p(M)$ satisfies the conditions of Proposition 1. From (25) for $j, r = 1, 2, ...,$ we get $\left(\triangle_M^2 + q\right)(w_j - w_r) = f_j - f_r$. Also, from (21), we get

$$\left\|q(w_j - w_r)\right\|_p \leq C\left\|f_j - f_r\right\|_p. \tag{26}$$

Since $(f_j)$ is a cauchy sequence in $L^p(M)$, from (26), it follows that $(qw_j)$ is also a cauchy sequence in $L^p(M)$, which implies $(qw_j)$ converges to $s \in L^p(M)$. Let $\Psi \in C_c^\infty(M)$, then $0 = (qw_j, \Psi) - (w_j, q\Psi) \to (s, \Psi) - (w, q\Psi) = (s - qw, \Psi)$. So $qw = s$ (because $C_c^\infty(M)$ is dense in $L^p(M)$). Hence, $qw_j \to qw$ in $L^p(M)$ as $j \to \infty$. But, we have $u = w$, so $qu = qw$. Since we have $\|qw_j\|_p \leq C\|f_j\|_p$, by taking the limit as $j \to \infty$, we obtain $\|qu\|_p \leq C\|f\|_p = C\|Au\|_p$, where $C \geq 0$ is a constant independent of $u$. This concludes the proof of the Theorem. □

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