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ORIGINAL ARTICLE

Commutative fundamental (*m*, *n*)-modules

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Abstract In this paper, we introduce the concept of fundamental relation θ^* on an (m, n)-hypermodule M as the smallest equivalence relation such that M/θ^* is a commutative (m, n)-module, and then some related properties are investigated.

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1. Introduction

The notion of an n-ary group was introduced by Dörnte [1], as a natural generalization of group. One can find the basic results on n-ary groups in Post [2]. The notion of n-ary hypergroup was first introduced by Davvaz and Vougiouklis [3] as a generalization of n-ary group, and studied mainly by many authors, for example see [4,5]. Let H be a non-empty set and h be a mapping h: $H \times H \to \wp^*(H)$, where $\wp^*(H)$ is the set of all non-empty subsets of H. Then, h is called a *binary hyperoperation* on H. We denote by H^n the cartesian product $H \times \cdots \times H$, where H appears n times and an element of H^n will be denoted by $(x_1, \ldots, x_n)$, where $x_i \in H$ for any i with $1 \leqslant i \leqslant n$. In general, a mapping h: $H^n \to \wp^*(H)$ is called an *n-ary hyperoperation* and n is called the *arity of the hyperoperation*. Let h be an n-ary hyperoperation on H and $A_1, \ldots, A_n$ be non-empty subsets of H. We define $h(A_1, \ldots, A_n) = \cup \{h(x_1, \ldots, x_n) | x_i \in A_i, i = 1, \ldots, n\}$. We shall use the following abbreviated notation: the sequence $x_i, x_{i+1}, \ldots, x_j$ will be denoted by x_i^j. Also, for every $a \in H$, we write $h(a, \ldots, a) = h\left(\begin{matrix}(n)\\ a\end{matrix}\right)$ and for $j < i$, x_i^j is the empty set. In this convention, $h(x_1, \ldots, x_i, y_{i+1}, \ldots, y_j, x_{j+1}, \ldots, x_n)$ will be written $h\left(x_1^i, y_{i+1}^j, x_{j+1}^n\right)$. If h is an n-ary groupoid and $t = l(n-1)+1$, then the t-ary hyperoperation $h_{(l)}$ given by

$$h_{(l)}\left(x_1^{l(n-1)+1}\right) = h\left(h\left(\ldots, h\left(h\left(x_1^n\right), x_{n+1}^{2n-1}\right), \ldots\right), x_{(l-1)(n-1)+2}^{l(n-1)+1}\right)$$

is denoted by $h_{(l)}$. A non-empty set H with an n-ary hyperoperation h: $H^n \to P^*(H)$ is called an *n-ary hypergroupoid* and is denoted by (H, h). An n-ary hypergroupoid (H, h) is an *n-ary semihypergroup* if the following associative axiom holds:

$$h\left(x_1^{i-1}, h\left(x_i^{n+i-1}\right), x_{n+i}^{2n-1}\right) = h\left(x_1^{j-1}, h\left(x_j^{n+j-1}\right), x_{n+j}^{2n-1}\right)$$

for every $i, j \in \{1, 2, \ldots, n\}$ and $x_1, x_2, \ldots, x_{2n-1} \in H$.

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An n-ary hypergroupoid (H, h) is called *commutative*, if for every $\sigma \in \mathbb{S}_n$, we have $h(x_1^n) = h\left(x_{\sigma(1)}^{\sigma(n)}\right)$. An n-ary semihypergroup (H, h), in which the equation $b \in h(a_1^{i-1}, x_i, a_{i+1}^n)$ has the solution $x_i \in H$ for every $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_n, b \in H$ and $1 \leqslant i \leqslant n$, is called an n- ary hypergroup. Furthermore, (m, n)-rings have been introduced by Crombez [6] and then investigated by Crombez and Timm [7], Dudek [8], Iancu [9]. Recently, the notion of (m, n)-hyperrings has been defined by Mirvakili and Davvaz [10] obtaining (m, n)-rings from (m, n)-hyperrings by fundamental relations. Also, the principal notions of hyperstructure theory can be found in [11,12].

Definition 1.1. Let R be a non-empty set, f be an m-ary hyperoperation on R and g be an n-ary hyperoperation on R. An *(m, n)-hyperring* is an algebraic hyperstructure (R, f, g), which satisfies the following axioms:

(1) (R, f) is an m-ary hypergroup,
(2) (R, g) is an n-ary semihypergroup,
(3) the n-ary hyperoperation g is distributive with respect to the m-ary hyperoperation f, i.e.,

$$g\left(a_1^{i-1}, f\left(x_1^m\right), a_{i+1}^n\right) = f\left(g\left(a_1^{i-1}, x_1, a_{i+1}^n\right), \ldots, g\left(a_1^{i-1}, x_m, a_{i+1}^n\right)\right),$$

for every $a_1^{i-1}, a_{i+1}^n, x_1^m \in R,\ 1 \leqslant i \leqslant n$.

(R, f, g) is called a *commutative (m, n)-hyperring* if (R, f) and (R, g) are commutative. A construction of an (m, n)-hyperring (R, f, g) of a hyperring $(R, +, \cdot)$ was presented by Mirvakili and Davvaz [10] as follows:

Example 1. Let $(R, +, \cdot)$ be a hyperring. Let f be an m-ary hyperoperation and g be an n-ary operation (clearly, any n-ary operation is an n-ary hyperoperation) on R as follows:

$$f\left(x_1^m\right) = \sum_{i=1}^{m} x_i, \quad \forall x_1^m \in R,$$

$$g(x_1^n) = \prod_{i=1}^{n} x_i, \quad \forall x_1^n \in R.$$

Then, (R, f, g) is an (m, n)-hyperring and denoted by $(R, f, g) = der_{(m,\ n)}(R, +, \cdot)$.

2. (*m*, *n*)-hypermodules

In [13], Anvariyeh et al. introduced the class of (m, n)-hypermodules over (m, n)-hyperrings. They defined the fundamental relation ϵ^* on (m, n)-hypermodules. In [14], Anvariyeh and Mirvakili considered a special kind of (m, n)-hypermodules, called canonical (m, n)-hypermodule, and a special kind of (m, n)-hyperrings, called Krasner (m, n)-hyperring [10]. Then, in [15], Belali et al. defined the class of free and cyclic canonical (m, n)-hypermodules over Krasner (m, n)-hyperrings. In this section, we recall the definition of (m, n)-hypermodules [13].

Definition 2.1. Let M be a non-empty set. Then, $M = (M, h, k)$ is an *(m, n)-hypermodule* over an (m, n)-hyperring R, if (M, h) is an m-ary hypergroup and the map

$$k : \underbrace{R \times \ldots \times R}_{n-1} \times M \to \wp^*(M)$$

satisfies the following conditions:

(1) $k\left(r_1^{n-1}, h\left(x_1^m\right)\right) = h\left(k\left(r_1^{n-1}, x_1\right), \ldots, k\left(r_1^{n-1}, x_m\right)\right)$,
(2) $k\left(r_1^{i-1}, f\left(s_1^m\right), r_{i+1}^{n-1}, x\right) = h\left(k\left(r_1^{i-1}, s_1, r_{i+1}^{n-1}, x\right), \ldots, k\left(r_1^{i-1}, s_m, r_{i+1}^{n-1}, x\right)\right)$,
(3) $k\left(r_1^{i-1}, g\left(r_i^{i+n-1}\right), r_{i+m}^{n+m-2}, x\right) = k\left(r_1^{n-1}, k\left(r_m^{n+m-2}, x\right)\right)$.

If k is a scalar n-ary hyperoperation, $S_1, \ldots, S_{n-1}$ are non-empty subsets of R and $M_1 \subseteq M$, we set $k(S_1, \ldots, S_{n-1}, M_1) = \cup \{k(r_1, \ldots, r_{n-1}, x) |\ r_i \in S_i,\ i = 1, \ldots, n-1,\ x \in M_1\}$. An (m, n)-hypermodule M is an R-hypermodule, if $m = n = 2$.

Example 2. Let $M = \{0,1,2\}$ and $(R, f, g) = der_{(3,2)}(\mathbb{Z}, +, \cdot)$ (see Example 1). We define the commutative hyperoperation h and hyperoperation k as follows:

$$h(0,0,0) = h(0,0,2) = h(0,2,2) = h(2,2,2) = \{0,2\},$$

$$h(0,0,1) = h(0,2,1) = h(2,2,1) = \{1\},$$

$$h(0,1,1) = h(2,1,1) = \{0,2\},$$

$$h(1,1,1) = \{1\},$$

and k: $R \times M \to \wp^*(M)$,

$$k(r,x) = \begin{cases} \{0,2\} & \text{if } r \in 2\mathbb{Z} \text{ or } x \in \{0,2\}, \\ \{1\} & \text{otherwise.} \end{cases}$$

Then, (M, h, k) is an (3, 2)-ary hypermodule over (3, 2)-ary hyperring (R, f, g).

Example 3. Let R be a hyperring and M be an R-hypermodule. Then, R with m-ary hyperoperation $f\left(r_1^m\right) = \sum_{i=1}^m r_i$, and n-ary hyperoperation $g\left(r_1^n\right) = \prod_{i=1}^n r_i$, is an (m, n)-hyperring. Also, M with hyperoperation h with $h\left(x_1^m\right) = \sum_{i=1}^m x_i$, where $x_i \in M$, is an m-hypergroup. Now, we define the scalar n-ary hyperoperation k by

$$k(r_1, \ldots, r_{n-1}, x) := \left(\prod_{i=1}^{n} r_i\right) \cdot x.$$

Then, M is an (m, n)-hypermodule over (m, n)-hyperring R.

Example 4. Let $(R, +, \cdot)$ be a hyperring and $(M, +)$ be an R-hypermodule. If N is a subhypermodule of M, then set:

$$h\left(x_1^m\right) = \sum_{i=1}^{m} x_i + N, \quad \forall x_1^m \in M,$$

$$f\left(r_1^m\right) = \sum_{i=1}^{m} r_i, \quad \forall r_1^m \in R,$$

$$g(x_1^n) = \prod_{i=1}^{n} r_i, \quad \forall r_1^n \in R,$$

$$k(r_1^{n-1}, x) = \left(\sum_{i=1}^{n-1} r_i\right) \cdot x + N, \quad \forall r_1^{n-1} \in R, \quad \forall x \in M.$$

Then, (M, h, k) is an (m, n)-hypermodule over (m, n)-hyperring (R, f, g).

Example 5. Let $(H, \cdot)$ be a commutative almost group (i.e., a semigroup $H = H^* \cup \{0\}$, where $(H^*, \cdot)$ is a group and 0 a two side absorbing element). Now, if $g(x_1^n) = \prod_{i=1}^n x_i$, then (H, g) is an n-ary group. For every $x_1^k \in H^*$, we define an m-ary hyperoperation f on H as follows:

$$f\left(x_1^k, \overset{(m-k)}{0}\right) = \begin{cases} 0 & k = 0, \\ \bigcup_{i=1}^{k}\{x_i\} & |\bigcup_{i=1}^{k} x_i| = k, \\ H - \{x_1\} & k = 2, \quad |\bigcup_{i=1}^{k} x_i| = 1, \\ H & k \geqslant 3, \quad |\bigcup_{i=1}^{k} x_i| < k, \end{cases}$$

f is a commutative hyperoperation and 0 is a scalar identity and $f\left(\overset{(m)}{0}\right) = 0$. Then, the hyperstructure (H, f, g) is an (m, n)-hyperring and therefore (H, f, g) is an (m, n)-hypermodule over the (m, n)-hyperring (H, f, g).

Leoreanu-Fotea and Corsini proved the following theorem in [16].

Theorem 2.2. *Let (H, f) be an n-ary semihypergroup (n-ary hypergroup) and e be a scalar neutral element of H. For all $x, y \in H$, we define: $x * y := f\left(x, y, \overset{(m-1)}{e}\right)$. Then, $(H, *)$ is a semihypergroup (hypergroup).*

Theorem 2.3. *Let (M, h, k) over R be an (m, n)-hypermodule such that h and f have zero scalar elements 0_R and 0_M, also 1_R be identity of g, such that:*

$$h\left(\overset{(i-1)}{0_M}, x, \overset{(m-i)}{0_M}\right) = x, \quad \forall x \in M$$

$$f\left(\overset{(i-1)}{0_R}, r, \overset{(m-i)}{0_R}\right) = r, \quad \forall r \in R$$

$$g\left(\overset{(i-1)}{1_R}, r, \overset{(n-i)}{1_R}\right) = r, \quad \forall r \in R$$

$$k\left(\overset{(n-1)}{1}, x\right) = x, \quad \forall x \in M.$$

Now, suppose that

$$x + y := h\left(x, y, \overset{(m-2)}{0_M}\right), \quad \forall x, y \in M$$

$$r + s := f\left(r, s, \overset{(m-2)}{0_R}\right), \quad \forall r, s \in R$$

$$r \cdot s = g\left(r, s, \overset{(n-2)}{1_R}\right), \quad \forall r, s \in R$$

$$r \circ x := k\left(\overset{(n-2)}{1_R}, r, x\right), \quad \forall r \in R \text{ and } x \in M.$$

Then, $(M, +, \circ)$ is a hypermodule with zero element 0_M over the hyperring $(R, +, \cdot)$ with zero scalar 0_R and identity scalar 1_R. Also,

$$der_{(m,n)}(M, +, \circ) = (M, f, k) \quad \text{and} \quad der_{(m,n)}(R, +, \cdot) = (R, f, g).$$

Proof. By Theorem 2.2, it is not difficult to see that $(M, +)$ is a hypergroup, $(R, +, \cdot)$ is a hyperring and M is a hypermodule over the hyperring R. □

Lemma 2.4. *Let (M, h, k) be an (m, n)-hypermodule over an (m, n)-hyperring R. Then, N is an (m, n)-subhypermodule M over the (m, n)-hyperring R if and only if the following conditions hold:*

(1) *The equation $b \in h(a_1^{i-1}, x_i, a_{i+1}^m)$ is solvable at the place $i = 1$ and $i = m$ or at least one place $1 < i < m$, for every $a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_m, b \in N$.*
(2) *For any $r_1, r_2, \ldots, r_{n-1} \in R$ and $y \in N$ imply that $k(r_1, r_2, \ldots, r_{n-1}, y) \subseteq N$.*

Proof. N is an m-ary hypergroup by Theorem 2.3 of [3]. Since k is a closed scalar n-ary hyperoperation on N, then N is an (m, n)-subhypermodule on (m, n)-hyperring R. □

Definition 2.5. Let (M_1, h_1, k_1) and (M_2, h_2, k_2) be two (m, n)-hypermodules over an (m, n)-hyperring R. A *homomorphism* from M_1 to M_2 is a mapping ϕ: $M_1 \to M_2$ such that

(1) $\phi(h_1(a_1, \ldots, a_m)) = h_2(\phi(a_1), \ldots, \phi(a_m))$,
(2) $\phi(k(r_1, \ldots, r_{n-1}, a)) = k(r_1, \ldots, r_{n-1}, \phi(a))$,

for all $a_1^m \in M_1$, $a \in M$ and $r_1^{n-1} \in R$.

Lemma 2.6. *Let (M_1, h_1, k_1) and (M_2, h_2, k_2) be two (m, n)-hypermodule over an (m, n)-hyperring R and ϕ: $M_1 \to M_2$ a homomorphism. Then,*

(1) *If S is an (m, n)-subhypermodule of M_1 over an (m, n)-hyperring R, then $\phi(S)$ is an (m, n)-subhypermodule of M_2.*
(2) *If K is an (m, n)-subhypermodule of M_2 over an (m, n)-hyperring R, such that $\phi^{-1}(K) \neq \emptyset$, then $\phi^{-1}(K)$ is an (m, n)-subhypermodule of M_1.*

Proof.

(1) We know $\phi(S)$ is an m-ary subhypergroup of M_2. Let $r_1, r_2, \ldots, r_{n-1} \in R$ and $y \in \phi(S)$. Then, there exists $x \in S$ such that $\phi(x) = y$. Hence, $k(r_1, \ldots, r_{n-1}, y) = k(r_1, \ldots, r_{n-1}, \phi(x)) = \phi(r_1, \ldots, r_{n-1}, x) \in \phi(S)$.
(2) The proof of this part is similar to (1). □

Let (M, h, k) be an (m, n)-hypermodule over an (m, n)-hyperring R. An equivalence relation ρ on M is called *compatible* if $a_1 \rho b_1, \ldots, a_m \rho b_m$, then for all $a \in h(a_1, \ldots, a_m)$ there exists $b \in h(b_1, \ldots, b_m)$ such that $a\rho b$, and if $r_1, \ldots, r_{n-1} \in R$, and $x\rho y$, then for all $a \in k(r_1, \ldots, r_{n-1}, x)$ there exists $b \in k(r_1, \ldots, r_{n-1}, y)$ such that $a\rho b$.

Let (M, h, k) be an (m, n)-hypermodule over an (m, n)-hyperring R and ρ be an equivalence relation on M. Then, ρ is a *strongly compatible relation* if $a_i \rho b_i$ for all $1 \leqslant i \leqslant m$, then $h(a_1, \ldots a_m)\ \bar{\bar{\rho}} h(b_1, \ldots, b_m)$, and for every $r_1, \ldots, r_{n-1} \in R$ and $x\rho y$, then $k(r_1, \ldots, r_{n-1}, x)\ \bar{\bar{\rho}}\ k(r_1, \ldots, r_{n-1}, y)$. Now, we recall the following theorem from [3].

Theorem 2.7. *Let (H, f) be an m-ary hypergroup and let ρ be an equivalence relation on H. Then, the relation ρ is strongly compatible if and only if the quotient $(H/\rho, f/\rho)$ is an m-ary group.*

Now, we introduce the strong compatible relation Γ on an (m, n)-hyperring R.

Definition 2.8. Let (R, f, g) be an (m, n)-hyperring. For every $k \in \mathbb{N}$ and $l_1^s \in \mathbb{N}$, when $s = k(m-1)+1$, we define the relation $\Gamma_{k;l_1^s}$, as follows: $x\ \Gamma_{k;l_1^s}\ y$ if and only if there exist $x_{i1}^{it_i} \in R$, where $t_i = l_i(n-1)+1$, $i = 1, \ldots, s$ such that $\{x, y\} \subseteq f_{(k)}(u_1, \ldots, u_s)$, where for every $i = 1, \ldots, s$, $u_i = g_{(l_i)}(x_{i1}^{it_i})$.

Now, set $\Gamma_k = \bigcup_{l_1^s \in \mathbb{N}} \Gamma_{k;l_1^s}$ and $\Gamma = \bigcup_{k \in \mathbb{N}^*} \Gamma_k$. Then, the relation Γ is reflexive and symmetric. Let Γ^* be the transitive closure of the relation Γ.

Definition 2.9. Let (R, f, g) be an (m, n)-hyperring. For every $k \in \mathbb{N}$ and $l_1^s \in \mathbb{N}$, when $s = k(m-1)+1$, we define the relation $\alpha_{k;l_1^s}$, as follows: $x\ \alpha_{k;l_1^s}\ y$ if and only if there exist $x_{i1}^{it_i} \in R$, $\sigma \in \mathbb{S}_n$ and $\sigma_i \in \mathbb{S}_{t_i}$, where $t_i = l_i(n-1)+1$, $i = 1, \ldots, s$ such that $x \in f_{(k)}(u_1, \ldots, u_s)$ and $y \in f_{(k)}(u'_{\sigma(1)}, \ldots, u'_{\sigma(s)})$, where for every $i = 1, \ldots, s$, $u_i = g_{(l_i)}(x_{i1}^{it_i})$ and $u'_i = g_{(l_i)}(x_{i\sigma_i(1)}^{i\sigma_i(t_i)})$.

Now, set $\alpha_k = \bigcup_{l_1^s \in \mathbb{N}} \alpha_{k;l_1^s}$ and $\alpha = \bigcup_{k \in \mathbb{N}^*} \alpha_k$. Then, the relation α is reflexive and symmetric. Let α^* be the transitive closure of relation α.

Theorem 2.10. [10]. *The relation Γ^* is a strongly compatible relation on both m-ary hypergroup (R, f) and n-ary semihypergroup (R, g) and the quotient $(R/\Gamma^*, f/\Gamma^*, g/\Gamma^*)$ is an (m, n)-ring.*

Similar to the proof of Theorem 2.10, we have:

Theorem 2.11. *The relation α^* is a strongly compatible relation on both m-ary hypergroup (R, f) and n-ary semihypergroup (R, g) and the quotient $(R/\alpha^*, f/\alpha^*, g/\alpha^*)$ is a commutative (m, n)-ring.*

Theorem 2.12. [13] *Let (M, h, k) be an (m, n)-hypermodule over an (m, n)-hyperring R and ρ be an equivalence relation on M. The following conditions are equivalent.*

(1) *The relation ρ is strongly compatible.*
(2) *If $r_1, \ldots, r_{n-1} \in R$, $x_1^m, a, b \in M$ and $a\rho b$, then for every $(i = 1, \ldots, m)$, we have $h(x_1^{i-1}, a, x_{i+1}^m)\ \bar{\bar{\rho}}\ h(x_1^{i-1}, b, x_{i+1}^m)$ and $k(r_1, \ldots, r_{n-1}, a)\ \bar{\bar{\rho}}\ k(r_1, \ldots, r_{n-1}, b)$.*
(3) *The quotient $(M/\rho, h/\rho, k/\rho)$ is an (m, n)-module over an (m, n)-hyperring R. In other words, M is an m-ary group and the scalar n-ary hyperoperation k is singleton.*

Theorem 2.13 [13]. *Let (M, h, k) be an (m, n)-hypermodule over (m, n)-hyperring (R, f, g) and δ be a strongly compatible relation on f and g. Let ρ be a strongly compatible relation on h such that $\rho(k(r_1^{n-1}, x_i)) = k(\delta(r_1), \ldots, \delta(r_{n-1}), \rho(x_i))$. Then, $(M/\rho, h/\rho, k/\rho)$ is an (m, n)-module on (m, n)-ring $(R/\delta, f/\delta, g/\delta)$.*

3. Fundamental and commutative fundamental (*m*, *n*)-modules

Fundamental relations have an important role in the multialgebra [17]. In [13], Anvariyeh et al. defined the fundamental relation ϵ^* on an (m, n)-hypermodule (M, h, k) such that $(M/\epsilon^*, h/\epsilon^*, k/\epsilon^*)$ is the smallest (m, n)-module over the (m, n)-ring $(R/\Gamma^*, f/\Gamma^*, g/\Gamma^*)$.

In this section, we define the fundamental relation θ^* on an (m, n)-hypermodule (M, h, k) such that $(M/\epsilon^*, h/\epsilon^*, k/\epsilon^*)$ is the smallest *commutative* (m, n)-module over the (m, n)-ring $(R/\Gamma^*, f/\Gamma^*, g/\Gamma^*)$.

Let R be a hyperring and M be a hypermodule over R. We recall the definition of relation ϵ on M as follows [20]:

$$x\epsilon y \iff x, y \in \sum_{i=1}^{n} m'_i; \quad m'_i = m_i \quad \text{or } m'_i = \sum_{j=1}^{n_i}\left(\prod_{k=1}^{k_{ij}} x_{ijk}\right) z_i,$$
$$m_i \in M, \quad x_{ijk} \in R, \quad z_i \in M.$$

The equivalence relation ϵ^* (the transitive closure of ϵ) was first introduced by Vougiouklis on hyperrings and studied mainly by many authors concerning hypermodules. Now, we recall the definition of relation θ on M as follows [18]: $x\theta y \iff \exists n \in \mathbb{N}$, $\exists(m_1, \ldots, m_n) \in M^n$, $\exists(k_1, k_2, \ldots, k_n) \in \mathbb{N}^n$, $\exists \sigma \in \mathbb{S}_n$, $\exists(x_{i1}, x_{i2}, \ldots, x_{ik}) \in R^{k_i}$, $\exists \sigma_i \in S_{n_i}$, $\exists \sigma_{ij} \in \mathbb{S}_{k_{ij}}$, such that

$$x \in \sum_{i=1}^{n} m'_i, \quad m'_i = m_i \quad \text{or } m'_i = \sum_{j=1}^{n_i}\left(\prod_{k=1}^{k_{ij}} x_{ijk}\right) m_i$$

and $y \in \sum_{i=1}^{n} m\prime_{\sigma(i)}$, where

$$m'_{\sigma(i)} = m_{\sigma(i)} \quad \text{if } m'_i = m_i,$$
$$m'_{\sigma(i)} = B_{\sigma(i)} m_{\sigma(i)} \quad \text{if } m'_i = \sum_{j=1}^{n_i}\left(\prod_{k=1}^{k_{ij}} x_{ijk}\right) m_i,$$

with

$$B_i = \sum_{j=1} n_i A_{i\sigma_i(j)}, \quad A_{ij} = \prod_{k=1}^{k_{ij}} x_{ij\sigma_{ij}(k)}.$$

The relation θ is reflexive and symmetric. We denote θ^* the transitive closure of θ.

Remark 1. If M is a hypermodule over a hyperring R, the fundamental relation ϵ^* on M, defined as the smallest equivalence relation such that the quotient M/ϵ^* is a module over the corresponding fundamental ring such that M/ϵ^* as a group, is not abelian [18–20]. But the quotient M/θ^* is a module over the corresponding fundamental ring such that M/θ^* is an abelian group.

Let M be an (m, n)-hypermodule over an (m, n)-hyperring R. We define relations $\epsilon$and θ on M.

Definition 3.1. Let M be an (m, n)-hypermodule over an (m, n)-hyperring R. Let $x, y \in M$. Then, $x\ \theta\ y$ if and only if there exist $a, b_{ij}, c_{ijk} \in \mathbb{N}^*$, $x_i \in M$ and $x_{ijkl} \in R$, $\sigma \in \mathbb{S}_r$, $\sigma_i \in \mathbb{S}_{n-1}$, $\sigma_{ij} \in \mathbb{S}_{s_{ij}}$ and $\sigma_{ijk} \in \mathbb{S}_{t_{ijk}}$ where $1 \leqslant i \leqslant r = a(m-1)+1$, $1 \leqslant j \leqslant n-1$, $1 \leqslant k \leqslant s_{ij} = b_{ij}(m-1)+1$ and $1 \leqslant l \leqslant t_{ijk} = c_{ijk}(n-1)+1$ such that

$$x \in h_{(a)}(u_1, \ldots, u_r) \quad \text{and } y \in h_{(a)}(u'_{\sigma(1)}, \ldots, u'_{\sigma(r)}),$$

where $u_i = m_i$ or $k(A_{i1}^{in-1}, x_i)$, with

$$A_{ij} = f_{(b_{ij})}(B_{ij1}^{ijs_{ij}}) \quad \text{and } B_{ijk} = g_{(c_{ijk})}\left(x_{ijk1}^{ijkt_{ijk}}\right)$$

and

$$u'_i = \begin{cases} m_i & \text{if } u_i = m_i \\ k(A'^{i\sigma_i(n-1)}_{i\sigma_i(1)}, x_i) & \text{if } u_i = k(A_{i1}^{in-1}, x_i), \end{cases}$$

where

$$A'_{ij} = f_{(b_{ij})}(B'^{ij\sigma_{ij}(s_{ij})}_{ij\sigma_{ij}(1)}) \text{ with } B'_{ijk} = g_{(c_{ijk})}\left(x^{ijk\sigma_{ijk}(t_{ijk})}_{ijk\sigma_{ijk}(1)}\right).$$

We say that $x \epsilon y$ if in Definition 3.1, $\sigma = id_{S_r}$, $\sigma_i = id_{S_{n-1}}$, $\sigma_{ij} = id_{S_{s_{ij}}}$ and $\sigma_{ijk} = id_{S_{t_{ijk}}}$. Relations θ and ϵ are reflexive and symmetric. Let θ^* and ϵ^* be their transitive closure, respectively.

Theorem 3.2. *The relation θ^* is a strongly compatible relation on M, as (m, n)-hypermodule, on both m-ary hyperoperation h and scalar n-ary hyperoperation k.*

Proof. Let $a_1\, \theta^*\, b_1, \ldots, a_m\, \theta^*\, b_m$. Then, $\theta^*(a_1) = \theta^*(b_1), \ldots, \theta^*(a_m) = \theta^*(b_m)$. For every $a \in h(a_1, \ldots, a_m)$ and $b \in h(b_1, \ldots, b_m)$, we have

$$\begin{aligned}\theta^*(a) &= \theta^*(h(a_1, \ldots, a_m)) = h/\theta^*(\theta^*(a_1), \ldots, \theta^*(a_m)) \\ &= h/\theta^*(\theta^*(b_1), \ldots, \theta^*(b_m)) = \theta^*(h(b_1, \ldots, b_m)) = \theta^*(b).\end{aligned}$$

Now, let $r_1, \ldots, r_{n-1} \in R$, $a_1, b_1 \in M$ and $a_1\theta^* b_1$. Then, for every $a \in k(r_1, \ldots, r_{n-1}, a_1)$ and $b \in k(r_1, \ldots, r_{n-1}, b_1)$, we have

$$\begin{aligned}\theta^*(a) &= \theta^*(k(r_1, \ldots, r_{n-1}, a_1)) = k/\theta^*(r_1, \ldots, r_{n-1}, \theta^*(a_1)) \\ &= k/\theta^* k(r_1, \ldots, r_{n-1}, \theta^*(b_1)) = \theta^*(b). \qquad \square\end{aligned}$$

Theorem 3.3. *Let (M, h, k) be an (m, n)-hypermodule over the (m, n)-hyperring R. Then, the quotient $(M/\theta^*, h/\theta^*, k/\theta^*)$ is an (m, n)-hypermodule over an (m, n)-hyperring R, where*

$$\begin{aligned}h/\theta^*(\theta^*(a_1), \ldots, \theta^*(a_m)) &:= \{\theta^*(a) | a \in h(a_1, \ldots, a_m)\} \\ &= \theta^*(h(a_1^m))\end{aligned}$$

and

$$\begin{aligned}k/\theta^*(r_1, \ldots, r_{n-1}, \theta^*(a)) &:= \{\theta^*(x) | x \in k(r_1, \ldots, r_{n-1}, a)\} \\ &= \theta^*(k(r_1^{n-1}, a)).\end{aligned}$$

Moreover, $(M/\theta^*, h/\theta^*)$ is an abelian group and for every $r_1, \ldots, r_{n-1} \in R$, $x \in M$ and $\tau \in \mathbb{S}_{n-1}$ we have $k/\theta^*(r_1, \ldots, r_{n-1}, \theta^*(a)) = k/\theta^*(r_{\tau(1)}, \ldots, r_{\tau(n-1)}, \theta^*(a))$.

Proof. $\theta^*$is a strongly compatible relation on M by Theorem 3.2. Now, by Theorem 2.12 and definition of relation θ, the proof is completed. $\square$

Example 6. Let (R, f, g) be a non-commutative (m, n)-ring. Then, (R, f, g) is an (m, n)-module over the (m, n)-ring (R, f, g). It easy to see that $\epsilon^* = \Gamma^* = \{(x, x) | x \in R\} \neq \theta^* = \alpha^*$.

Example 7. Let (G, f) be a non-commutative m-ary group and $a \in G$. Let H be a non-empty set such that $H \cap G = \emptyset$. Let f_H be an m-ary hyperoperation define on $G \cup H$ as follows:

$$f_H(x_1^m) = \begin{cases} f(y_1^m) & \text{if } f(y_1^m) \neq a \\ \{a\} \cup H & \text{if } f(y_1^m) = a \end{cases} \quad \text{for all } x_1^m \in G \cup H,$$

where $y_i = x_i$ if $x_i \in G$ and $y_i = a$ if $x_i \in H$. Then, $(R = G \cup H, f_H)$ is an m-ary hypergroup. Now, we define an n-ary hyperoperation g_H as follows:

$$g_H(x_1^n) = \{a\} \cup H, \quad \text{for all } x_1^n \in R.$$

It is not difficult to see that (R, f_H, g_H) is an (m, n)-hyperring. Let $M = R$, $h = f_G$ and $k = g_H$, then (M, h, k) is an (m, n)-hypermodule over the (m, n)-hyperring R and

$$\{(x, x) | x \in M\} \neq \epsilon^* = \Gamma^* \neq \theta^* = \alpha^*.$$

We consider the natural map π: $M \to M/\theta^*$, where $\pi(x) = \theta^*(x)$.

Theorem 3.4. [13]*Let (M_1, h_1, k_1) and (M_2, h_2, k_2) be two (m, n)-hypermodules over an (m, n)-hyperring R, and let ϕ: $M_1 \to M_2$ be a homomorphism. Then, there exists a compatible relation ρ on M_1 and a homomorphism ψ: $M_1/\rho \to M_2$ such that $\psi \circ \pi = \phi$.*

Theorem 3.5. [13]*Let ρ and ϑ be compatible relations on (m, n)-hypermodule (M, h, k) over an (m, n)-hyperring R, such that $\rho \subseteq \vartheta$. Then, there exists a compatible relation μ on $(M/\rho, h/\rho, k/\rho)$ such that $(M/\rho)/\mu$ is isomorphic to M/ϑ, as (m, n)-hypermodules.*

Let (M_1, h_1, k_1) and (M_2, h_2, k_2) be two (m, n)-hypermodules over an (m, n)-hyperring R. Define the direct hyperproduct $(M_1 \times M_2, h_1 \times h_2, k_1 \times k_2)$ to be the (m, n)-hypermodule whose universe is the set $M_1 \times M_2$ and such that for $a_i \in M_i$, $a'_i \in M_2$, $1 \leqslant i \leqslant m$,

$$\begin{aligned}&(h_1 \times h_2)((a_1, a'_1), \ldots, (a_m, a'_m)) \\ &\quad = \{(a, a') | a \in h_1(a_1, \ldots, a_m), a' \in h_2(a'_1, \ldots, a'_m)\},\end{aligned}$$

and

$$\begin{aligned}(k_1 \times k_2)(r_1, \ldots, r_{n-1}, (x, x')) = \{(a, a') | a &\in k_1(r_1, \ldots, r_{n-1}, x), \\ a' &\in k_2(r_1, \ldots, r_{n-1}, x')\}.\end{aligned}$$

The mapping π_i: $M_1 \times M_2 \to M_i$, $i = 1, 2$, defined by $\pi_i((a_1, a_2)) = a_i$, is called the *projection map* on the ith coordinate of $M_1 \times M_2$, also the mapping π_i: $M_1 \times M_2 \to M_i$ is an onto homomorphism.

If (M, h, k) is an (m, n)-hypermodule, then $\hat{\theta}$ denoted the *transitive closure* of the relation $\theta = \cup_{p \geqslant 0} \theta_p$, where θ_0 is the diagonal, i.e., $\theta_0 = \{(x, x) | x \in M\}$ and for every integer $p \geqslant 1$, θ_p is the relation defined as follows:

$$x\theta_p y \quad \text{if and only if } x \in h_{(p)}(u_1^r), \quad y \in h_{(p)}(u'_{\sigma(1)^{\sigma(r)}}), \quad \text{with}$$
$$r = h(m-1) + 1,$$

where u_i and u'_i are defined in Definition 3.1. If $x\theta_0 y$ (i.e., $x = y$) then, we write $\{x, y\} \subseteq u_{(0)}$. We define θ^* as the smallest equivalence relation such that the quotient $(M/\theta^*, h/\theta^*, k/\theta^*)$ is an (m, n)-module over an (m, n)-hyperring R, where M/θ^* is the set of all equivalence classes. The θ^* is called the *commutative fundamental equivalence relation*.

Lemma 3.6. *Let (M, h, k) be an (m, n)-hypermodule over an (m, n)-hyperring R. Then, for every $p \in \mathbb{N}^*$, we have $\theta_p \subseteq \theta_{p+1}$.*

Proof. Let $x\theta_p y$. Then, there exists $p \in \mathbb{N}$, and $u_1, \ldots, u_r$, where $r = p(m-1)+1$, such that $x \in h_{(p)}(u_1^r)$ and $y \in h_{(p)}(u'_{\sigma(1)^{\sigma(r)}})$. By reproducibility of h, there exist $v_1, \ldots, v_m$, such that $u_1 \subseteq h(v_1, \ldots, v_m)$. If $\sigma(t) = 1$, then

$$x \in h_{(p)}(u_1^r) = h_{(p)}(u_1 \ldots, u_r) \subseteq h_{(p)}(h(v_1, \ldots, v_m), u_2, \ldots u_r) = h_{(p+1)}(v_1^p, u_2^r),$$

$$\begin{aligned} y \in h_{(p)}\left(u'_{\sigma(1)^{\sigma(r)}}\right) &= h_{(p)}\left(u'_{\sigma(1)} \ldots, u'_{\sigma(t-1)}, u'_{\sigma(t)}, u'_{\sigma(t+1)}, \ldots, u'_{\sigma(r)}\right) \\ &\subseteq h_{(p)}(u'_{\sigma(1)} \ldots, u'_{\sigma(t-1)}, h(v_1, \ldots, v_m), \\ &\quad u'_{\sigma(t+1)}, \ldots, u'_{\sigma(r)}) = h_{(p+1)}(u'_{\sigma(1)} \ldots, u'_{\sigma(t-1)}, \\ &\quad v_1, \ldots, v_m, u'_{\sigma(t+1)}, \ldots, u'_{\sigma(r)}). \end{aligned}$$

This means that $x\theta_{p+1}y$. □

Corollary 3.7. *Let (M, h, k) be an (m, n)-hypermodule over an (m, n)-hyperring R. Then, for every $p \in \mathbb{N}^*$, we have $\theta_p^* \subseteq \theta_{p+1}^*$.*

Theorem 3.8. *The fundamental relation θ^* is the transitive closure of the relation θ, i.e., $(\theta^* = \hat{\theta})$.*

Proof. Similar to the proof of Theorem 2.10 of [3], we know that the quotient $M/\hat{\theta}$ is an m-ary group, where $h/\hat{\theta}$ is defined in the usual manner

$$h/\hat{\theta}(\hat{\theta}(x_1), \ldots, \hat{\theta}(x_m)) = \{\hat{\theta}(y) | y \in h(\hat{\theta}(x_1), \ldots, \hat{\theta}(x_m))\}$$

for all $x_1, \ldots, x_m \in M$.

Now, we prove that $M/\hat{\theta}$ is an (m, n)-module over an (m, n)-hyperring R. The scalar n-ary hyperoperation $k/\hat{\theta}$ in $M/\hat{\theta}$ is defined in the usual manner:

$$k/\hat{\theta}(r_1, \ldots, r_{n-1}, \hat{\theta}(x)) = \{\hat{\theta}(y) | y \in k(r_1, \ldots, r_{n-1}, x)\},$$

for all $r_1, \ldots, r_{n-1} \in H$ and $x \in M$. Suppose $a \in \hat{\theta}(x)$. Then, we have $a\hat{\theta}x$, if there exist $x_1, \ldots, x_m$ with $x_1 = a, \ldots, x_m = x$ such that $\{x_i, x_{i+1}\} \subseteq h_{(i)}$. So every element $z \in k(r_1, \ldots, r_{n-1}, x_i)$ is equivalent to every element to $k(r_1, \ldots, r_{n-1}, x_{i+1})$. Therefore, $k/\theta^*(r_1, \ldots, r_{n-1}, \theta^*(x))$ is a singleton. So, we can write $k/\theta^*(r_1, \ldots, r_{n-1}, \theta^*(x)) = \theta^*(y)$ for all $y \in k(r_1, \ldots, r_{n-1}, \theta^*(x))$.

Moreover, since k has n-ary hypermodule scalar properties, consequently, $k/\hat{\theta}$ has (m, n)-hypermodule scalar properties.

Now, let θ be an equivalence relation on M such that M/θ is (m, n)-hypermodule over an (m, n)-hyperring R. Then, for all $x_1, \ldots, x_m \in M$, we have $h/\theta(\theta(x_1), \ldots, \theta(x_m)) = \theta(y)$ for all $y \in h(\theta(x_1), \ldots, \theta(x_m))$. Also $k/\theta(r_1, \ldots, r_{n-1}, \theta(x)) = \theta(z)$, for all $z \in k(r_1, \ldots, r_{n-1}, \theta(x))$. But also, for every $x_1, \ldots, x_m$, $x \in M$, $r_1, \ldots, r_{n-1} \in R$, $A_i \subseteq \theta(x_i)$,$(i = 1, \ldots, m)$ and $A \subseteq \theta(x)$, we have

$$h/\theta(\theta(x_1), \ldots, \theta(x_m)) = \theta(h(x_1, \ldots, x_m)) = \theta(h(A_1, \ldots, A_m))$$

and

$$\begin{aligned} k/\theta(r_1, \ldots, r_{n-1}, \theta(x)) &= \theta(k(r_1, \ldots, r_{n-1}, x)) \\ &= \theta(k(r_1, \ldots, r_{n-1}, A)). \end{aligned}$$

Therefore, $\theta(a) = \theta(u_{(i)})$ for all $i \geqslant 0$ and for all $a \in h_u$ or k. So for every $a \in M$, $x \in \theta(a)$ which implies $x \in \theta(a)$. But θ is transitively closed, so we obtain $x \in \theta^*(a)$ which implies $x \in \theta(a)$. Hence, the relation θ^* is the smallest equivalence relation on M such that M/θ^* is an (m, n)-hypermodule over an (m, n)-hyperring R. □

Theorem 3.9. *Let (M, h, k) be an (m, n)-hypermodule over (m, n)-hyperring (R, f, g). Then, $(M/\theta^*, h/\theta^*, k/\theta^*)$ is a commutative (m, n)-module on a commutative (m, n)-ring $(R/\alpha^*, f/\alpha^*, g/\alpha^*)$.*

Proof. By Theorem 3.2, θ^* is a strongly compatible relation on M, and similar to the proof of Theorem 4.1 of [3], $(M/\theta^*, h/\theta^*)$ is an m-ary group. Also, by Theorem 2.11, R/α^*, $(f/\alpha^*, g/\alpha^*)$ is a commutative (m, n)-ring. Now, let $r_1, \ldots, r_{n-1} \in R$, $x \in M$ and define $k_{\theta^*}(\alpha^*(r_1), \ldots, \alpha^*(r_{n-1}), \theta^*(x)) := k(\alpha^*(r_1), \ldots, \alpha^*(r_{n-1}), \theta^*(x))$. If $x \in h_a(u_1, \ldots, u_r)$ and $r_i \in f_{k_i}(u'_1, \ldots, u'_s)$. Then,

$$\begin{aligned} k(\alpha^*(r_1) \ldots, \alpha^*(r_{n-1}), \theta^*(x)) &\subseteq k(f_{k_1}, \ldots, f_{k_{n-1}}, h_a(u_1, \ldots, u_r)) \\ &= h_a(k(f_{k_1}, \ldots, f_{k_{n-1}}, u_1), \ldots, k(f_{k_1}, \ldots, f_{k_{n-1}}, u_r)). \end{aligned}$$

So, for every $r'_1\alpha^* r_1, \ldots . r'_{n-1}\alpha^* r_{n-1}$ and $y\theta^* x$, we have $k(\alpha^*(r'_1) \ldots, \alpha^*(r'_{n-1}), \theta^*(y)) \subseteq h_a(k(f_{k_1}, \ldots, f_{k_{n-1}}, u_1), \ldots, k(f_{k_1}, \ldots, f_{k_{n-1}}, u_r))$.

Since M is an (m, n)-hypermodule on (m, n)-hyperring R, the properties of M as an (m, n)-hypermodule guarantee that the m-ary group M/θ^* is an $(m, n) - ary$ R/α^*-module. □

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