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ORIGINAL ARTICLE

On left (θ, ϕ)-derivations in *BCI*-algebras

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θ-ideal

Abstract The notion of (regular) left (θ, ϕ)-derivations of a *BCI*-algebra is introduced, some useful examples are discussed, and related properties are investigated. Conditions for a left (θ, ϕ)-derivation to be regular are provided. The concepts of a $d_{(\theta,\phi)}$-invariant left (θ, ϕ)-derivation and θ-ideal are introduced, and their relations are discussed. Furthermore, some more interesting results are established.

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1. Introduction

Several authors [2–5] have studied derivations in rings and near-rings. Jun and Xin [6] applied the notion of derivation in ring and near-ring theory to *BCI*-algebras, and as a result, they introduced a new concept, called a (regular) derivation, in *BCI*-algebras. Using this concept as defined, they investigated some of its properties. Using the notion of a regular derivation, they also established characterizations of a *p*-semisimple *BCI*-algebra. For a self-map d of a *BCI*-algebra, they defined a d-invariant ideal and gave conditions for an ideal to be d-invariant. After the work of Jun and Xin [6], many research articles have been appeared on the derivations of BCI-algebras, and a greater interest has been devoted to the study of derivation in BCI-algebras (see [1,7–11]). In [1], Abujabal and Al-Shehri introduced the notion of left derivation of a *BCI*-algebra and investigated some related properties.

The aim of this paper is to consider the generalizations of some results of Abujabal and Al-Shehri [1]. We introduce the notion of left (θ, ϕ)-derivations of a *BCI*-algebra X and investigate related properties. We provide conditions for a left (θ, ϕ)-derivation to be regular. We also introduce the concepts of a $d_{(\theta,\phi)}$-invariant left (θ, ϕ)-derivation and θ-ideal, and then, we investigate their relations. Finally, we establish some more interesting results.

2. Preliminaries

We begin with the following definitions and properties that will be needed in the sequel.

A nonempty set X with a constant 0 and a binary operation $*$ is called a *BCI*-algebra if for all $x, y, z \in X$, the following conditions hold:

(I) $((x * y) * (x * z)) * (z * y) = 0$,
(II) $(x * (x * y)) * y = 0$,

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(III) $x * x = 0$,
(IV) $x * y = 0$ and $y * x = 0$ imply $x = y$.

Define a binary relation $\leqslant$on X by letting $x * y = 0$ if and only if $x \leqslant y$. Then $(X, \leqslant)$ is a partially ordered set. A BCI-algebra X satisfying $0 \leqslant x$ for all $x \in X$, is called BCK-algebra.

A *BCI*-algebra X has the following properties: for all x, y, $z \in X$

(a1) $x * 0 = x$.
(a2) $(x * y) * z = (x * z) * y$.
(a3) $x \leqslant y$ implies $x * z \leqslant y * z$ and $z * y \leqslant z * x$.
(a4) $(x * z) * (y * z) \leqslant x * y$.
(a5) $x * (x * (x * y)) = x * y$.
(a6) $0 * (x * y) = (0 * x) * (0 * y)$.
(a7) $x * 0 = 0$ implies $x = 0$.

For a *BCI*-algebra X, denote by X_+ (resp. $G(X)$) the *BCK*-part (resp. the *BCI*-G part) of X, i.e., X_+ is the set of all $x \in X$ such that $0 \leqslant x$ (resp. $G(X) := \{x \in X | 0 * x = x\}$). Note that $G(X) \cap X_+ = \{0\}$ (see [12]). If $X_+ = \{0\}$, then X is called a *p-semisimple BCI-algebra*. In a p-semisimple *BCI*-algebra X, the following hold:

(a8) $(x * z) * (y * z) = x * y$.
(a9) $0 * (0 * x) = x$ for all $x \in X$.
(a10) $x * (0 * y) = y * (0 * x)$.
(a11) $x * y = 0$ implies $x = y$.
(a12) $x * a = x * b$ implies $a = b$.
(a13) $a * x = b * x$ implies $a = b$.
(a14) $a * (a * x) = x$.

Let X be a p-semisimple *BCI*-algebra. We define addition "+" as $x + y = x * (0 * y)$ for all x, $y \in X$. Then $(X, +)$ is an abelian group with identity 0 and $x - y = x * y$. Conversely let $(X, +)$ be an abelian group with identity 0 and let $x * y = x - y$. Then X is a p-semisimple *BCI*-algebra and $x + y = x * (0 * y)$ for all $x, y \in X$ (see [13]).

For a *BCI*-algebra X we denote $x \wedge y = y * (y * x)$, in particular $0 * (0 * x) = a_x$, and $L_p(X) := \{a \in X | x * a = 0 \Rightarrow x = a, \forall x \in X\}$. We call the elements of $L_p(X)$ the p-atoms of X. For any $a \in X$, let $V(a) := \{x \in X | a * x = 0\}$, which is called the *branch* of X with respect to a. It follows that $x * y \in V(a * b)$ whenever $x \in V(a)$ and $y \in V(b)$ for all x, $y \in X$ and all $a, b \in L_p(X)$. Note that $L_p(X) = \{x \in X | a_x = x\}$, which is the p-semisimple part of X, and X is a p-semisimple *BCI*-algebra if and only if $L_p(X) = X$ (see [14, Proposition 3.2]). Note also that $a_x \in L_p(X)$, i.e., $0 * (0 * a_x) = a_x$, which implies that $a_x * y \in L_p(X)$ for all $y \in X$. It is clear that $G(X) \subset L_p(X)$, and $x * (x * a) = a$ and $a * x \in L_p(X)$ for all $a \in L_p(X)$ and all $x \in X$. A BCI-algebra X is said to be torsion free if $x + x = 0 \Rightarrow x = 0$ for all $x \in X$ [10]. For more details, refer to [6,12–15].

3. Left (θ, ϕ)-derivations

In what follows θ, ϕ are endomorphisms and I_X is the identity map of a *BCI*-algebra X unless otherwise specified.

Definition 3.1 [1]. A self-map D of a *BCI*-algebra X is called a *left derivation* of X if it satisfies:

$$(\forall x, y \in X)(D(x * y) = (x * D(y)) \wedge (y * D(x))).$$

Definition 3.2. A self-map $d_{(\theta,\phi)}$ of a *BCI*-algebra X is called a *left (θ, ϕ)-derivation* of X if it satisfies:

$$(\forall x, y \in X)(d_{(\theta,\phi)}(x * y) = (\theta(x) * d_{(\theta,\phi)}(y)) \wedge (\phi(y) * d_{(\theta,\phi)}(x))).$$

Note that if $\theta = \phi = I_X$, then the left (θ, ϕ)-derivation of a *BCI*-algebra X is a left derivation of a *BCI*-algebra X, In this case, $d_{(\theta,\ \phi)}$ is denoted by D.

Example 3.3. Consider a *BCI*-algebra $X = \{0, a, b\}$ with the following Cayley table:

$*$	0	a	b
0	0	0	b
a	a	0	b
b	b	b	0

(1) Define a map

$$d_{(\theta,\phi)} : X \to X, x \mapsto \begin{cases} b & \text{if } x \in \{0, a\}, \\ 0 & \text{if } x = b, \end{cases}$$

and define two endomorphisms

$$\theta : X \to X, x \mapsto \begin{cases} 0 & \text{if } x \in \{0, a\}, \\ b & \text{if } x = b, \end{cases}$$

and

$$\phi : X \to X, x \mapsto \begin{cases} 0 & \text{if } x \in \{0, b\}, \\ a & \text{if } x = a. \end{cases}$$

It is routine to verify that $d_{(\theta,\phi)}$ is a left (θ, ϕ)-derivation of X.

(2) Define a map

$$d_{(\theta,\phi)} : X \to X, x \mapsto \begin{cases} 0 & \text{if } x \in \{0, b\}, \\ a & \text{if } x = a, \end{cases}$$

and define two endomorphisms

$$\theta : X \to X, x \mapsto \begin{cases} a & \text{if } x \in \{0, a\}, \\ 0 & \text{if } x = b, \end{cases}$$

and

$$\phi : X \to X, x \mapsto \begin{cases} b & \text{if } x \in \{0, b\}, \\ a & \text{if } x = a. \end{cases}$$

It is routine to verify that $d_{(\theta,\phi)}$ is a left (θ, ϕ)-derivation of X.

Proposition 3.4. *Let $d_{(\theta,\phi)}$ be a left (θ, ϕ)-derivation of a BCI-algebra X. Then*

(1) $(\forall x \in X)(x \in L_p(X) \Rightarrow d_{(\theta,\ \phi)}(x) \in L_p(X))$.
(2) $(\forall x \in L_p(X))(d_{(\theta,\phi)}(x) = 0 + d_{(\theta,\ \phi)}(x))$.
(3) $(\forall x, y \in L_p(X))(d_{(\theta,\ \phi)}(x + y) = \theta(x) + d_{(\theta,\phi)}(y))$.
(4) $(\forall x \in X)(x \in G(X) \Rightarrow d_{(\theta,\ \phi)}(x) \in G(X))$.

Proof.

(1) For any $x \in L_p(X)$, we have

$$d_{(\theta,\phi)}(x) = d_{(\theta,\phi)}(0*(0*x))$$
$$= (\theta(0)*d_{(\theta,\phi)}(0*x)) \wedge (\phi(0*x)*d_{(\theta,\phi)}(0))$$
$$= (0*d_{(\theta,\phi)}(0*x)) \wedge (\phi(0*x)*d_{(\theta,\phi)}(0))$$
$$= (\phi(0*x)*d_{(\theta,\phi)}(0))*((\phi(0*x)*d_{(\theta,\phi)}(0))*(0*d_{(\theta,\phi)}(0*x)))$$
$$= 0*d_{(\theta,\phi)}(0*x) \in L_p(X).$$

(2) Let for all $x \in L_p(X)$. By (1), we have $d_{(\theta,\ \phi)}(x) \in L_p(X)$. Then

$$d_{(\theta,\phi)}(x) = 0*(0*d_{(\theta,\phi)}(x)) = 0 + d_{(\theta,\phi)}(x).$$

(3) For any $x, y \in L_p(X)$, we have

$$d_{(\theta,\phi)}(x+y) = d_{(\theta,\phi)}(x*(0*y))$$
$$= (\theta(x)*d_{(\theta,\phi)}(0*y)) \wedge (\phi(0*y)*d_{(\theta,\phi)}(x))$$
$$= (\phi(0*y)*d_{(\theta,\phi)}(x))*((\phi(0*y)*d_{(\theta,\phi)}(x))*(\theta(x)*d_{(\theta,\phi)}(0*y)))$$
$$= \theta(x)*d_{(\theta,\phi)}(0*y)$$
$$= \theta(x)*((\theta(0)*d_{(\theta,\phi)}(y)) \wedge (\phi(y)*d_{(\theta,\phi)}(0)))$$
$$= \theta(x)*(0*d_{(\theta,\phi)}(y))$$
$$= \theta(x)+d_{(\theta,\phi)}(y).$$

(4) Let $x \in G(X)$. Then $0*x = x$, and so

$$d_{(\theta,\phi)}(x) = d_{(\theta,\phi)}(0*x)$$
$$= (\theta(0)*d_{(\theta,\phi)}(x)) \wedge (\phi(x)*d_{(\theta,\phi)}(0))$$
$$= (\phi(x)*d_{(\theta,\phi)}(0))*((\phi(x)*d_{(\theta,\phi)}(0))*(0*d_{(\theta,\phi)}(x)))$$
$$= 0*d_{(\theta,\phi)}(x)$$

since $0*d_{(\theta,\phi)}(x) \in L_p(X)$. Hence $d_{(\theta,\ \phi)}(x) \in G(X)$. □

If we take $\theta = \phi = I_X$ in Proposition 3.4, then we have the following corollary.

Corollary 3.5 [1]. *Let D be a left derivation of a BCI-algebra X. Then*

(1) $(\forall x \in X)\ (x \in L_p(X) \Rightarrow D(x) \in L_p(X))$.
(2) $(\forall x \in X)(D(x) = 0 + D(x))$.
(3) $(\forall x,y \in L_p(X))(D(x+y) = x + D(y))$.
(4) $(\forall x \in X)(x \in G(X) \Rightarrow D(x) \in G(X))$.

Proposition 3.6. *Let $d_{(\theta,\phi)}$ be a left (θ,ϕ)-derivation of a BCI-algebra X. Then*

(1) $(\forall x \in X)(x \in L_p(X) \Rightarrow d_{(\theta,\ \phi)}(x) = \theta(x)*d_{(\theta,\phi)}(0) = \theta(x) + d_{(\theta,\ \phi)}(0))$.
(2) $(\forall x,\ y \in L_p(X))(d_{(\theta,\ \phi)}(x+y) = d_{(\theta,\phi)}(x) + d_{(\theta,\phi)}(y) - d_{(\theta,\ \phi)}(0))$.
(3) $(\forall x,\ y \in X)(d_{(\theta,\phi)}(x*y) \leqslant \theta(x)*d_{(\theta,\phi)}(y))$.
(4) *If θ is the identity map on X, then $d_{(\theta,\phi)}$ is identity on $L_p(X)$ if and only if $d_{(\theta,\phi)}(0) = 0$.*

Proof.

(1) For any $x \in L_p(X)$, we have

$$d_{(\theta,\phi)}(x) = d_{(\theta,\phi)}(x*0) = (\theta(x)*d_{(\theta,\phi)}(0)) \wedge (\phi(0)*d_{(\theta,\phi)}(x))$$
$$= (\theta(x)*d_{(\theta,\phi)}(0)) \wedge (0*d_{(\theta,\phi)}(x))$$
$$= (0*d_{(\theta,\phi)}(x))*((0*d_{(\theta,\phi)}(x))*(\theta(x)*d_{(\theta,\phi)}(0)))$$
$$= (0*d_{(\theta,\phi)}(x))*((0*(\theta(x)*d_{(\theta,\phi)}(0)))*d_{(\theta,\phi)}(x))$$
$$= 0*(0*(\theta(x)*d_{(\theta,\phi)}(0)))$$
$$= \theta(x)*d_{(\theta,\phi)}(0) = \theta(x)*(0*d_{(\theta,\phi)}(0))$$
$$= \theta(x)+d_{(\theta,\phi)}(0)$$

since $\theta(x)*d_{(\theta,\phi)}(0) \in L_p(X)$ and $d_{(\theta,\phi)}(0) \in G(X)$.

(2) If $x,\ y \in L_p(X)$, then $x + y \in L_p(X)$. Using (1), we have

$$d_{(\theta,\phi)}(x+y) = \theta(x+y) + d_{(\theta,\phi)}(0) = \theta(x) + \theta(y) + d_{(\theta,\phi)}(0)$$
$$= \theta(x) + d_{(\theta,\phi)}(0) + \theta(y) + d_{(\theta,\phi)}(0) - d_{(\theta,\phi)}(0)$$
$$= d_{(\theta,\phi)}(x) + d_{(\theta,\phi)}(y) - d_{(\theta,\phi)}(0).$$

(3) For any $x,\ y \in X$, we have

$$d_{(\theta,\phi)}(x*y) = (\theta(x)*d_{(\theta,\phi)}(y)) \wedge (\phi(y)*d_{(\theta,\phi)}(x))$$
$$= (\phi(y)*d_{(\theta,\phi)}(x))*((\phi(y)*d_{(\theta,\phi)}(x))*(\theta(x)*d_{(\theta,\phi)}(y)))$$
$$\leqslant \theta(x)*d_{(\theta,\phi)}(y).$$

(4) It follows from (1). This completes the proof. □

Definition 3.7. A left (θ,ϕ)-derivation $d_{(\theta,\phi)}$ of a *BCI*-algebra X is said to be *regular* if $d_{(\theta,\ \phi)}(0) = 0$.

Example 3.8.

(1) The left (θ,ϕ)-derivation $d_{(\theta,\phi)}$ of X in Example 3.3(1) is not regular.
(2) The left (θ,ϕ)-derivation $d_{(\theta,\phi)}$ of X in Example 3.3(2) is regular.

Theorem 3.9. *If X is a BCK-algebra, then every left (θ,ϕ)-derivation of X is regular.*

Proof. Let $d_{(\theta,\phi)}$ be a left (θ,ϕ)-derivation of a *BCK*-algebra X. Then by Proposition 3.6(3), we have $d_{(\theta,\phi)}(0) = d_{(\theta,\phi)}(0*x) \leqslant \theta(0)*d_{(\theta,\phi)}(x) = 0$ and we have $0 \leqslant d_{(\theta,\ \phi)}(0)$, so we obtain $d_{(\theta,\phi)}(0) = 0$. Hence $d_{(\theta,\phi)}$ is regular. □

In a *BCI*-algebra, Theorem 3.9 is not true as seen in the following example:

Example 3.10. In Example 3.3(1), $d_{(\theta,\phi)}$ is a left (θ,ϕ)-derivation of a *BCI*-algebra X which is not regular.

Theorem 3.11. *Let $d_{(\theta,\phi)}$ be a regular left (θ,ϕ)-derivation of a BCI-algebra X. Then*

(1) *Both $\theta(x)$ and $d_{(\theta,\phi)}(x)$ belong to the same branch for all $x \in X$.*
(2) $(\forall x \in X)(d_{(\theta,\phi)}(x) \leqslant \theta(x))$.
(3) $(\forall x,y \in X)(d_{(\theta,\ \phi)}(x)*\theta(y) \leqslant \theta(x)*d_{(\theta,\phi)}(y))$.

Proof.

(1) For any $x \in X$, we get

$$\begin{aligned} 0 = d_{(\theta,\phi)}(0) &= d_{(\theta,\phi)}(a_x * x) \\ &= (\theta(a_x) * d_{(\theta,\phi)}(x)) \wedge (\phi(x) * d_{(\theta,\phi)}(a_x)) \\ &= (\phi(x) * d_{(\theta,\phi)}(a_x)) * ((\phi(x) * d_{(\theta,\phi)}(a_x)) * (\theta(a_x) * d_{(\theta,\phi)}(x))) \\ &= \theta(a_x) * d_{(\theta,\phi)}(x) \end{aligned}$$

since $\theta(a_x) * d_{(\theta,\phi)}(x) \in L_p(X)$. Hence $\theta(a_x) \leqslant d_{(\theta,\phi)}(x)$, and so $d_{(\theta,\ \phi)}(x) \in V(\theta(a_x))$. Obviously, $\theta(x) \in V(\theta(a_x))$.

(2) Since $d_{(\theta,\phi)}$ is regular, $d_{(\theta,\phi)}(0) = 0$. Then

$$\begin{aligned} d_{(\theta,\phi)}(x) &= d_{(\theta,\phi)}(x * 0) \\ &= (\theta(x) * d_{(\theta,\phi)}(0)) \wedge (\phi(0) * d_{(\theta,\phi)}(x)) \\ &= (\theta(x) * 0) \wedge (0 * d_{(\theta,\phi)}(x)) \\ &= (0 * d_{(\theta,\phi)}(x)) * ((0 * d_{(\theta,\phi)}(x)) * \theta(x)) \\ &\leqslant \theta(x). \end{aligned}$$

(3) Since $d_{(\theta,\phi)}(x) \leqslant \theta(x)$ for all $x \in X$ by (2). Using (a3), we have

$$d_{(\theta,\phi)}(x) * \theta(y) \leqslant \theta(x) * \theta(y) \leqslant \theta(x) * d_{(\theta,\phi)}(y).$$

This completes the proof. □

Theorem 3.12. *For any left (θ,ϕ)-derivation $d_{(\theta,\phi)}$ of a BCI-algebra X, the set*

$$d^{-1}_{(\theta,\phi)}(0) := \{x \in X | d_{(\theta,\phi)}(x) = 0\}$$

is a subalgebra of X if $\theta(x) = 0$ for all $x \in X$. Moreover, if θ is one-one and $d_{(\theta,\phi)}$ is regular, then $d^{-1}_{(\theta,\phi)}(0) \subseteq X_+$.

Proof. Assume that $\theta(x) = 0$ for all $x \in X$. Let $x, y \in d^{-1}_{(\theta,\phi)}(0)$. Then $d_{(\theta,\phi)}(x) = 0 = d_{(\theta,\phi)}(y)$, and so

$$d_{(\theta,\phi)}(x * y) \leqslant \theta(x) * d_{(\theta,\phi)}(y) = 0 * 0 = 0$$

by Proposition 3.6(3). Hence $d_{(\theta,\ \phi)}(x * y) = 0$ by (a7), that is, $x * y \in d^{-1}_{(\theta,\phi)}(0)$. Hence $d^{-1}_{(\theta,\phi)}(0)$ is a subalgebra of X.

Assume that θ is one-to-one and $d_{(\theta,\phi)}$ is regular. Let $x \in d^{-1}_{(\theta,\phi)}(0)$. Then $0 = d_{(\theta,\phi)}(x) \leqslant \theta(x)$ by Theorem 3.11(2), which implies that $\theta(x) \in X_+$, that is, $0 * \theta(x) = 0$. It follows that $\theta(0 * x) = \theta(0)$ so that $0 * x = 0$ since θ is one-one. Therefore, $x \in X_+$, and thus $d^{-1}_{(\theta,\phi)}(0) \subseteq X_+$. This completes the proof. □

Definition 3.13. For a left (θ,ϕ)-derivation $d_{(\theta,\phi)}$ of a *BCI*-algebra X, we say that an ideal A of X is a θ-*ideal* (resp. ϕ-ideal) if $\theta(A) \subseteq A$ (resp. $\phi(A) \subseteq A$).

Definition 3.14. For a left (θ,ϕ)-derivation $d_{(\theta,\phi)}$ of a *BCI*-algebra X, we say that an ideal A of X is $d_{(\theta,\phi)}$-*invariant* if $d_{(\theta,\phi)}(A) \subseteq A$.

Example 3.15. (1) Let $d_{(\theta,\phi)}$ be a left (θ,ϕ)-derivation of X which is described in Example 3.3 (1). We know that $A := \{0,a\}$ is both a θ-ideal and a ϕ-ideal of X. But $A := \{0,a\}$ is an ideal of X which is not $d_{(\theta,\phi)}$-invariant.

(2) Let $d_{(\theta,\phi)}$ be a left (θ,ϕ)-derivation of X which is described in Example 3.3(2). We know that $A := \{0,a\}$ is both a θ-ideal and a ϕ-ideal of X. Also, $A := \{0,a\}$ is a $d_{(\theta,\phi)}$-invariant ideal of X.

Theorem 3.16. *Let $d_{(\theta,\phi)}$ be a left (θ,ϕ)-derivation of a BCI-algebra X. Then $d_{(\theta,\phi)}$ is regular if and only if every θ-ideal of X is $d_{(\theta,\ \phi)}$-invariant.*

Proof. Let A be a θ-ideal of X. Suppose $d_{(\theta,\phi)}$ is regular, then it follows from Theorem 3.11(2) that $d_{(\theta,\phi)}(x) \leqslant \theta(x)$ for all $x \in X$ which implies $d_{(\theta,\phi)}(x) * \theta(x) = 0$. Let $y \in X$ be such that $y \in d_{(\theta,\phi)}(A)$. Then $y = d_{(\theta,\phi)}(x)$ for some $x \in A$. Thus

$$y * \theta(x) = d_{(\theta,\phi)}(x) * \theta(x) = 0 \in A.$$

Note that $\theta(x) \in \theta(A) \subseteq A$. Since A is an ideal of X, it follows that $y \in A$ so that $d_{(\theta,\phi)}(A) \subseteq A$. Therefore, A is $d_{(\theta,\phi)}$-invariant.

Conversely, suppose that every θ-ideal of X is $d_{(\theta,\phi)}$-invariant. Since the zero ideal $\{0\}$ is clearly θ-ideal and $d_{(\theta,\phi)}$-invariant, we have $d_{(\theta,\phi)}(\{0\}) \subseteq \{0\}$, and so $d_{(\theta,\phi)}(0) = 0$. Hence $d_{(\theta,\phi)}$ is regular. This completes the proof. □

If we take $\theta = \phi = I_X$ in Theorem 3.16, then we have the following corollary.

Corollary 3.17 [1]. *Let D be a left derivation of a BCI-algebra X. Then D is regular if and only if every ideal of X is D-invariant.*

Proposition 3.18. *Let $d_{(\theta,\phi)}$ be a left (θ,ϕ)-derivation of a p-semisimple BCI-algebra X. Then $(\forall x,\ y \in X)\ (d_{(\theta,\phi)}(x * y) = \theta(x) * d_{(\theta,\phi)}(y))$.*

Proof. Let X be a p-semisimple *BCI*-algebra. Then for any x, $y \in X$, we have

$$\begin{aligned} &d_{(\theta,\phi)}(x * y) = (\theta(x) * d_{(\theta,\phi)}(y)) \wedge (\phi(y) * d_{(\theta,\phi)}(x)) \\ &= \theta(x) * d_{(\theta,\phi)}(y). \end{aligned}$$

This completes the proof. □

If we take $\theta = \phi = I_X$ in Proposition 3.18, then we have the following corollary.

Corollary 3.19 [1]. *Let D be a left derivation of a p-semisimple BCI-algebra X. Then $(\forall x, y \in X)(D(x * y) = x * D(y))$.*

Theorem 3.20. *Let X be a torsion free BCI-algebra and $d_{(\theta,\phi)}$ be a left (θ,ϕ)-derivation on X such that $\theta \circ d_{(\theta,\phi)} = d_{(\theta,\phi)}$. If $d^2_{(\theta,\phi)} = 0$ on $Lp(X)$, then $d_{(\theta,\phi)} = 0$ on $Lp(X)$.*

Proof. Let us suppose $d^2_{(\theta,\phi)} = 0$ on $Lp(X)$. Let $x \in Lp(X)$, then $x + x \in Lp(X)$, $d_{(\theta,\phi)}(x + x) \in Lp(X)$ by Proposition 3.4(1). Using Proposition 3.6(1) and (2), we have

$$\begin{aligned} 0 = d^2_{(\theta,\phi)}(x + x) &= d_{(\theta,\phi)}(d_{(\theta,\phi)}(x + x)) \\ &= d_{(\theta,\phi)}(0) + \theta(d_{(\theta,\phi)}(x + x)) = d_{(\theta,\phi)}(0) + d_{(\theta,\phi)}(x + x) \\ &= d_{(\theta,\phi)}(0) + d_{(\theta,\phi)}(x) + d_{(\theta,\phi)}(x) - d_{(\theta,\phi)}(0) \\ &= d_{(\theta,\phi)}(x) + d_{(\theta,\phi)}(x). \end{aligned}$$

Since X is a torsion free. Therefore, $d_{(\theta,\phi)}(x) = 0$ for all $x \in Lp(X)$ implying thereby $d_{(\theta,\phi)} = 0$. This completes the proof. □

Theorem 3.21. *Let X be a torsion free BCI-algebra and let $d_{(\theta,\phi)}$, $d'_{(\theta,\phi)}$ be two left (θ,ϕ)-derivations on X such that $\theta\circ d'_{(\theta,\phi)} = d'_{(\theta,\phi)}$. If $d_{(\theta,\phi)}\circ d'_{(\theta,\phi)} = 0$ on $Lp(X)$, then $d'_{(\theta,\phi)} = 0$ on $Lp(X)$.*

Proof. Let us suppose $d_{(\theta,\phi)}\circ d'_{(\theta,\phi)} = 0$ on $Lp(X)$. Let $x\in Lp(X)$, then $x + x\in Lp(X)$, $d_{(\theta,\phi)}(x + x)\in Lp(X)$ by Proposition 3.4(1). Using Proposition 3.6(1) and (2), we have

$$\begin{aligned}
0 &= (d_{(\theta,\phi)}\circ d'_{(\theta,\phi)})(x+x) = d_{(\theta,\phi)}(d'_{(\theta,\phi)}(x+x)) = d_{(\theta,\phi)}(0)+\theta(d'_{(\theta,\phi)}(x+x))\\
&= d_{(\theta,\phi)}(0)+d'_{(\theta,\phi)}(x+x) = d_{(\theta,\phi)}(0)+\left(d'_{(\theta,\phi)}(x)+d'_{(\theta,\phi)}(x)-d'_{(\theta,\phi)}(0)\right)\\
&= \left(d_{(\theta,\phi)}(0)-d'_{(\theta,\phi)}(0)\right)+\left(d'_{(\theta,\phi)}(x)+d'_{(\theta,\phi)}(x)\right)\\
&= \left((d_{(\theta,\phi)}(0)*d'_{(\theta,\phi)}(0))\right)+\left(d'_{(\theta,\phi)}(x)+d'_{(\theta,\phi)}(x)\right)\\
&= \left(d_{(\theta,\phi)}(0)*\left(0*d'_{(\theta,\phi)}(0)\right)\right)+\left(d'_{(\theta,\phi)}(x)+d'_{(\theta,\phi)}(x)\right)\\
&= \left(d_{(\theta,\phi)}(0)+d'_{(\theta,\phi)}(0)\right)+\left(d'_{(\theta,\phi)}(x)+d'_{(\theta,\phi)}(x)\right)\\
&= \left(d_{(\theta,\phi)}(0)+\theta d'_{(\theta,\phi)}(0)\right)+\left(d'_{(\theta,\phi)}(x)+d'_{(\theta,\phi)}(x)\right)\\
&= d_{(\theta,\phi)}\left(d'_{(\theta,\phi)}(0)\right)+\left(d'_{(\theta,\phi)}(x)+d'_{(\theta,\phi)}(x)\right)\\
&= (d_{(\theta,\phi)}\circ d'_{(\theta,\phi)})(0)+\left(d'_{(\theta,\phi)}(x)+d'_{(\theta,\phi)}(x)\right) = d'_{(\theta,\phi)}(x)+d'_{(\theta,\phi)}(x).
\end{aligned}$$

Since X is a torsion free. Therefore, $d'_{(\theta,\phi)}(x) = 0$ for all $x\in Lp(X)$ and so $d'_{(\theta,\phi)} = 0$. This completes the proof. □

Proposition 3.22. *Let $d_{(\theta,\phi)}$ be a left (θ,ϕ)-derivation of a BCI-algebra X. If $d^2_{(\theta,\phi)} = 0$ on $L_p(X)$, then $(\theta\circ d_{(\theta,\phi)})(x) = \frac{1}{2}((\theta\circ d_{(\theta,\phi)})(0) - d_{(\theta,\phi)}(0))$ for all $x\in L_p(X)$.*

Proof. Assume that $d^2_{(\theta,\phi)} = 0$ on $L_p(X)$. Let $x\in L_p(X)$. Then $x + x\in L_p(X)$ by Proposition 3.4(1). Using Proposition 3.6(1) and (2), we have

$$\begin{aligned}
0 &= d^2_{(\theta,\phi)}(x+x) = d_{(\theta,\phi)}(d_{(\theta,\phi)}(x+x)) = d_{(\theta,\phi)}(0)+\theta(d_{(\theta,\phi)}(x+x))\\
&= d_{(\theta,\phi)}(0)+\theta(d_{(\theta,\phi)}(x)+d_{(\theta,\phi)}(x)-d_{(\theta,\phi)}(0)) = d_{(\theta,\phi)}(0)+2\theta(d_{(\theta,\phi)}(x))-\theta(d_{(\theta,\phi)}(0)).
\end{aligned}$$

Hence $(\theta\circ d_{(\theta,\phi)})(x) = \frac{1}{2}((\theta\circ d_{(\theta,\phi)})(0) - d_{(\theta,\phi)}(0))$ for all $x\in L_p(X)$.

This completes the proof. □

Proposition 3.23. *Let $d_{(\theta,\phi)}$ and $d'_{(\theta,\phi)}$ be two left (θ,ϕ)-derivations and a of a BCI-algebra X. If $d_{(\theta,\phi)}\circ d_{(\theta,\phi)} = 0$ on $L_p(X)$, then $(\theta\circ d'_{(\theta,\phi)})(x) = \frac{1}{2}\left((\theta\circ d'_{(\theta,\phi)})(0) - d_{(\theta,\phi)}(0)\right)$ for all $x\in L_p(X)$.*

Proof. Let $x\in L_p(X)$. Then $x + x\in L_p(X)$, and so $d'_{(\theta,\phi)}(x + x)\in L_p(X)$ by Proposition 3.4(1). It follows from Propositions 3.6(1) and (2) that

$$\begin{aligned}
0 &= \left(d_{(\theta,\phi)}\circ d'_{(\theta,\phi)}\right)(x+x) = d_{(\theta,\phi)}\left(d'_{(\theta,\phi)}(x+x)\right) = d_{(\theta,\phi)}(0)+\theta\left(d'_{(\theta,\phi)}(x+x)\right)\\
&= d_{(\theta,\phi)}(0)+\theta\left(d'_{(\theta,\phi)}(x)+d'_{(\theta,\phi)}(x)-d'_{(\theta,\phi)}(0)\right) = d_{(\theta,\phi)}(0)+2\theta\left(d'_{(\theta,\phi)}(x)\right)-\theta\left(d'_{(\theta,\phi)}(0)\right)
\end{aligned}$$

so that $\theta\left(d'_{(\theta,\phi)}(x)\right) = \frac{1}{2}\left((\theta\circ d'_{(\theta,\phi)})(0) - d_{(\theta,\phi)}(0)\right)$ for all $x\in L_p(X)$.

This completes the proof. □

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