Journal of the Egyptian Mathematical Society (2014) 22, 162–166

Egyptian Mathematical Society

Journal of the Egyptian Mathematical Society

www.etms-eg.org
www.elsevier.com/locate/joems

ORIGINAL ARTICLE

Positive solutions of a quadratic integro-differential equation

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Received 6 May 2013; revised 16 July 2013; accepted 25 July 2013
Available online 29 September 2013

KEYWORDS
Integro-differential quadratic equation;
Measure of noncompactness;
Positive absolute continuous solutions

Abstract In this paper, we study the existence of at least one positive and nondecreasing solution for the initial value problem of a quadratic integro-differential equation by applying the technique of measure of noncompactness. Some examples will be included to illustrate the obtained results.

MATHEMATICS SUBJECT CLASSIFICATION: 11D09; 47Gxx; 47H10; 45G10

1. Introduction

The study of initial value problems of nonlinear quadratic functional differential and integral equations is initiated in the works of Dhage [1] and Dhage and O'Regan [2]. The theory of quadratic integral equations is also intensively studied and finds numerous applications in describing real world problems (see [3–21] for instance). Many authors studied the existence of solutions for several classes of nonlinear quadratic integral equations with nonsingular kernels (see e.g. [4–8] and the references therein). The quadratic integral equation can be very often encountered in many applications (see [11–16] and the references therein).

Many papers studied the existence of integrable solutions (see [13,15,21]) by applying the technique of measure of noncompactnes.

In this paper, we deal with the quadratic integro differential equation

$$x(t) = \int_0^t f_1(s, x'(s))\, ds \int_0^t f_2(s, x'(s))\, ds \quad a.e., \quad t \in (0,1] \tag{1}$$

$$\text{with} \quad x(0) = x_0 \tag{2}$$

We prove the existence of at least one solution $x \in AC(0,1]$ (absolutely continuous on $(0,1]$) of the quadratic integro differential Eq. (1) with the initial condition (2) where the functions $f_i(t, x(t))$, $i = 1, 2$ are L_1-Carathèodory functions. Our proof depends on the measure of noncompactness. In fact, our result in this paper is motivated by the extension of the work of El-Sayed and Hashem [13].

2. Definitions and auxiliary facts

In this section, we collect some definitions and results needed in our further investigations.

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Peer review under responsibility of Egyptian Mathematical Society.

Production and hosting by Elsevier

1110-256X
http://dx.doi.org/10.1016/j.joems.2013.07.014

Let $L_1 = L_1(I)$, $I = [0,1]$ be denoted the space of lebesgue integrable functions on I and the norm in $L_1(I)$ is defined by

$$\|x\|_{L_1} = \int_0^1 |x(t)|dt$$

Assume that the function f: $I \times R \to R$ satisfies Carathèodory condition i.e., it is measurable in t for any x and continuous in x for almost all t. Then to every function $x(t)$ being measurable on the interval I we may assign the function

$$(Fx)(t) = f(t, x(t)), \quad t \in I.$$

The operator F defined in such a way is called the superposition operator. This operator is one of the simplest and most important operators investigated in nonlinear functional analysis. For this operator we have the following theorem due to Krasnosel'skii [22].

Theorem 1. *The superposition operator F maps L_1 into itself if and only if*

$$|f(t,x)| \leqslant c(t) + k|x| \text{ for all } \quad t \in I$$

and $x \in R$, where $c(t)$ is a function from L_1 and k is a nonnegative constant.

Now let E be a Banach space with zero element θ and let X be a nonempty bounded subset of E. Moreover denote by $B_r = B(\theta, r)$ the closed ball in E centered at θ and with radius r. In the sequel we shall need some criteria for compactness in measure; the complete description of compactness in measure was given by Fre'chet [22], but the following sufficient condition will be more convenient for our purposes (see [22]).

Theorem 2. *Let X be a bounded subset of L_1. Assume that there is a family of subsets $(\Omega_c)_{0 \leqslant c \leqslant b-a}$ of the interval (a,b) such that meas $\Omega_c = c$ for every $c \in [0, b-a]$, and for every $x \in X$, $x(t_1) \leqslant x(t_2)$, $(t_1 \in \Omega_c,\ t_2 \notin \Omega_c)$, then the set X is compact in measure.*

The measure of weak noncompactness defined by De Blasi [3,20] is given by,

$$\beta(X) = \inf(r > 0 \\ : \text{there exists a weakly compact subset } Y \text{ of } E \text{ such that } X \subset Y + K_r)$$

The function $\beta(X)$ possesses several useful properties which may be found in [20]. The convenient formula for the function $\beta(X)$ in L_1 was given by Appel and De Pascale (see [3])

$$\beta(X) = \lim_{\epsilon \to 0}\left(sup_{x \in X}\left(sup\left[\int_D |x(t)|dt : D \subset [a,b], \text{meas } D \leqslant \epsilon\right]\right)\right), \tag{3}$$

where the symbol meas$D$ stands for Lebesgue measure of the set D.

Next, we shall also use the notion of the Hausdorff measure of noncompactness χ (see [22]) defined by

$$\chi(X) = inf(r > 0 \\ : \text{ there exists a finite subset } Y \text{ of } E \text{ such that } X \subset Y + K_r)$$

In the case when the set X is compact in measure, the Hausdorff and De Blasi measures of noncompactness will be identical. Namely we have (see [3,20])

Theorem 3. *Let X be an arbitrary nonempty bounded subset of L_1. If X is compact in measure then $\beta(X) = \chi(X)$.*

Finally, we will recall the fixed point theorem due to Darbo [7].

Theorem 4. *Let Q be a nonempty, bounded, closed and convex subset of E and let H: $Q \to Q$ be a continuous transformation which is a contraction with respect to the Hausdorff measure of noncompactness χ, i.e., there exists a constant $\alpha \in [0,1)$ such that $\chi(HX) \leqslant \alpha\chi(X)$ for any nonempty subset X of Q. Then H has at least one fixed point in the set Q.*

3. Existence of solutions

Firstly, we study the existence of solutions of the quadratic integral equation

$$x(t) = f_1(t, x(t)) \int_0^t f_2(s, x(s))\, ds + f_2(t, x(t)) \times \int_0^t f_1(s, x(s))\, ds \tag{4}$$

Let the integral operator H_i be defined as

$$(H_i x)(t) = \int_0^t f_i(s, x(s))\, ds, \quad i = 1, 2.$$

Then Eq. (4) may be written in operator form as:

$$(Ax)(t) = (F_1 x)(t) \cdot (H_2 x)(t) + (F_2 x)(t) \cdot (H_1 x)(t)$$

where $(F_i x)(t) = f_i(t, x(t))$, $i = 1, 2$.

Consider the assumptions:

(i) f_i: $I \times R_+ \to R_+$ satisfy Carathèodory condition (i.e. measurable in t for all $x \in R_+$ and continuous in x for all $t \in [0,1]$) and there exist two functions a_1, $a_2 \in L_1$ and constants b_1, $b_2 > 0$ such that

$$f_i(t,x) \leqslant a_i(t) + b_i|x| \quad \forall (t,x) \in I \times R_+.$$

Moreover, $f_i(t,x)$, $i = 1, 2$ are a.e. nondecreasing in both variables;

(ii) Let $d > \sqrt{16\ b_1 b_2 \|a_1\| \cdot \|a_2\|}$,where $d = 1 - 2b_1\|a_2\| - 2b_2\|a_1\|$.

Now let r be a positive root of the equation

$$2b_1 b_2 r^2 - (1 - 2b_1\|a_2\| - 2b_2\|a_1\|)\ r + 2\|a_1\| \cdot \|a_2\| = 0.$$

Define the set

$$B_r = \{x \in L_1 : \|x\| \leqslant r\}.$$

For the existence of at least one L_1-positive solution of the quadratic integral Eq. (4) we have the following theorem.

Theorem 5. *Let the assumptions (i) and (ii) be satisfied. If $2rb_1b_2 < 1$, then the quadratic integral Eq. (4) has at least one solution $x \in L_1$ which is positive and a.e. nondecreasing on I.*

Proof. Take an arbitrary $x \in L_1$, then, we get

$$|(Ax)(t)| \leqslant (a_1(t)+b_1|x(t)|)\int_0^t (a_2(s)+b_2|x(s)|)\,ds + (a_2(t) + b_2|x(t)|)\int_0^t (a_1(s)+b_1|x(s)|)\,ds$$

which implies that

$$\begin{aligned}
\|(Ax)(t)\| &= \int_0^1 |(Ax)(t)|\,dt\\
&\leqslant \int_0^1 a_1(t)\int_0^t a_2(s)\,ds\,dt + b_2\int_0^1 a_1(t)\int_0^t |x(s)|\,ds\,dt\\
&\quad + b_1\int_0^1 |x(t)|\int_0^t a_2(s)\,ds\,dt + b_1 b_2\\
&\quad \times \int_0^1 |x(t)|\int_0^t |x(s)|\,ds\,dt + \int_0^1 a_2(t)\int_0^t a_1(s)\,ds\,dt\\
&\quad + b_1\int_0^1 a_2(t)\int_0^t |x(s)|\,ds\,dt + b_2\\
&\quad \times \int_0^1 |x(t)|\int_0^t a_1(s)\,ds\,dt + b_1 b_2\\
&\quad \times \int_0^1 |x(t)|\int_0^t |x(s)|\,ds\,dt\\
&\leqslant \int_0^1 a_2(s)\int_s^1 a_1(t)dtds + b_2\int_0^1 |x(s)|\int_s^1 a_1(t)dtds\\
&\quad + b_1\int_0^1 a_2(s)\int_s^1 |x(t)|\,dt\,ds + 2b_1 b_2\\
&\quad \times \int_0^1 |x(s)|\int_s^1 |x(t)|dtds + \int_0^1 a_1(s)\int_s^1 a_2(t)\,dt\,ds\\
&\quad + b_1\int_0^1 |x(s)|\int_s^1 a_2(t)dtds + b_2\int_0^1 a_1(s)\\
&\quad \times \int_s^1 |x(t)|\,dt\,ds\\
&\leqslant \|a_1\|\int_0^1 a_2(s)\ ds + b_2\|a_1\|\int_0^1 |x(s)|ds\\
&\quad + b_1\|x\|\int_0^1 a_2(s)\ ds + 2b_1 b_2\|x\|\int_0^1 |x(s)|\,ds\\
&\quad + \|a_2\|\int_0^1 a_1(s)\ ds + b_1\|a_2\|\int_0^1 |x(s)|\,ds\\
&\quad + b_2\|x\|\int_0^1 a_1(s)\,ds\\
&\leqslant 2\|a_1\|\|a_2\| + 2b_1\|x\|\|a_2\| + 2b_2\|a_1\|\|x\|\\
&\quad + 2b_1 b_2\|x\|^2\\
&\leqslant r.
\end{aligned}$$

From this estimate we show that the operator A maps the ball B_r into itself with

$$r = \frac{d-\sqrt{d^2-16b_1b_2\|a_1\|\cdot\|a_2\|}}{2b_1b_2}.$$

From assumption (ii) we have

$$0 < d^2 - 16b_1b_2\|a_1\|\cdot\|a_2\| < d^2,$$

which implies that

$$0 < \sqrt{d^2-16b_1b_2\|a_1\|\cdot\|a_2\|} < d.$$

Then d is positive which implies that r is a positive constant.

Now, let Q_r denote the subset of $B_r \in L_1$ consisting of all functions which are a.e. nondecreasing on I.

The set Q_r is nonempty, bounded, convex and closed (see Banaś [22, pp. 780]). Moreover this set is compact in measure (see Lemma 2 in [23, pp. 63]).

From assumption (i) we deduce that the operator A maps Q_r into itself. Since the operator $(F_i x)(t) = f_i(t, x(t))$ is continuous (Theorem 1 in Section 2), then the operator H_i is continuous and hence the product $F_i \cdot H_i$ is continuous. Thus the operator A is continuous on Q_r.

Let X be a nonempty subset of Q_r. Fix $\epsilon > 0$ and take a measurable subset $D \subset I$ such that meas $D \leqslant \epsilon$. Then, for any $x \in X$, using the same reasoning as in [22,23], we get

$$\begin{aligned}
\|Ax\|_{L_1(D)} &= \int_D |(Ax)(t)|\,dt\\
&\leqslant \int_D a_1(t)\int_0^t a_2(s)\,ds\,dt + \int_D a_2(t)\int_0^t a_1(s)\,ds\,dt + b_2\\
&\quad \times \int_D a_1(t)\int_0^t |x(s)|\,ds\,dt + b_1\int_D a_2(t)\\
&\quad \times \int_0^t |x(s)|\,ds\,dt + b_1\int_D |x(t)|\int_0^t a_2(s)\,ds\,dt + b_2\\
&\quad \times \int_D |x(t)|\int_0^t a_1(s)\,ds\,dt + 2b_1 b_2\\
&\quad \times \int_D |x(t)|\int_0^t |x(s)|\,ds\,dt\\
&\leqslant \int_D a_2(s)\int_D a_1(t)dtds + \int_D a_1(s)\int_D a_2(t)dtds + b_2\\
&\quad \times \int_D |x(s)|\int_D a_1(t)\,dt\,ds + b_1\int_D |x(s)|\int_D a_2(t)\,dt\,ds\\
&\quad + b_1\int_D a_2(s)\int_D |x(t)|dtds + b_2\int_D a_1(s)\\
&\quad \times \int_D |x(t)|\,dt\,ds + 2b_1 b_2\int_D |x(s)|\int_D |x(t)|\,dt\,ds\\
&\leqslant \|a_1\|_{L_1(D)}\int_D a_2(s)ds + \|a_2\|_{L_1(D)}\int_D a_1(s)ds + b_1\\
&\quad \times \int_D a_2(s)\int_D |x(t)|\,dt\,ds + b_2\int_D a_1(s)\int_D |x(t)|\,dt\,ds\\
&\quad + b_2\|a_1\|_{L_1(D)}\int_D |x(s)|ds + b_1\|a_2\|_{L_1(D)}\int_D |x(s)|\,ds\\
&\quad + 2b_1 b_2\int_D |x(s)|\int_D |x(t)|\,dt\,ds\\
&\leqslant 2\|a_1\|_{L_1(D)}\|a_2\|_{L_1(D)} + 2b_1\|x\|_{L_1(D)}\|a_2\|_{L_1(D)}\\
&\quad + 2b_2\|a_1\|_{L_1(D)}\|x\|_{L_1(D)} + 2b_1 b_2\|x\|_{L_1(D)}\|x\|_{L_1(D)}\\
&\leqslant 2\|a_1\|_{L_1(D)}\|a_2\|_{L_1(D)} + rb_1\|a_2\|_{L_1(D)} + rb_2\|a_1\|_{L_1(D)}\\
&\quad + 2rb_1 b_2\|x\|_{L_1(D)}.
\end{aligned}$$

Since

$$\lim_{\epsilon\to 0}\{\sup\{\int_D |a_i(t)|dt : D \subset I,\ \text{meas } D < \epsilon\}\} = 0, \quad i = 1, 2.$$

We obtain

$$\beta(Ax(t)) \leqslant 2rb_1b_2\beta(x(t)).$$

This implies

$$\beta(AX) \leqslant 2rb_1b_2\beta(X), \tag{5}$$

where β is the De Blasi measure of weak noncompactness.

Keeping in mind Theorem 3 we can write (5) in the form

$$\chi(AX) \leqslant 2rb_1b_2\chi(X),$$

where χ is the Hausedorff measure of noncompactness.

Since $2r\ b_1b_2 < 1$, from Theorem 4 follows that A is contraction with respect to the measure of noncompactness χ. Thus A has at least one fixed point in Q_r which is a solution of the quadratic integral Eq. (4). □

Definition 6. By a solution of the problem of quadratic integro-differential Eqs. (1) and (2) we mean a function $x \in AC(0,1]$ and this function satisfies (1) and (2).

Theorem 7. *Let the assumption of Theorem 5 are satisfied, then there exists at least one solution $x \in AC(0,1]$ of the quadratic integro-differential Eqs. (1) and (2) which is positive and nondecreasing on I.*

Proof. Differentiation both sides of (1), we obtain

$$x'(t) = f_1(t,x'(t))\int_0^t f_2(s,x'(s))\,ds + f_2(t,x'(t))\int_0^t f_1(s,x'(s))\,ds$$

put $x'(t) = u(t) \in L_1$, then (1) will be similar to (4), and,

$$x(t) = x(0) + \int_0^t u(s)\,ds \in AC(0,1],$$

and then from Theorem 5 there exist at least one positive and nondecreasing solution of (1) and (2). □

4. Examples

In this section we provide some examples illustrating our result obtained in Theorem 7.

Example 8. Consider the problem

$$\begin{cases} x(t) = \left(\int_0^t f(s,x'(s)ds)\right)^2, & a.e\ t \in (0,1] \\ x(0) = x_0, \end{cases}$$

then this problem has at least one positive and nondecreasing solution $x \in AC(0,1]$, by taking $f_1(t,x(t)) = f_2(t,x(t))$ in Eq. (1).

Example 9. Consider the quadratic integro-differential problem

$$\begin{cases} x(t) = \int_0^t \left(\frac{s}{10} + \frac{1}{3-s}x'(s)\right)ds \int_0^t \left(-\frac{1}{6}\ln(1-s) + \frac{1}{3-s}x'(s)\right)ds, & a.e.\ t \in (0,1] \\ x(0) = x_0, \end{cases}$$

observe that the above problem is a special case of (1) and (2). Indeed if we put

$$f_1(t,y) = \frac{t}{10} + \frac{1}{3-t}y(t), \quad f_2(t,y) = -\frac{1}{6}\ln(1-t) + \frac{1}{3-t}y(t).$$

Then we can easy check that the assumptions of Theorem 7 are satisfied, then the problem has at least one positive and nondecreasing positive solution $x \in AC(0,1]$.

Acknowledgement

The author is extending his heartfelt thanks to the reviewers for their valuable suggestions for the improvement of the article.

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