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On stability of the functional equation of p-Wright affine functions in (2,α)-Banach spaces

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Abstract
Stability of functional equations has recent applications in many fields. We show that the stability results obtained by J. Brzdęk and concerning the functional equation of the p-Wright affine function:

$$f(px_1 + (1-p)x_2) + f((1-p)x_1 + px_2) = f(x_1) + f(x_2),$$

can be proved also in (2,α)-Banach spaces, for some real number $\alpha \in (0, 1)$. This is done using some fixed-point theorem.

Keywords: Hyers-Ulam stability, p-Wright convexity, Affine function, Banach spaces

Introduction

The rabid development of the theory of functional equations has been strongly promoted by its applications in various fields, e.g., networks and communication (see, e.g., [4, 16, 17, 21, 31]). They have applications in computer graphics [29], in information theory [2, 28], in digital filtering [34], and in decision theory [1, 35]. Stability of functional equations is nowadays a popular subject with many interesting applications (see, e.g., [5–8, 12, 23, 30] for more details). Stability can be seen from different perspectives, see [30], and hundreds of researchers are dealing with such amazing topic. It has applications in optimization theory (see, e.g., [26]), it is somehow related to the notion of shadowing (see, e.g., [22]), and it has applications in the economy (see [13]).

The starting point of the stability of functional equations was due to S.M. Ulam who posed an open problem in 1940. The problem posed by Ulam can be stated as follows (see, e.g., [30]):

Let G_1 be a group and (G_2, d) a metric group. Given $\varepsilon > 0$, does there exist $\delta > 0$ such that if $g : G_1 \to G_2$ satisfies

$$d(g(xy), g(x)g(y)) < \delta$$

for all $x, y \in G_1$, then a homomorphism $f : G_1 \to G_2$ exists such that

$$d(g(x), f(x)) < \varepsilon,$$

for all $x, y \in G_1$?

It should be noted that Hyers's introduced a partial answer to Ulam's problem in Banach spaces(see, e.g., [23]).

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Stability is useful because it can be considered as an efficient tool for evaluating the error people usually face when replacing functions that satisfy some equations only approximately, by the exact solutions to those equations. Roughly speaking, an equation is said to be stable in some class of functions if any function from that class, satisfying the equation approximately (in some sense), is near (in some way) to an exact solution of the equation. In the last few decades, several stability problems of various (functional, difference, differential, integral) equations have been investigated by many mathematicians (see, e.g., [9–11, 24, 27] for more details), but mainly in classical spaces.

Since the notion of an approximate solution and the idea of nearness of two functions can be understood in many ways, depending on the needs and tools available in a particular situation. One of such non-classical measures of a distance can be introduced by the notion of a 2-norm. As far as we know, the concept of the linear 2-normed space was introduced first by Gähler et al. in [20], and it seems that the first work on the Hyers-Ulam stability of functional equations in complete 2-normed spaces (that is, 2-Banach spaces), see, e.g, [19]. See also [14, 33] for some details in 2-Banach spaces. In this article, we investigate the stability of the functional equation of the p-Wright affine functions investigated in [3] but in $(2,\alpha)$-Banach spaces.

The article is organized as follows: in the "Preliminaries" section, we recall some definitions and the functional equation of our interest; the "Fixed-point theorem" section introduces the fixed-point theorem used in the stability; in the "Stability" section, we investigate the stability of the functional equation of the p-Wright affine functions; the "An observation on superstability" section introduces a simple observation on superstability; and the "Conclusion" section concludes our work.

Preliminaries

Throughout the article, we use $\mathbb{R}_+$ to denote the set of nonnegative reals, $\mathbb{R}$ denotes the set of reals, $\mathbb{N}$ denotes the set of positive integers, and $\mathbb{K}$ to denote the field of real or complex numbers. Let $0 < p < 1$ be a fixed real number. We say that a function f:

$$f : I \longmapsto \mathbb{R},$$

mapping a real nonempty interval I into the set of reals $\mathbb{R}$ is p-Wright convex provided (see, e.g., [15])

$$f(px_1 + (1-p)x_2) + f((1-p)x_1 + px_2) \leq f(x_1) + f(x_2), \;\; x_1, x_2 \in I.$$

If f satisfies the functional equation

$$f(px_1 + (1-p)x_2) + f((1-p)x_1 + px_2) = f(x_1) + f(x_2), \tag{1}$$

then we say that it is p-Wright affine (see [15]). Note that for $p = 1/2$, Eq. (1) becomes the Jensen's functional equation

$$f(\frac{x_1 + x_2}{2}) = \frac{f(x_1) + f(x_2)}{2}.$$

For $p = 1/3$, Eq. (1) takes the form

$$f(2x_1 + x_2) + f(x_1 + 2x_2) = f(3x_1) + f(3x_2),$$

which has been investigated by Najati and Park in [32]. The cases of more arbitrary p were studied in [15] (see also [25]). We prove some results concerning the Hyers-Ulam stability of (1). The method of the proof of the main result corresponds to some observations in

[12] and the main tool in it is a fixed point. Let us recall first (see, for instance, [18]) some definitions.

Definition 1 *By a linear 2-normed space, we mean a pair* $(X, \|., .\|)$ *such that* X *is an at least two-dimensional real linear space and*

$$\|\cdot,\cdot\| : X \times X \to \mathbb{R}$$

is a function satisfying the following conditions:

(1) $\|x_1, x_2\| = 0$ *if and only if* x_1 *and* x_2 *are linearly dependent;*
(2) $\|x_1, x_2\| = \|x_2, x_1\|$ *for* $x_1, x_2 \in X$
(3) $\|x_1, x_2 + x_3\| \leq \|x_1, x_2\| + \|x_1, x_3\|$ *for* $x_i \in X, i = 1, 2, 3$
(4) $\|\beta x_1, x_2\| = |\beta|\|x_1, x_2\|$ *for* $\beta \in \mathbb{R}$ *and* $x_1, x_2 \in X$.

A generalized version of a linear 2-normed spaces is the $(2, \alpha)$-normed space defined in the following manner:

Definition 2 *Let* α *be a fixed real number with* $0 < \alpha \leq 1$, *and let* X *be a linear space over* $\mathbb{K}$ *with* $\dim X > 1$. *A function*

$$\|., .\|_\alpha : X \times X \to \mathbb{R}_+$$

is called a $(2, \alpha)$*-norm on* X *if and only if it satisfies the following conditions:*

(C1) $\|x_1, x_2\|_\alpha = 0$ *if and only if* x_1 *and* x_2 *are linearly dependent;*
(C2) $\|x_1, x_2\|_\alpha = \|x_2, x_1\|_\alpha$ *for* $x_1, x_2 \in X$
(C3) $\|x_1, x_2 + x_3\|_\alpha \leq \|x_1, x_2\|_\alpha + \|x_1, x_3\|_\alpha$ *for* $x_i \in X, i = 1, 2, 3$
(C4) $\|\lambda x_1, x_2\|_\alpha = |\lambda|^\alpha \|x_1, x_2\|_\alpha$ *for* $\lambda \in \mathbb{R}$ *and* $x_1, x_2 \in X$

The pair $(X, \|., .\|_\alpha)$ is called a $(2, \alpha)$-normed space.

Definition 3 *A sequence* $(x_n)_{n\in\mathbb{N}}$ *of elements of a linear* $(2, \alpha)$*-normed space* X *is called a Cauchy sequence if there are linearly independent* $y, z \in X$ *such that*

$$\lim_{n,m\to\infty} \|x_n - x_m, z\|_\alpha = 0 = \lim_{n,m\to\infty} \|x_n - x_m, y\|_\alpha,$$

whereas $(x_n)_{n\in\mathbb{N}}$ *is said to be convergent if there exists an* $x \in X$ *(called a limit of this sequence and denoted by* $\lim_{n\to\infty} x_n$*) with*

$$\lim_{n,m\to\infty} \|x_n - x, y\|_\alpha = 0, y \in X.$$

A linear $(2, \alpha)$-normed space in which every Cauchy sequence is convergent is called a $(2, \alpha)$-Banach space.

Let us also mention that in linear $(2, \alpha)$-normed spaces, every convergent sequence has exactly one limit and the standard properties of the limit of a sum and a scalar product are valid. Next, it is easily seen that we have the following property.

Lemma 1 *If* X *is a linear* $(2, \alpha)$*-normed space,* $x, y, z \in X$, y, z *are linearly independent, and*

$$\|x, y\|_\alpha = 0 = \|x, z\|_\alpha,$$

then $x = 0$.

Let us yet recall a lemma from [33].

Lemma 2 *If* X *is a linear* $(2,\alpha)$*-normed space and* $(x_n)_{n\in\mathbb{N}}$ *is a convergent sequence of elements of* X*, then*

$$\lim_{n\to\infty} \|x_n, z\|_\alpha = \|\lim_{n\to\infty} x_n, z\|_\alpha, z \in X.$$

We introduce a simple example of a $(2,\alpha)$-normed space.

Example 1 *For* $x = (x_1, x_2)$, $y = (y_1, y_2) \in X = \mathbb{R}^2$, *the* $(2,\alpha)$*-norm on* X *is defined by*

$$\|x, y\|_\alpha = |x_1 y_2 - x_2 y_1|^\alpha,$$

where α *is a fixed real number with* $0 < \alpha \leq 1$.

The main tool used in this article is the following fixed-point theorem. It is a version of theorem 1 in [10]. In order to write it, we need the following assumptions.

Fixed-point theorem

Let us introduce the following three assumptions:

(A1) E is a nonempty set, $(Y, \|.,.\|_\alpha)$ is a $(2,\alpha)$-Banach space, Y_0 is a subset of Y containing two linearly independent vectors, $j \in \mathbb{N}$,

$$f_i : E \to E,\ g_i : Y_0 \to Y_0,\ \text{and}\ L_i : E \times Y_0 \to \mathbb{R}_+\ \text{for}\ i = 1, \cdots, j;$$

(A2) $\mathrm{T} : Y^E \to Y^E$ is an operator satisfying the inequality

$$\|\mathrm{T}\xi(x) - \mathrm{T}\mu(x), y\|_\alpha \leq \sum_{i=1}^{j} L_i(x,y)\|\xi(f_i(x)) - \mu(f_i(x)), g_i(y)\|_\alpha, \quad \xi, \mu \in Y^E,\ x \in E, y \in Y_0 \tag{2}$$

(A3) $\Lambda : \mathbb{R}^{E\times Y_0} \to \mathbb{R}^{E\times Y_0}$ is an operator defined by

$$\Lambda\delta(x,y) := \sum_{i=1}^{j} L_i(x,y)\delta(f_i(x), g_i(y)), \delta \in \mathbb{R}^{E\times Y_0}, \quad x \in E,\ \ y \in Y_0 \tag{3}$$

Now, its the position to present the abovementioned fixed-point theorem.

Theorem 1 *Let hypotheses (A1)–(A3) hold and function*

$$\varepsilon : E \times Y_0 \to \mathbb{R}_+ \ \textit{and}\ \varphi : E \to Y$$

fulfill the following two conditions:

$$\|\mathrm{T}\varphi(x) - \varphi(x), y\|_\alpha \leq \varepsilon(x,y),\ \ x \in E,\ \ y \in Y_0 \tag{4}$$

$$\varepsilon^*(x,y) := \sum_{i=1}^{\infty}(\Lambda^l \varepsilon)(x,y) < \infty,\ \ x \in E, y \in Y_0 \tag{5}$$

Then, there exists a unique fixed point ψ of T for which

$$\|\varphi(x) - \psi(x), y\|_\alpha \leq \varepsilon^*(x, y), \;\; x \in E, \;\; y \in Y_0. \tag{6}$$

Moreover,

$$\psi(x) = \lim_{l \to \infty} (\mathrm{T}^l \varphi)(x), \;\; x \in E. \tag{7}$$

We skip the proof as it is illustrated in [12].

Stability

In this section, we introduce the main result in this article that concerns the stability of Eq. (1); it corresponds in particular to some results in [12].

Theorem 2 *Let (A1) be valid, $p \in \mathbb{K}, A, k \in (0, \infty)$, $|p|^{\alpha k} + |1-p|^{\alpha k} < 1$, and*

$$g : E \to Y$$

satisfy

$$\begin{aligned} &\|g(px_1 + (1-p)x_2) + g((1-p)x_1 + px_2) - g(x_1) - g(x_2), y\|_\alpha \\ &\qquad \leq A\left(\|x_1, y\|_\alpha^k + \|x_2, y\|_\alpha^k\right), \;\; x_1, x_2 \in E, y \in Y_0. \end{aligned} \tag{8}$$

Then there exists a unique solution $G : X \to Y$ of Eq. (1) such that

$$\|g(x) - G(x), y\|_\alpha \leq \frac{A\|x, y\|_\alpha^k}{1 - |p|^{\alpha k} - |1-p|^{\alpha k}}, x \in E \tag{9}$$

and G is given by

$$G(x) := g(0) + \lim_{n \to \infty} \left(\mathrm{T}^n g_0\right)(x), \;\; x \in E, \tag{10}$$

where g_0 and T are defined by (13) and (14). Moreover, G is the unique solution of Eq. (1) such that there exists a constant $M \in (0, \infty)$ with

$$\|g(x) - G(x), y\|_\alpha \leq M\|x, y\|_\alpha^k, \;\; x \in E, y \in Y_0 \tag{11}$$

Proof Note that (8) with $x_2 = 0$ gives

$$\begin{aligned} &\|g(px_1) + g((1-p)x_1) - g(x_1) - g(0), y\|_\alpha \\ &\quad \leq A\left(\|x_1, y\|_\alpha^k + \|y\|_\alpha^k\right), \;\; x_1 \in E, y \in Y_0 \end{aligned} \tag{12}$$

Write

$$g_0(x_1) = g(x_1) - g(0), \;\; x_1 \in E \tag{13}$$

and

$$\mathrm{T}\xi(x_1) = \xi(px_1) + \xi((1-p)x_1), \;\; x_1 \in E, \xi \in Y^E. \tag{14}$$

Then (12) implies the inequality

$$\begin{aligned} &\|g_0(px_1) + g_0((1-p)x_1) \;\; - g(x_1) - g(0), y\|_\alpha \\ &\qquad\qquad \leq A(\|x_1, y\|_\alpha^k), \;\; x_1 \in E \end{aligned} \tag{15}$$

which means that

$$\|\mathrm{T}g_0(x_1) - g_0(x_1), y\|_\alpha \leq A\left(\|x_1, y\|_\alpha^k\right), \;\; x_1 \in E. \tag{16}$$

Further, note that (A3) holds with $k = 2, f_1(x) = px, f_2(x) = (1-p)x, L_i(x) = 1$ for $i = 1, 2, x \in E$. Define Λ as in (A3). Clearly, with $\varepsilon(x) := A(\|x_1, y\|_\alpha^k)$ for $x \in E$, we have

$$\begin{aligned}\varepsilon^*(x_1) &:= \sum_{n=0}^{\infty}(\Lambda^n\varepsilon)(x_1)\\ &\leq A(\|x_1, y\|_\alpha^k)\sum_{n=0}^{\infty}(|p|^{\alpha k}+|1-p|^{\alpha k})^n\\ &= \frac{A(\|x_1, y\|_\alpha^k)}{1-|p|^{\alpha k}-|1-p|^{\alpha k}}, \ x_1 \in E. \end{aligned} \tag{17}$$

Hence, according to Theorem 2, there exists a unique solution $G_0 : X \to Y$ of the equation

$$G_0(x_1) = G_0(px_1) + G_0((1-p)x_1), \ x_1 \in E \tag{18}$$

such that

$$\|g_0(x_1) - G_0(x_1), y\|_\alpha \leq \frac{A\left(\|x_1, y\|_\alpha^k\right)}{1-|p|^{\alpha k}-|1-p|^{\alpha k}}, \ x_1 \in E; \tag{19}$$

moreover,

$$G_0(x_1) := \lim_{n\to\infty}(\mathrm{T}^n g_0)(x_1), \ x_1 \in E. \tag{20}$$

Now we show that, for every $x_1, x_2 \in E, n \in \mathbb{N}_0$ (nonnegative integers),

$$\begin{aligned}&\|\mathrm{T}^n g_0(px_1+(1-p)x_2)+\mathrm{T}^n g_0((1-p)x_1+px_2)-\mathrm{T}^n g(x_1)-\mathrm{T}^n g(x_2), y\|_\alpha\\ &\quad\leq A(|p|^{\alpha k}+|1-p|^{\alpha k})^n\left(\|x_1, y\|_\alpha^k+\|x_2, y\|_\alpha^k\right), x_1, x_2 \in E, y \in Y_0 \end{aligned}\tag{21}$$

It is easy to see that the case $n = 0$ is just (8). Next, fix $m \in \mathbb{N}_0$ and assume that (21) holds for every $x_1, x_2 \in E$ with $n = m$. Then

$$\begin{aligned}&\|\mathrm{T}^{m+1}g_0(px_1+(1-p)x_2) \ +\mathrm{T}^{m+1}g_0((1-p)x_1+px_2)\\ &\quad -\mathrm{T}^{m+1}g(x_1)-\mathrm{T}^{m+1}\ g(x_2), y\|_\alpha = \|\mathrm{T}^m g_0(p(px_1+(1-p)x_2))\\ &\quad +\mathrm{T}^m g_0((1-p)(px_1 \ +(1-p)x_2))+\mathrm{T}^m g_0(p((1-p)x_1+px_2))\\ &\quad +\mathrm{T}^m\ g_0((1-p)(1-p)x_1+px_2))-\mathrm{T}^m g_0(px_1)\\ &\quad -\mathrm{T}^m\ g_0((1-p)x_1)-\mathrm{T}^m g_0(px_2)-\mathrm{T}^m g_0((1-p)x_2), y\|_\alpha \end{aligned}\tag{22}$$

which is clearly

$$\begin{aligned}&\leq \|\mathrm{T}^m g_0(ppx_1+(1-p)px_2)+\mathrm{T}^m g_0((1-p)px_1+ppx_2)-\mathrm{T}^m g_0(px_1)\\ &\quad -\mathrm{T}^m g_0(px_2), y\|_\alpha+\|\mathrm{T}\ ^m g_0(p(1-p)x_1+(1-p)(1-p)x_2)\\ &\quad +\mathrm{T}^m g_0((1-p)(1-p)\ x_1+p(1-p)x_2)\\ &\quad -\mathrm{T}^m g_0((1-p)x_1)\ -\mathrm{T}^m g_0(p(1-p)x_2), y\|_\alpha\\ &\quad \leq A(|p|^{\alpha k}+|1-p|^{\alpha k})^m\left((p\|x_1, y\|_\alpha)^k+(p\|x_2, y\|_\alpha)^k\right)\\ &\quad +(|p|^{\alpha k}+|1-p|^{\alpha k})\ ^m\left(((1-p)\|x_1, y\|_\alpha)^k+((1-p)\|x_2, y\|_\alpha)^k\right)\\ &\quad =(|p|^{\alpha k}+|1-p|^{\alpha k})\ ^m\left((\|x_1, y\|_\alpha)^k+(\|x_2, y\|_\alpha)^k\right),\\ &\qquad x_1, x_2 \in E, y \in Y_0. \end{aligned}\tag{23}$$

Thus, by induction, we have shown that (21) holds for every $x_1, x_2 \in E$ and $n \in \mathbb{N}_0$. Letting $n \to \infty$ in (21), we obtain that

$$G_0(px_1 + (1-p)x_2) + G_0((1-p)x_1 + px_2) = G_0(x_1) + G_0(x_2), \; x_1, x_2 \in E \tag{24}$$

Write $G(x_1) := G_0(x_1) + g(0)$ for $x_1 \in E$. Then it is easily seen that

$$G(px_1 + (1-p)x_2) + G((1-p)x_1 + px_2) = G(x_1) + G(x_2), \; x_1, x_2 \in E \tag{25}$$

and (9) holds. It remains to show the uniqueness of G. So suppose that $M_0 \in (0, \infty)$ and $G_1 : X \to Y$ is a solution to (1) with

$$\|g(x_1) - G_1(x_1), y\|_\alpha \le M_0 \|x_1, y\|_\alpha, \; x_1 \in E, y \in Y_0 \tag{26}$$

Note that

$$G(0) = g(0) = G_1(0),$$

$$G_1(px_1) + G_1((1-p)x_1) = G_1(x_1) + G_1(0), \; x_1 \in E, \tag{27}$$

$$G(px_1) + G((1-p)x_1) = G(x_1) + G(0), \; x_1 \in E, \tag{28}$$

and, by (9),

$$\|G(x_1) - G_1(x_1), y\|_\alpha \le \frac{(M+A)\|x_1, y\|_\alpha^k}{1 - |p|^{\alpha k} - |1-p|^{\alpha k}}$$

$$= (M+A)\|x_1, y\|_\alpha^k \sum_{n=j}^{\infty} (|p|^{\alpha k} + |1-p|^{\alpha k})^n, \; x_1 \in E \tag{29}$$

The case $j = 0$ is exactly (29). So fix $l \in \mathbb{N}_0$ and assume that (29) holds for $j = l$. Then, in view of (27) and (28),

$$\begin{aligned}
&\|G(x_1) - G_1(x_1), y\|_\alpha \\
&= \|G(px_1) + G((1-p)x_1) - G_1(px_1) - G_1((1-p)x_1), y\|_\alpha, \\
&\le \|G(px_1) - G_1(px_1), y\|_\alpha + \|G((1-p)x_1) - G_1((1-p)x_1), y\|_\alpha \\
&\le (M+A)(\|p\|_\alpha^k \|x_1, y\|_\alpha^k + \|(1-p)\|_\alpha^k \|x_1, y\|_\alpha^k) \sum_{n=l}^{\infty} (|p|^{\alpha k} + |1-p|^{\alpha k})^n, \\
&\le (M+A)\|x_1, y\|_\alpha^k \sum_{n=l+1}^{\infty} (|p|^{\alpha k} + |1-p|^{\alpha k})^n, \; x_1 \in E, y \in Y_0
\end{aligned}$$

Thus, we have shown (29). Now, letting $j \to \infty$ in (29), we get $G_1 = G$. □

An observation on superstability

The following is a very simple observation on the superstability of Eq. (1) complements Theorem 2.

Theorem 3 *Let (A1) be valid, $p \in \mathbb{F}, A, k \in (0, \infty)$, $|p|^{2\alpha k} + |1-p|^{2\alpha k} < 1$, and*

$$g : E \to Y$$

satisfy

$$\|g(px_1+(1-p)x_2)+g((1-p)x_1+px_2)-g(x_1)-g(x_2),y\|_\alpha \leq A\|x_1,y\|_\alpha^k\|x_2,y\|_\alpha^k, \quad (30)$$

for every $x_1, x_2 \in E, y \in Y_0$. Then g is a solution to (1).

Proof It is easy to see that (30) with $x_2 = 0$ gives

$$g(x_1) = g(px_1) + g((1-p)x_1) - g(0), \; x_1 \in E \quad (31)$$

We show that, for every $x_1, x_2 \in E, y \in Y_0, n \in \mathbb{N}_0$,

$$\begin{aligned} &\|g(px_1 + (1-p)x_2) + g((1-p)x_1 + px_2) - g(x_1) - g(x_2), y\|_\alpha \\ &\quad \leq A\left(|p|^{\alpha 2k} + |1-p|^{\alpha 2k}\right)^n \|x_1, y\|_\alpha^k \|x_2, y\|_\alpha^k, \end{aligned} \quad (32)$$

It is easy to see that the case $n = 0$ is just (30). Next, fix $m \in \mathbb{N}_0$ and assume that (32) holds for every $x_1, x_2 \in E$, with $n = m$. Then, by (31),

$$\begin{aligned} &\|g(px_1 + (1-p)x_2) + g((1-p)x_1 + px_2) - g(x_1) - g(x_2), y\|_\alpha \\ &= \|g(p(px_1 + (1-p)x_2)) + g((1-p)(px_1 + (1-p)x_2)) \\ &\quad + g(p((1-p)x_1 + px_2)) + g((1-p)((1-p)x_1 + px_2)) \\ &\quad - g(px_1) - g((1-p)x_1) - g(px_2) - g((1-p)x_2), y\|_\alpha \end{aligned} \quad (33)$$

which is clearly

$$\begin{aligned} &\leq A\left(|p|^{\alpha 2k} + |1-p|^{\alpha 2k}\right)^m \|p\|_\alpha^k \|x_1, y\|_\alpha^k \|p\|_\alpha^k \|x_2, y\|_\alpha^k \\ &\quad + A\left(|p|^{\alpha 2k} + |1-p|^{\alpha 2k}\right)^m \|1-p\|_\alpha^k \|x_1, y\|_\alpha^k \|1-p\|_\alpha^k \|x_2, y\|_\alpha^k \\ &= A\left(|p|^{\alpha 2k} + |1-p|^{\alpha 2k}\right)^{m+1} \|x_1, y\|_\alpha^k \|x_2, y\|_\alpha^k \end{aligned} \quad (34)$$

for every $x_1, x_2 \in E, y \in Y_0$. Therefore, by induction, we have shown that (32) holds for every $x_1, x_2 \in E$ and $n \in \mathbb{N}_0$. Letting $n \to \infty$ in (32), we obtain that g is a solution to (1). □

Conclusion

In this paper, we managed to generalize some recent results concerning the stability of the functional equation of the p-Wright affine functions in (2,α)-Banach spaces, for some real number $\alpha \in (0, 1]$. The main tool in the investigation is a fixed-point theory. This work may be further generalized to be in (n,α)-Banach spaces, for some natural number n.

Authors' contributions
The author worked on the results and also read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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