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Complete decomposable *MS*-algebras

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Abstract

According to the characterization of decomposable *MS*-algebras in terms of triples (M, D, φ), where M is a de Morgan algebra, D is a distributive lattice with 1 and φ is a (0,1)-homomorphism of M into $F(D)$, the filter lattice of D, we characterize complete decomposable *MS*-algebras in terms of complete decomposable *MS*-triples. Also, we describe the complete homomorphisms of complete decomposable *MS*-algebras by means of complete decomposable *MS*-triples.

Keywords: *MS*-algebras, Complete lattice, Complete decomposable *MS*-algebras, Complete decomposable *MS*-triples, Triple homomorphisms, Complete homomorphisms

AMS Mathematics Subject Classification (2010): Primary 06D30; Secondary 06D15.

Introduction

Morgan Stone algebras (or simply *MS*-algebras) are introduced and characterized by T.S. Blyth and J.C. Varlet [1] as a generalization of both de Morgan algebras and Stone algebras. In [2], T.S. Blyth and J.C. Varlet described the lattice $\Lambda(\mathbf{MS})$ of subclasses of the class **MS** of all *MS*-algebras. A. Badawy, D. Guffova, and M. Haviar [3] introduced and characterized decomposable *MS*-algebras by means of decomposable *MS*-triples. Moreover, they constructed a one-to-one correspondence between decomposable *MS*-algebras and decomposable *MS*-triples. A. Badawy and R. El-Fawal [4] studied many properties of decomposable *MS*-algebras in terms of decomposable *MS*-triples as homomorphisms and subalgebras. Also, they formulated and solved some fill in problems concerning homomorphisms and subalgebras of decomposable *MS*-algebras. A. Badawy [5] introduced the notion of d_L-filters of principal *MS*-algebras. Recently, A. Badawy [6] studied the relationship between de Morgan filters and congruences of decomposable *MS*-algebras. Also, many properties of ideals of *MS*-algebras are given in [7] and [8].

Several authors studied complete p-algebras, like C.C. Chain and G. Grätzer [9] for Stone algebras, S. El-Assar, and M. Atallah [10] for distributive p-algebras and P. Mederly [11] for modular p-algebras.

In this paper, we introduce complete decomposable *MS*-algebras and complete decomposable *MS*-triples. We show that a decomposable *MS*-algebra L constructed from the decomposable *MS*-triple (M, D, φ) is complete if and only if the triple (M, D, φ) is complete. Also, a description of complete homomorphisms of decomposable *MS*-algebras is given in terms of complete decomposable *MS*-triples.

Preliminaries

In this section, we present definitions and main results which are needed through this paper. We refer the reader to [1–4, 12–15] for more details.

A de Morgan algebra is an algebra $(L; \vee, \wedge, ^{-}, 0, 1)$ of type (2,2,1,0,0) where $(L; \vee, \wedge, 0, 1)$ is a bounded distributive lattice and the unary operation of involution $^{-}$ satisfies

$\overline{\overline{x}} = x, \overline{(x \vee y)} = \overline{x} \wedge \overline{y}, \overline{(x \wedge y)} = \overline{x} \vee \overline{y}.$

An *MS*-algebra is an algebra $(L; \vee, \wedge, ^{\circ}, 0, 1)$ of type (2,2,1,0,0) where $(L; \vee, \wedge, 0, 1)$ is a bounded distributive lattice and the unary operation $^{\circ}$ satisfies

$x \leq x^{\circ\circ}, (x \wedge y)^{\circ} = x^{\circ} \vee y^{\circ}, 1^{\circ} = 0.$

The following Theorem gives the basic properties of *MS*-algebras.

Theorem 1 *([1, 12]). For any two elements a, b of an MS-algebra L, we have*
(1) $0^{\circ} = 1$,
(2) $a \leq b \Rightarrow b^{\circ} \leq a^{\circ}$,
(3) $a^{\circ\circ\circ} = a^{\circ}$,
(4) $(a \vee b)^{\circ} = a^{\circ} \wedge b^{\circ}$,
(5) $(a \vee b)^{\circ\circ} = a^{\circ\circ} \vee b^{\circ\circ}$,
(6) $(a \wedge b)^{\circ\circ} = a^{\circ\circ} \wedge b^{\circ\circ}$.

Lemma 1 *([1, 3]). Let L be an MS-algebra. Then*
(1) $L^{\circ\circ} = \{x \in L : x = x^{\circ\circ}\}$ *is a de Morgan subalgebra of L,*
(2) $D(L) = \{x \in L : x^{\circ} = 0\}$ *is a filter (filter of dense elements) of L.*

For any lattice L, let $F(L)$ denotes the set of all filters of L. It is known that, $(F(L); \wedge, \vee)$ is a distributive lattice if and only if L is a distributive lattice, where the operation $\wedge$ and $\vee$ are given by

$F \wedge G = F \cap G$ and $F \vee G = \left\{x \in L : x \geq f \wedge g, f \in F, g \in G\right\}$, respectively for every $F, G \in F(L)$.

Also, $[a) = \{x \in L : x \geq a\}$ is a principal filter of L generated by a.

Definition 1 [9]. *Let* $L = (L; \vee, \wedge, 0_L, 1_L)$ *and* $L_1 = (L_1; \vee, \wedge, 0_{L_1}, 1_{L_1})$ *be bounded lattices. The map* $h : L \to L_1$ *is called (0,1)-lattice homomorphism if*
(1) $0_L h = 0_{L_1}$ *and* $1_L h = 1_{L_1}$,
(2) h preserves joins, that is, $(x \vee y)h = xh \vee yh$ *for every* $x, y \in L$,
(3) h preserves meets, that is, $(x \wedge y)h = xh \wedge yh$ *for every* $x, y \in L$.

Definition 2 [14] *A (0,1)-lattice homomorphism* $h : L \to L_1$ *of an MS-algebra L into an MS-algebra* L_1 *is called a homomorphism if* $x^{\circ}h = xh^{\circ}$ *for all* $x \in L$. *If L and* L_1 *are de Morgan algebras, then h is called a de Morgan homomorphism.*

Definition 3 [3] *An MS-algebra L is called decomposable MS-algebra if for every* $x \in L$ *there exists* $d \in D(L)$ *such that* $x = x^{\circ\circ} \wedge d$.

Definition 4 [3] *A decomposable MS-triple is* (M, D, φ), *where*
(i) $(M; \vee, \wedge, ^{-}, 0, 1)$ *is a de Morgan algebra,*
(ii) $(D; \vee, \wedge, 1)$ *is a distributive lattice with 1,*

(iii) φ *is a* $(0,1)$*-homomorphism from* M *into* $F(D)$ *such that for every element* $a \in M$ *and for every* $y \in D$ *there exists an element* $t \in D$ *with* $a\varphi \cap [y) = [t)$.

Theorem 2 [3] *(Construction Theorem) Let* (M, D, φ) *be a decomposable MS-triple. Then*

$$L = \{(a, \bar{a}\varphi \vee [x)) : a \in M, x \in D\}$$

is a decomposable MS-algebra, if we define

$$(a, \bar{a}\varphi \vee [x)) \vee (b, \bar{b}\varphi \vee [y)) = \left(a \vee b, \overline{(a \vee b)}\varphi \vee [t)\right) \text{ for some } t \in D,$$

$$(a, \bar{a}\varphi \vee [x)) \wedge (b, \bar{b}\varphi \vee [y)) = \left(a \wedge b, \overline{(a \wedge b)}\varphi \vee [x \wedge y)\right),$$

$$(a, \bar{a}\varphi \vee [x))^{\circ} = (\bar{a}, a\varphi),$$

$$1_L = (1, [1)),$$

$$0_L = (0, D).$$

Conversely, every decomposable MS-algebra L can be associated with the decomposable MS-triple $(L^{\circ\circ}, D(L), \varphi(L))$*, where*
$a\varphi(L) = [a^{\circ})(L), a \in L^{\circ\circ}$.

The decomposable MS-algebra L constructed in Theorem 2 is called the decomposable MS-algebra associated with the decomposable MS-triple (M, D, φ) and the construction of L described in Theorem 2 is called a decomposable MS-construction.

Corollary 1 [3] *Let L be a decomposable MS-algebra associated with the decomposable MS-triple* (M, D, φ)*. Then*
(1) $L^{\circ\circ} = \{(a, \bar{a}\varphi) : a \in M\}$,
(2) $D(L) = \{(1, [x)) : x \in D\}$,
(3) $D \cong D(L)$ *and* $M \cong L^{\circ\circ}$,
(4) The order of L is given as follows: $(a, \bar{a}\varphi \vee [x)) \leq (b, \bar{b}\varphi \vee [y))$ *iff* $a \leq b$ *and* $\bar{a}\varphi \vee [x) \supseteq \bar{b}\varphi \vee [y)$.

Definition 5 [14] *A lattice L is called complete if* $\inf_L H$ *and* $\sup_L H$ *exist for each* $\phi \neq H \subseteq L$.

Definition 6 [14] *A lattice L is called conditionally complete if every upper bounded subset of L has a supermum in L and every lower bounded subset of L has an infimum in L.*

An MS-algebra L is called complete if it is complete as a lattice.

Definition 7 [14] *A lattice homomorphism* $h : L \to L_1$ *of a complete lattice L into a complete lattice* L_1 *is called complete if*
$(\inf_L H)h = \inf_{L_1} Hh$ *and* $(\sup_L H)h = \sup_{L_1} Hh$ *for each* $\phi \neq H \subseteq L$.

A homomorphism $h : L \to L_1$ of a complete MS-algebra L into a complete MS-algebra L_1 is called complete if it is complete as a lattice homomorphism.

Characterization of complete decomposable *MS*-algebras via triples

In this section, we introduce and characterize complete decomposable *MS*-triples of complete decomposable *MS*-algebras.

Let L be a decomposable *MS*-algebra L. For $\phi \neq N \subseteq L$, define N° as follows:
$N^{\circ} = \{n^{\circ} : n \in N\}$.

Lemma 2 *If L is a complete decomposable MS-algebra, then for* $\phi \neq N \subseteq L$, $\phi \neq C \subseteq L^{\circ\circ}$ *and* $\phi \neq E \subseteq D(L)$, *we have*
(1) $(\sup_L N)^{\circ} = \inf_L N^{\circ}$,
(2) $\sup_{L^{\circ\circ}} C = (\sup_L C)^{\circ\circ} = (\inf_L C^{\circ})^{\circ}$,
(3) $\inf_{L^{\circ\circ}} C = \inf_L C$,
(4) $\inf_{D(L)} E = \inf_L E$ *and* $\sup_{D(L)} E = \sup_L E$.

Proof (1). Let $x = \sup_L N$. Then $x \geq n$ for all $n \in N$ implies $x^{\circ} \leq n^{\circ}$. Hence x° is a lower bound of N°. Let y be a lower bound of N°. Then $y \leq n^{\circ}$ for all $n \in N$ implies $y^{\circ} \geq n^{\circ\circ} \geq n$. So, y° is an upper bound of N. Thus $x \leq y^{\circ}$ as $x = \sup_L N$. This gives $x^{\circ} \geq y^{\circ\circ} \geq y$. Therefore $x^{\circ} = \inf_L N^{\circ} = (\sup_L N)^{\circ}$.
(2) Let $\sup_L C = x$. Then $x^{\circ\circ} = (\sup_L C)^{\circ\circ}$. We have to show that $x^{\circ\circ} = \sup_{L^{\circ\circ}} C$. Since $\sup_L C = x$, then $x \geq c$ for all $c \in C$. so, $x^{\circ\circ} \geq c^{\circ\circ} = c$ for all $c \in C$. Therefore $x^{\circ\circ}$ is an upper bound of C. Let y be another upper bound of C in $L^{\circ\circ}$. Then $y \geq c$ for all $c \in C$. Thus $y^{\circ\circ} \geq c^{\circ\circ} = c$. Hence $y^{\circ\circ}$ is an upper bound of C. So $y^{\circ\circ} \geq x$ as $x = \sup_L C$. It follows that $y = y^{\circ\circ} \geq x^{\circ\circ}$. Hence $x^{\circ\circ}$ is the least upper bound of C. Since $x^{\circ\circ} \in L^{\circ\circ}$, then $x^{\circ\circ} = \sup_{L^{\circ\circ}} C$. By (1) we have $(\sup_L C)^{\circ\circ} = (\inf_L C^{\circ})^{\circ}$.
(3) Let $x = \inf_L C$. Then $x \leq c$ for all $c \in C$. Then $x^{\circ\circ} \leq c^{\circ\circ} = c$. Hence $x^{\circ\circ}$ is a lower bound of C. Thus $x \geq x^{\circ\circ}$ as $x = \inf_L C$. But $x \leq x^{\circ\circ}$. Then $x^{\circ\circ} = x$ and $x \in L^{\circ\circ}$. Thus $\inf_{L^{\circ\circ}} C = x$.
(4) Let $x = \inf_L E$ and $y = \inf_{D(L)} E$. Then $x \leq e$ and $y \leq e$ for all $e \in E$ imply that $x = y$. Now we prove $\sup_{D(L)} E = \sup_L E$. Let $y = \sup_L E$. Then $y \geq e$ for all $e \in E$. It follows that $y^{\circ} \leq e^{\circ} = 0$. Then $y \in D(L)$ implies $y = \sup_{D(L)} E$. □

Let (M, D, φ) be a decomposable *MS*-triple. For any $\emptyset \neq E \subseteq D$, consider the set M_E as follows:
$M_E = \{a \in M : \bar{a}\varphi \vee [z) \supset E \text{ for some } z \in D\}$.

Lemma 3 *Let* (M, D, φ) *be a decomposable MS-triple. For any* $\emptyset \neq E \subseteq D$, *we have*
(1) M_E *is an ideal of* M,
(2) $[E) = \cup\{[t) : t \in E\}$,
(2) $M_E = M_{[E)}$.

Proof (1). Let $a, b \in M_E$. Then $\bar{a}\varphi \vee [z_1) \supset E$ and $\bar{b}\varphi \vee [z_2) \supset E$ for some $z_1, z_2 \in D$. Hence $E \subset (\bar{a}\varphi \vee [z_1)) \cap (\bar{b}\varphi \vee [z_2)) = \overline{(a \vee b)}\varphi \vee [t)$ for some $t \in D$ (see Theorem 2). It follows that $a \vee b \in M_E$. Now, let $a \in M_E$ and $c \in M$. Then, $\exists z \in D$ such that $\bar{a}\varphi \vee [z) \supset E$. Since $a \wedge c \leq a$, then $\overline{a \wedge c} \geq \bar{a}$. This gives $\overline{(a \wedge c)}\varphi \supseteq \bar{a}\varphi$. It follows that $\overline{(a \wedge c)}\varphi \vee [z) \supseteq \bar{a}\varphi \vee [z) \supset E$. Then $a \wedge c \in M_E$. Consequently, M_E is an ideal of M.
(2) Obvious.

(3) Clearly, $M_{[E)} \subseteq M_E$. Let $a \in M_E$. Then, $\exists z \in D$ such that $\bar{a}\varphi \vee [z) \supset E$. Since $\bar{a}\varphi \vee [z)$ is a filter of D and $[E)$ is the smallest filter of D containing E, then $\bar{a}\varphi \vee [z) \supset [E)$. Hence, $a \in M_{[E)}$ and $M_E \subseteq M_{[E)}$. Therefore ,$M_E = M_{[E)}$. □

Definition 8 *A complete decomposable MS-triple is a decomposable MS-triple (M, D, φ) satisfying the following conditions:*
(i) M is complete,
(ii) D is conditionally complete,
(iii) For each $\emptyset \neq E \subseteq D$, the set M_E has the greatest element in M.

Theorem 3 *Let L be a complete decomposable MS-algebra constructed from the decomposable MS-triple (M, D, φ). Then, the triple (M, D, φ) is complete.*

Proof Since L is associated with the decomposable MS-triple (M, D, φ), then by Theorem 2, we have
$L = \{(a, \bar{a}\varphi \vee [x)) : a \in M, x \in D\}$.
Corollary 1(1)-(3), gives
$L^{\circ\circ} = \{(a, \bar{a}\varphi) : a \in M\} \cong M$ and $D(L) = \{(1, [x)) : x \in D\} \cong D$.
We have to prove that a decomposable MS-triple (M, D, φ) is complete. So we proceed to prove (i)–(iii) of Definition 8. For (i), let $\emptyset \neq C \subseteq M$. Consider a subset $\acute{C} = \{(c, \bar{c}\varphi) : c \in C\}$ of $L^{\circ\circ}$ corresponding to C. Since L is complete, then $\inf_L \acute{C} = (a, \bar{a}\varphi \vee [x))$ for some $(a, \bar{a}\varphi \vee [x)) \in L$. Thus, $(a, \bar{a}\varphi \vee [x)) \leq (c, c\varphi)$ for all $c \in C$. Then $a \leq c$ for all $c \in C$ implies that a is a lower bound of C. We verify that a is the greatest lower bound of C in M. Let b be a lower bound of C. Then $b \leq c$ for all $c \in C$. This gives $\bar{b}\varphi \supseteq \bar{c}\varphi$. Therefore, $(b, \bar{b}\varphi) \leq (c, \bar{c}\varphi)$ for all $c \in C$ and $(b, b\varphi)$ is a lower bound of $\acute{C}$. Then $(a, \bar{a}\varphi \vee [x)) \geq (b, b\varphi)$ as $\inf_L C = (a, \bar{a}\varphi \vee [x))$. Consequently, $a \geq b$ and $a = \inf_M C$. Since $a = \inf_M C$ and M is bounded above by 1, then, M is complete.
Now we prove (ii). Let $\phi \neq E \subseteq D$. Consider $\acute{E} \subseteq D(L)$ corresponding to E. Then
$\acute{E} = \{(1, [e)) : e \in D\}$.
Let z be a lower bound of E. Since L is complete, then $\inf_L \acute{E}$ exists. Let $\inf_L \acute{E} = (a, \bar{a}\varphi \vee [x))$. Since $z \leq e$ for all $e \in E$ as z is a lower bound of E. Then, $[z) \supseteq [e)$ and $(1, [z)) \leq (1, [e))$. Thus, $(1, z)$ is a lower bound of $\acute{E}$. Then, $(a, \bar{a}\varphi \vee [x)) \geq (1, [z))$ because of $\inf_L \acute{E} = (a, \bar{a}\varphi \vee [x))$. This implies that $a \geq 1$ and $\bar{a}\varphi \vee [x) \subseteq [z)$. Consequently, $a = 1$ and $\bar{a}\varphi \vee [x) = 0\varphi \vee [x) = [x)$. Thus $[x) \subseteq [z)$ implies $x \geq z$. This shows that x is the greatest lower bound of E in D and $x = \inf_D E$. Using a similar way, we can show that, if E has an upper bound, then $\sup_D E$ exists. Therefore, D is a conditionally complete lattice as required.
Now we prove (iii). Let $\emptyset \neq E \subseteq D$. Consider $\acute{E} \subseteq D(L)$ corresponding to E. Then

$$\acute{E} = \{(1, [x)) : x \in E\}.$$

Since L is complete, then $\inf_L \acute{E}$ exists. Let $(b, \bar{b}\varphi \vee [z)) = \inf_L \acute{E}$. We show that b is the largest element of M_E. Since $(b, \bar{b}\varphi \vee [z)) = \inf_L \acute{E}$, then $(b, \bar{b}\varphi \vee [z)) \leq (1, [x))$, $\forall x \in E$. This gives $b \leq 1$ and $\bar{b}\varphi \vee [z) \supseteq [x)$, $\forall x \in E$. Therefore, $\bar{b}\varphi \vee [z) \supseteq \cup_{x \in E}[x) = [E) \supset E$. Thus, $b \in M_E$. Now, let $c \in M_E$. Then $\bar{c}\varphi \vee [y) \supset E$ for some $y \in D$. It follows that $\bar{c}\varphi \vee [y) \supseteq [E) \supseteq [x)$ for all $x \in E$. Hence, $(1, [x)) \leq (c, \bar{c}\varphi \vee [y))$ for all $x \in E$. Thus, $(c, \bar{c}\varphi \vee [y))$ is a lower bound of $\acute{E}$ and therefore $(c, \bar{c}\varphi \vee [y)) \leq (b, \bar{b}\varphi \vee [z))$. Then, $c \leq b$.

This deduce that b is the largest element of M_E in M. Therefore, (M, D, φ) is a complete decomposable MS-triple. □

The converse of the above theorem is given in the following.

Theorem 4 *Let L be a decomposable MS-algebra constructed from the complete decomposable MS-triple* (M, D, φ)*. Then L is complete.*

Proof Let (M, D, φ) be a complete decomposable MS-triple. Then –(iii) of Definition 8 hold. Let $\emptyset \neq N \subseteq L$, where L is constructed as in construction Theorem from the decomposable MS-triple (M, D, φ) as follows:

$L = \{(a, \bar{a}\varphi \vee [x)) : a \in M, x \in D\}$.

Since L is bounded, it is enough to show the existence of $\inf_L N$. Denote $a = \inf_M N^{\circ\circ}$ and $F = \cup\{[t) : (c, \bar{c}\varphi \vee [t)) \in N \text{for some } c \in M\}$ ($\cup$ means the union in $F(D)$). Let $b = \max M_F$. Now, we prove that there exists an element $z \in D$ such that $\bar{b}\varphi \vee [z) \supset F$ and if $\bar{b}\varphi \vee [y) \supset F$ for some $y \in D$ then $\bar{b}\varphi \vee [y) \supseteq \bar{b}\varphi \vee [z)$. For this purpose, consider the following set:

$\left\{x_\gamma : \gamma \in \Gamma \text{for all} x_\gamma \text{with} \bar{b}\varphi \vee [x_\gamma) \supset F\right\}$.

Thus, we have to find a $z \in D$ with $\bar{b}\varphi \vee [y) \supset F$ and $\bar{b}\varphi \vee [y) \supseteq \bar{b}\varphi \vee [z)$ for all $\gamma \in \Gamma$. The set $\left\{x_\gamma : \gamma \in \Gamma \text{for all} x_\gamma \text{with} \bar{b}\varphi \vee [x_\gamma) \supset F\right\}$ is bounded from above. Then, by (ii), there exists $s = \sup_D\{x_\gamma : \gamma \in \Gamma\}$. We prove that $\cap_{\gamma \in \Gamma}[x_\gamma) = [s)$.

$$\begin{aligned} y \in \cap_{\gamma\in\Gamma}[x_\gamma) &\Leftrightarrow y \in [x_\gamma), \ \forall \gamma \in \Gamma \\ &\Leftrightarrow y \geq x_\gamma, \ \forall \gamma \in \Gamma \\ &\Leftrightarrow y \text{ is an upper bound of } \{x_\gamma : \gamma \in \Gamma\} \\ &\Leftrightarrow y \geq s \text{ as } s = \sup_D\{x_\gamma : \gamma \in \Gamma\} \\ &\Leftrightarrow y \in [s). \end{aligned}$$

Then it is sufficient to prove the following equality.

$$\cap_{\gamma\in\Gamma} (\bar{b}\varphi \vee [x_\gamma)) = \bar{b}\varphi \vee \cap_{\gamma\in\Gamma}[x_\gamma) = \bar{b}\varphi \vee [s). \tag{1}$$

Let $t \in \bar{b}\varphi \vee [s)$. Then

$$\begin{aligned} t \in \bar{b}\varphi \vee [s) &\Rightarrow t \geq t_1 \wedge s \text{ where } t_1 \in \bar{b}\varphi \\ &\Rightarrow t \geq t_1 \wedge (s \vee x_\gamma) \text{ as } s \geq x_\gamma \text{for all } \gamma \in \Gamma \\ &\Rightarrow t \geq (t_1 \wedge s) \vee (t_1 \wedge x_\gamma) \\ &\Rightarrow t \geq t_1 \wedge x_\gamma \\ &\Rightarrow t \in \bar{b}\varphi \vee [x_\gamma) \text{ for all } \gamma \in \Gamma. \end{aligned}$$

Then $\bar{b}\varphi \vee \cap_{\gamma\in\Gamma}[x_\gamma) \subseteq \bar{b}\varphi \vee [x_\gamma)$ implies $\bar{b}\varphi \vee \cap_{\gamma\in\Gamma}[x_\gamma) \subseteq \cap_{\gamma\in\Gamma}(\bar{b}\varphi \vee [x_\gamma))$. Conversely, let $y \in \cap_{\gamma\in\Gamma}(\bar{b}\varphi \vee [x_\gamma))$. Then $y \in \bar{b}\varphi \vee [x_\gamma)$ for all $\gamma \in \Gamma$. Hence $y \geq t \wedge z$ for $t \in \bar{b}\varphi$ and $z \in [x_\gamma)$ for all $\gamma \in \Gamma$. It follows that $z \geq x_\gamma$ for all $\gamma \in \Gamma$. This means that z is an upper bound of the set $\{x_\gamma : \gamma \in \Gamma\}$. Then $s \leq z$ as $s = \sup_D\{x_\gamma : \gamma \in \Gamma\}$. Now

$$\begin{aligned} y &\geq t \wedge z \\ &= t \wedge (s \vee z) \text{ as } s \leq z \\ &= (t \wedge s) \vee (t \wedge z) \text{ by distributivity of } D \\ &\geq t \wedge s \in \bar{b}\varphi \vee [s). \end{aligned}$$

Then $y \in \bar{b}\varphi \vee [s)$. Therefore, $\cap_{\gamma \in \Gamma}(\bar{b}\varphi \vee [x_\gamma)) \subseteq \bar{b}\varphi \vee [s)$.

We prove the existence of $\inf_L N$. First, we claim that
$i = \left(a \wedge b, \overline{(a \wedge b)}\varphi \vee [z)\right) = \inf_L N$ (we put then$z = s$).
First, we show that i is a lower bound of N. Let $(f, \bar{f}\varphi \vee [y)) \in N$. Since $a = \inf_M N^{\circ\circ}$, we get $a \leq f$. So, $a \wedge b \leq a \leq f$. Then $a \wedge b \leq f$ implies that $\overline{a \wedge b} \geq \bar{f}$. Consequently, $\overline{(a \wedge b)}\varphi = \bar{a}\varphi \vee \bar{b}\varphi \supseteq \bar{f}\varphi$. Moreover, $[y) \subseteq F \subseteq \bar{b}\varphi \vee [z)$ as $y \in F$. Then

$$\begin{aligned} \overline{(a \wedge b)}\varphi \vee [z) &= (\bar{a} \vee \bar{b})\varphi \vee [z) \\ &= (\bar{a}\varphi \vee \bar{b}\varphi) \vee (\bar{b}\varphi \vee [z)) \\ &\supseteq \bar{f}\varphi \vee [y). \end{aligned}$$

Then $(a \wedge b, \overline{(a \wedge b)}\varphi \vee [z)) \leq (f, \bar{f} \vee [y))$ for all $(f, \bar{f} \vee [y)) \in N$. Therefore, i is a lower bound of N. It remains to show that i is the greatest lower bound of N. Let $(c, \bar{c}\varphi \vee [x))$ be a lower bound of N. Then, $(c, \bar{c}\varphi \vee [x)) \leq (f, \bar{f}\varphi \vee [y))$, $\forall (f, \bar{f}\varphi \vee [y)) \in N$. So, $c \leq f$, $\forall f \in N^{\circ\circ}$. Then c is a lower bound of $N^{\circ\circ}$. Thus $c \leq a$ as $a = \inf\limits_{M} N^{\circ\circ}$. On the other hand, $\bar{c}\varphi \vee [x) \supseteq \bar{f}\varphi \vee [y)$, $\forall (f, \bar{f}\varphi \vee [y)) \in N$. So, $\bar{c}\varphi \vee [x) \supseteq [y)$, $\forall y \in F$. Therefore, $\bar{c}\varphi \vee [x) \supseteq F$. Hence, $\bar{c}\varphi \vee [x) \supseteq \bar{b}\varphi \vee [z)$ by using equality (1). Then $\bar{c}\varphi \vee [x) \supseteq F$ implies that $c \in M_F$. So, $c \leq b$ as $b = \max\limits_{M} M_F \in M$. Now, we have $c \leq a$ and $c \leq b$. Then $c \leq a \wedge b$. Moreover, we have $\bar{c}\varphi \supseteq \bar{a}\varphi$ because of $c \leq a$. Also, $\bar{c}\varphi \vee [x) \supseteq \bar{b}\varphi \vee [z)$. So, $\bar{c}\varphi \vee [x) \supseteq \bar{a}\varphi \vee \bar{b}\varphi \vee [z) = \overline{(a \wedge b)}\varphi \vee [z)$. Therefore, $(c, \bar{c}\varphi \vee [x)) \leq i$. Then $i = \inf_L N$ and L is complete. □

Corollary 2 *If M and D are complete, then so is L.*

Proof . We need only to prove that the condition (*iii*) of Definition 8 holds. Let $E \subseteq D$ and $t = \inf_D E$. Then, $[t) = [\inf_D E) \supseteq E$. So, $(1, \bar{1}\varphi \vee [t)) = (1, [t)) \in L$. Therefore, $1 \in M_E$. Hence, by the above Theorem, L is complete. □

Corollary 3 *If M is finite and D is conditionally complete, then L is complete.*

Proof Since M is finite and M_E is an ideal of M (see Lemma 1(1)), then M is complete and M_E is a principal ideal of M. Therefore, M_E contains the greatest element in M. So, the conditions (*i*)–(*iii*) of Definition 8 are satisfied and consequently, L is complete. □

Combining Theorems 3 and 4, we get the following theorem.

Theorem 5 *Let L be a decomposable MS-algebra constructed from the decomposable MS-triple* (M, D, φ). *Then L is complete if and only if* (M, D, φ) *is complete.*

Let L be a complete decomposable *MS*-algebra. In the proof of Theorem 4 arbitrary meets in L are described. In the following Lemma, we describe joins in L.

Lemma 4 *Let L be a complete decomposable MS-algebra constructed from the decomposable MS-triple (M, D, φ). Let $\phi \neq N \subseteq L$ and $a = \sup_M N^{\circ\circ}$. Then there exists an element $z \in D$ such that $[z) = \bigcap\{\bar{c}\varphi\vee[t) : (c, \bar{c}\varphi\vee[t)) \in N\} \cap a\varphi$ and $\sup N = (a, \bar{a}\varphi\vee[z))$.*

Proof Let $\phi \neq N \subseteq L$ and $\sup_L N = (b, \bar{b}\varphi\vee[z))$. We can assume that $z \in a\varphi$. We prove that $b = a = \sup_M N^{\circ\circ}$. Using Lemma 2(2), we get

$\sup_M N^{\circ\circ} = (\sup_L N)^{\circ\circ} = (b, \bar{b}\varphi\vee[z))^{\circ\circ} = (b, \bar{b}\varphi)$.

But $a = (a, \bar{a}\varphi) = \sup_M N^{\circ\circ}$. Then $b = a$. Hence, $\bar{a}\varphi\vee[z)$ is the greatest filter of the form $\bar{a}\varphi\vee[x)$, $x \in D$ with

$\bar{a}\varphi\vee[z)) \subset \bar{c}\varphi\vee[t)$ for each $(c, \bar{c}\varphi\vee[t)) \in N$.

The last condition is equivalent to

$[z) \subset \bigcap\{\bar{c}\varphi\vee[t) : (c, \bar{c}\varphi\vee[t)) \in N\} \cap a\varphi$.

Let $\bigcap\{\bar{c}\varphi\vee[t) : (c, \bar{c}\varphi\vee[t)) \in N\} \cap a\varphi = R$. If $[z) \neq R$, then there is $y \in R$, $y \ngeq z$. It follows that $y\wedge z < z$ and $y\wedge z \in R$. Then $[z) \subset [y\wedge z)$ implies $\bar{a}\varphi\vee[z) \subset \bar{a}\varphi\vee[y\wedge z)$. Since $y\wedge z \in R$ then $[y\wedge z) \subset \bar{c}\varphi\vee[t)$ for all $(c, \bar{c}\varphi\vee[t)) \in N$. Since $a \geq c$ (as $a = \sup_M N^{\circ\circ}$) then $\bar{a} \leq \bar{c}$. It follows that $\bar{a}\varphi \leq \bar{c}\varphi$. Therefore, $\bar{a}\varphi\vee[y \wedge z) \subset \bar{c}\varphi\vee[t)$ for all $(c, \bar{c}\varphi\vee[t)) \in N$. Consequently,

$\bar{a}\varphi\vee[z) \subset \bar{a}\varphi\vee[y \wedge z) \subset \bar{c}\varphi\vee[t)$ for all $(c, \bar{c}\varphi\vee[t)) \in N$,

which contradicts the maximality of $\bar{a}\varphi\vee[z)$. □

Complete homomorphisms via complete triple homomorphisms

In this section, we introduce complete triple homomorphisms of complete decomposable *MS*-algebras. Then, we characterize complete homomorphisms of complete decomposable *MS*-algebras in terms of complete triple homomorphisms. For this purpose, we recall from [4], the notion of triple homomorphism of decomposable *MS*-triples and related properties which will be used in rest of the paper.

Definition 9 [4] *Let (M, D, φ) and (M_1, D_1, φ_1) be decomposable MS-triples. A triple homomorphism of the triple (M, D, φ) into (M_1, D_1, φ_1) is a pair (f, g), where f is a homomorphism of M into M_1, g is a homomorphism of D into D_1 preserving 1 such that for every $a \in M$,*

$$a\varphi g \subseteq af\varphi_1 \tag{2}$$

Lemma 5 [4] *Let (f, g) be a triple homomorphism of a decomposable MS-triple (M, D, φ) into a decomposable MS-triple (M_1, D_1, φ_1). Let $a, b \in M$ and $x, y, t \in D$. Then*
(i) $a\varphi\cap[y) = [t)$ implies $af\varphi_1\cap[yg) = [tg)$,
(ii) $\left(\bar{a}f\varphi_1\vee[xg)\right) \cap (\bar{b}f\varphi_1\vee[yg)) = \overline{(a \vee b)}f\varphi_1\vee[tg)$.

Theorem 6 [4] *Let L and L_1 be decomposable MS-algebras, (M, D, φ) and (M_1, D_1, φ_1) be the associated decomposable MS-triples, respectively. Let h be a homomorphism of L into L_1 and h_M, h_D the restrictions of h to M and D, respectively. Then (h_M, h_D) is a triple homomorphism of the decomposable MS-triples. Conversely, every triple homomorphism (f, g) of the decomposable MS-triples uniquely determines a homomorphism h of L into L_1 with $h_M = f, h_D = g$ by the following rule:*

$$xh = x^{\circ\circ}f \wedge dg, \text{for all } x \in L, \tag{3}$$

where $x = x^{\circ\circ} \wedge d$ for some $d \in D(L)$.

If L and L_1 are represented as in the construction Theorem then (3) reads

$$(a, \bar{a}\varphi \vee [x))h = (af, \overline{(af)}\varphi \vee [xg)) \text{ for all } (a, \bar{a}\varphi \vee [x)) \in L. \tag{4}$$

In the following, we will write $L = (M, D, \varphi)$ to indicate that (M, D, φ) is the decomposable MS-triple associated with L, that is, $L^{\circ\circ} = M$, $D(L) = D$, and $\varphi(L) = \varphi$. Let $L = (M, D, \varphi)$ and $L_1 = (M_1, D_1, \varphi_1)$ be decomposable MS-algebras, we will write $h = (f, g)$ to indicate that $(f, g) : (M, D, \varphi) \to (M_1, D_1, \varphi_1)$ is the triple homomorphism of decomposable MS-triples corresponding to the homomorphism h of L into L_1.

Lemma 6 *Let $h = (f, g)$ be a homomorphism of a decomposable MS-algebra L onto a decomposable MS-algebra L_1. Then for each $a \in L^{\circ\circ}$, we have*
$a\varphi g = af\varphi_1$.

Proof We have, $a\varphi g \subseteq af\varphi_1$ by (2). It remains to show that $af\varphi_1 \subseteq a\varphi g$. Let $y \in af\varphi_1$. Then
$y \in [(af)^\circ) \cap D(L_1) = [(ah)^\circ) \cap D(L_1)$ implies $y \in [(ah)^\circ)$ and $y \in D(L_1)$.
Then $y \geq (ah)^\circ = a^\circ h$. Since h is onto, then $g : D(L) \to D(L_1)$ is also onto. Hence, there exists $x \in D(L)$ such that $xh = y$. Evidently, $a^\circ \vee x \in [a^\circ) \cap D(L)$ and
$(a^\circ \vee x)h = a^\circ h \vee xh = xh$ as $xh = y \geq a^\circ h$.
Therefore, $y \in [a^\circ h) \cap D(L_1) = ([a^\circ)h \cap Dg) = ([a^\circ) \cap D)g = a\varphi g$. □

Now, we introduce the concept of complete triple homomorphism.

Definition 10 *A triple homomorphism (f, g) of a decomposable MS-triple (M, D, φ) into a decomposable MS-triple (M_1, D_1, φ_1) is called complete if the following conditions are satisfied*
(i) f is a complete homomorphism of M and M_1,
(ii) g is a complete homomorphism of D and D_1,
(iii) $(\max M_E)f = \max M_{1Eg}$ for each $\phi \neq E \subseteq D$.

Remark 1 *First, we observe that the map $g : D \to D_1$ is a complete means that $(\sup_D E)g = \sup_{D_1} Eg$ for any $E \subseteq D$ and if $\inf_D E$ and $\inf_{D_1} Mg$ exist then $(\inf_D E)g = \inf_{D_1} Eg$.*

Theorem 7 *Let $L = (M, D, \varphi)$ and $L_1 = (M_1, D_1, \varphi_1)$ be complete decomposable MS-algebras and let $h = (f, g)$ be a homomorphism of L onto L_1. Then h is complete if and only if (f, g) is complete.*

Proof The decomposable MS-triples (M, D, φ) and (M_1, D_1, φ_1) are associated with L and L_1, respectively. Let $h = (f, g)$ be a complete homomorphism of L onto L_1. Then f is a de Morgan homomorphism of M onto M_1 and g is a lattice homomorphism of D onto D_1 preserving 1. We have to verify that f and g are complete. Let $\phi \neq N \subseteq M$. Then

$$\left(\inf_M N\right)f = \left(\inf_L N\right)f = \left(\inf_L N\right)h = \inf_{L_1} Nh = \inf_{L_1} Nf = \inf_{M_1} Nf \text{ by Lemma 2(3)},$$
$$\left(\sup_M N\right)f = \left(\sup_L N\right)^{\circ\circ}f = \left(\left(\sup_L N\right)h\right)^{\circ\circ} = \left(\sup_{L_1} Nh\right)^{\circ\circ} = \sup_{M_1} Nf \text{ by Lemma 2(2)}.$$

Thus, f is complete. We prove that g is complete. Let $\phi \neq E \subseteq D$. Then
$(\sup_D E)g = (\sup_L E)g = (\sup_L N)h = \sup_{L_1} Nh = \sup_{D_1} Eg$ by Lemma 2(4).
If $\inf_D E$ and $\inf_{D_1} Eg$ exist, then
$(inf_D E)g = (\inf_L E)g = (\inf_L N)h = \inf_{L_1} Nh = \inf_{D_1} Eg$ by Lemma 2(4).
Now, we prove (iii). Let $\phi \neq E \subseteq D$. Consider E corresponding the set $\acute{E}$ on $D(L)$, where $\acute{E} = \{(1, [x)) : x \in E\} \subseteq D(L)$.
By (4), we have
$\acute{E}h = \{(1, [xg)) : x \in E\} \subseteq D(L_1)$.
Since h is complete, then $(\inf_L E)h = \inf_{L_1} Eh$ for each $\phi \neq E \subseteq L$. Hence, $(\inf_L E)^{\circ\circ} = \max M_E$ (see the proof of Theorem 3) and similarly $(\inf_{L_1} Eh)^{\circ\circ} = \max M_{1Eg}$. Conversely, assume that (i)–(iii) hold and let $h = (f, g)$ be a homomorphism of L onto L_1. We have to show that h is complete. First we prove that for $\phi \neq H \subseteq L$, $(\inf_L H)h = \inf_{L_1} Hh$ holds. Consider $E = \bigcup\{[t) : (c, \bar{c}\varphi\vee[x)) \in M\}$. Let $\max M_E = b$ and $\inf_M H^{\circ\circ} = a$. Then according to the proof of Theorem 4, we get
$i = \left(a \wedge b, \overline{(a \wedge b)}\varphi\vee[z)\right) = \inf_L H$, where $z = \sup_D \left\{x_\gamma : \bar{b}\varphi\vee[x_\gamma) \supset E\right\}$. Using (4), we have
$Hh = \left\{(cf, \bar{cf}\varphi\vee[xg)) : (c, \bar{c}\varphi\vee[x)) \in H\right\}$,
and
$ih = \left((a \wedge b)f, \overline{(a \wedge b)f}\varphi\vee[zg)\right) = (\inf_L H)h$.
Now, $\inf_{L_1}(Hf)^{\circ\circ} = (\inf_M H^{\circ\circ})f = af$ by (i) and $\max M_{1Eg} = (\max M_E)f = bf$ by (iii). Since L_1 is complete and $Hh \subset L_1$ then again according to the proof of Theorem 4, we get
$\inf_{L_1} Hh = \left((a \wedge b)f, \overline{(a \wedge b)f}\varphi\vee[z_1)\right) = ih$, where $z_1 = \sup\left\{x_\gamma g : \gamma \in \Gamma\right\} = (\sup\{x_\gamma : \gamma \in \Gamma\})g = zg$ as g is an onto homomorphism. Therefore, $\inf_L Mh = (\inf_{L_1} M)h$.
Now, we prove that $(\sup_L H)h = \sup_{L_1} Hh$. By Lemma 4, $\sup_L(M) = (a, \bar{a}\varphi\vee[z))$, where $a = \sup_M H^{\circ\circ}$ and $[z) = \bigcap\{[\bar{c}\varphi\vee[t) : (c, \bar{c}\varphi\vee[t)) \in H\} \cap a\varphi$. Then $\sup_{L_1} Hh = (a_1, \bar{a}_1\varphi_1\vee[z_1))$, where $a_1 = \sup_{M_1}(Hh)^{\circ\circ} = \sup_{L_1}(Hh)^{\circ\circ} = \sup_{L_1} H^{\circ\circ}h = (\sup_L M^{\circ\circ})h = (\sup_M H^{\circ\circ})h = ah = af$ (by using Lemma 2(2) and (i) of Definition 9) and $[z_1) = \bigcap\left\{\bar{cf}\varphi_1\vee[tg) : (c, \bar{c}\varphi\vee[t)) \in H\right\} \cap a_1\varphi_1$. We show that $zg = z_1$. We have $cf\varphi_1 = c\varphi g$ by Lemma 6 and $\bar{c}\varphi g\vee[tg) = (\bar{c}\varphi\vee[t))g$ by Lemma 5(1). Then

$$\begin{aligned}[z_1) &= \bigcap\{\bar{c}\varphi\vee[t))g : (c, \bar{c}\varphi\vee[t)) \in H\} \cap a\varphi g\\ &= \left(\bigcap\{\bar{c}\varphi\vee[t) : (c, \bar{c}\varphi\vee[t)) \in H\} \cap a\varphi\right)g\\ &= [zg)\end{aligned}$$

which implies $z_1 = zg$. Therefore, $(\sup_L H)h = \sup_{L_1} Hh$ and h is complete. □

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