New algorithm for calculating chromatic index of graphs and its applications

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Abstract

The problem of edge coloring is one of the fundamental problems in the graph theory. Graph coloring problem arises in several ways like to allocate scheduling and assignments. To follow this line of investigation, we design a new algorithm called "RF algorithm" to color the edges of a graph. In addition, we reinstate some classical results by applying the RF algorithm.

Keywords: Graph, Edge coloring, Chromatic index, Incidence matrix

2010 Mathematics subject classification: 05C15, 05C99, 05C90, 05C30.

Introduction

The problem of edge coloring appeared with the four-color problem. In 1880, Tait wrote the first paper dealing with the problem of edge coloring. Tait proved that only three colors are used to color the edges of every 3-connected planar graph. An s-edge coloring, s is a positive integer, is a way to color edges with s colors. The chromatic index $\chi'(x)$ is the minimum number of different colors needed to color edges such that any two adjacent edges are colored by different colors (for more details, see [1, 3–5, 7–9, 11–14]). König has proved, in 1916, that $\chi'(x) = \Delta(x)$ for every bipartite graph. $\frac{3\Delta(x)}{2}$ is the most number of colors that any graph can be edge-colored with, this was given by Shannon. Moreover, Vizing obtained for any simple graph $\Delta(x) + 1 \geq \chi'(x)$. Omai et al. [10] determined the AVD-chromatic index for the powers of paths. Further, Lehner [6] established a result state that if every non-trivial automorphism of a countable graph G with distinguishing index $D'(G)$ moves infinitely many edges, then $D'(G) \leq 2$. Grzesik and Khachatrian [2] proved that $k_{1, m, n}$ is interval colorable iff $\gcd(m + 1, n + 1) = 1$.

The above discussions motivate us to design a new algorithm to calculate the chromatic index of the graph. The main objective of our algorithm is to find a coloring that uses the smallest possible number of distinct colors. Some examples are given in support of our algorithm.

The main results

In this article, a new algorithm, RF coloring algorithm, will be designed to evaluate chromatic index of loopness graph. RF coloring algorithm is introduced as follows:

Consider a graph G of order n and size m. List its vertices as $v_1, v_2, v_3, \ldots, v_n$ and its edges as $e_1, e_2, e_3, \ldots, e_m$.

Step1. Write the incidence matrix of the graph G

$$I(G) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}.$$

Step2. In this step, we constitute the RF coloring matrix from the incidence matrix as follows:

(a) In the first row of the RF matrix, put

$$a_{11}^* = \begin{cases} 0 \text{ if } a_{11} = 0 \\ \\ 1 \text{ if } a_{11} = 1 \end{cases}$$

After that put

$$a_{12}^* = \begin{cases} 0 & \text{if} & a_{12} = 0 \\ 1 & \text{if} & a_{12} = 1, a_{11} = 0 \\ 2 & \text{if} & a_{11} = a_{12} = 1 \end{cases}$$

In the same way, for entry a_{1j}^*, where $1 < j \leq m$, put

$$a_{1j}^* = \begin{cases} 0 & \text{if } a_{1j} = 0 \\ \\ \max\left\{a_{11}^*, a_{12}^*, \ldots, a_{1(j-1)}^*\right\} + 1 & \text{if } a_{1j} = 1 \end{cases}$$

(b) Any column k has the entry $a_{1k}^* = h$ put

$$a_{ik}^* = \begin{cases} h & \text{if } a_{ik} = 1 \\ \\ 0 & \text{if } a_{ik} = 0 \end{cases}$$

where $1 < i \leq n$ and $1 < k \leq m$.

(c) Now start from the second row put

$$\begin{cases} a_{2l}^* = 0 & \text{if} & a_{2l} = 0 \\ \min\left(\left\{a_{1j}^* : a_{1j}^* \neq 0,\right\} \backslash \left\{a_{2j}^*, a_{sj}^* : a_{2j}^* \neq 0, a_{sj}^* \neq 0\right\}\right) & \text{if} & a_{2l} = 1, a_{Sl} = 1, \left\{a_{1j}^* : a_{1j}^* \neq 0\right\} \backslash \left\{a_{2j}^*, a_{Sj}^* : a_{2j}^* \neq 0 \neq a_{Sj}^*\right\} \neq \hat{\hat{o}} \\ \max\left(\left\{a_{1j}^*, a_{2j}^*, a_{sj}^* : a_{1j}^* \neq 0, a_{2j}^* \neq 0 \neq a_{sj}^*,\right\}\right) + 1 & \text{if} & a_{2l} = 1, a_{Sl} = 1, \left\{a_{1j}^* : a_{1j}^* \neq 0\right\} \backslash \left\{a_{2j}^*, a_{Sj}^* : a_{2j}^* \neq 0 \neq a_{Sj}^*\right\} = \hat{\hat{o}} \end{cases}$$

where $1 \leq l \leq m$, $1 \leq j \leq m$ and $1 \leq s \leq n$.

(d) Any column l has the entry $a^*_{2l} = f$ put

$$a^*_{il} = \begin{cases} f & \text{if } a_{il} = 1 \\ \\ 0 & \text{if } a_{il} = 0 \end{cases}$$

(e) Again repeat steps (c) and (d) to complete the RF coloring matrix

$$\text{RCI}(G) = \begin{bmatrix} a^*_{11} & a^*_{12} & \cdots & a^*_{1m} \\ a^*_{21} & a^*_{22} & \cdots & a^*_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a^*_{n1} & a^*_{n2} & \cdots & a^*_{nm} \end{bmatrix}$$

Step3. The greatest number in the RF coloring matrix is the chromatic index of the graph G and the value of entry a^*_{ij} is the color of the edge e_j where $a^*_{ij} \neq 0$.

In the following, we give examples solved by the new algorithm:

Example1. Let G be a graph shown below (Fig. 1).

The chromatic index of the graph G will be calculated by using the RF coloring algorithm as follows:

$$I(G) = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow[\substack{\text{Since } a_{11} = 1 \\ \text{Put } a^*_{11} = 1}]{} \begin{bmatrix} 1 & & & \\ & & & \\ & & & \\ & & & \end{bmatrix} \xrightarrow[\substack{\text{Since } a_{11} = 1 \\ a_{12} = 1 \\ \text{Put } a^*_{12} = 2}]{} \begin{bmatrix} 1 & 2 & & \\ & & & \\ & & & \\ & & & \end{bmatrix}$$

Fig. 1 The graph G
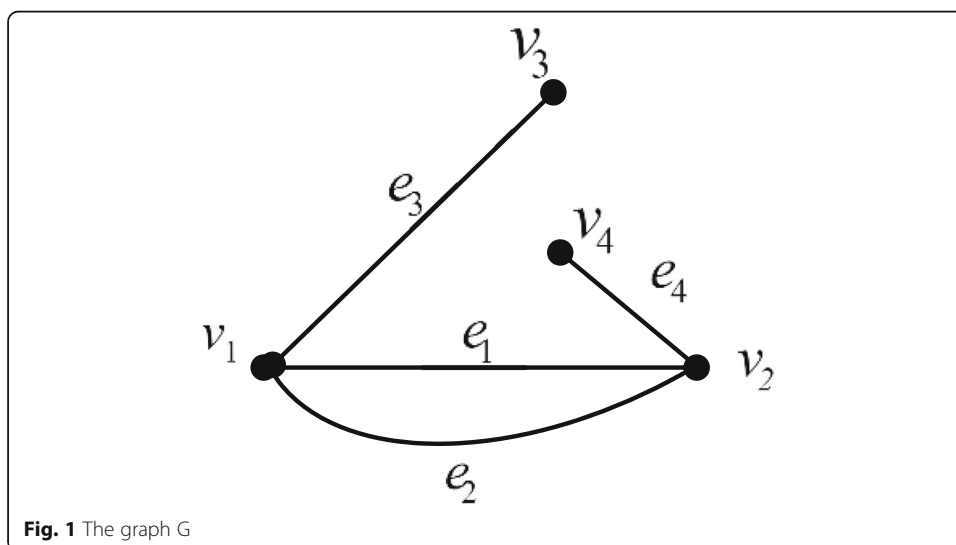
Since $a_{11} = a_{12} = a_{13} = 1$ Put $a_{13}^* = 3$ $\begin{bmatrix} 1 & 2 & 3 \\ & & \\ & & \\ & & \end{bmatrix}$ $\xrightarrow{}$ $\xrightarrow[\text{Put } a_{14}^* = 0]{\text{Since } a_{14} = 0}$ $\begin{bmatrix} 1 & 2 & 3 & 0 \\ & & & \\ & & & \\ & & & \end{bmatrix}$ $\xrightarrow[\substack{\text{Put } a_{21}^* = 1, \\ a_{31}^* = a_{41}^* = 0}]{\text{Column 1}}$ $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 1 & & & \\ 0 & & & \\ 0 & & & \end{bmatrix}$

$\xrightarrow[\substack{\text{Put } a_{22}^* = 2, \\ a_{32}^* = a_{42}^* = 0}]{\text{Column 2}}$ $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 1 & 2 & & \\ 0 & 0 & & \\ 0 & 0 & & \end{bmatrix}$ $\xrightarrow[\substack{\text{Put } a_{33}^* = 3, \\ a_{23}^* = a_{43}^* = 0}]{\text{Column3}}$

$\begin{bmatrix} 1 & 2 & 3 & 0 \\ 1 & 2 & 0 & \\ 0 & 0 & 3 & \\ 0 & 0 & 0 & \end{bmatrix}$ $\xrightarrow[\text{then } a_{24}^* = \ \max\{1,2,3\}\backslash\{1,2\} = 3]{\substack{2^{\text{nd}}\text{row} \\ \text{Since } a_{24} = 1}}$ $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 1 & 2 & 0 & 3 \\ 0 & 0 & 3 & \\ 0 & 0 & 0 & \end{bmatrix}$

$\xrightarrow[a_{43}^* = 0]{\substack{\text{Column 4} \\ \text{Put } a_{44}^* = 3,}}$ $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 1 & 2 & 0 & 3 \\ 0 & 0 & 3 & 0 \\ 1 & 2 & 0 & 3 \end{bmatrix}$

Then, the RF coloring matrix is given by

$$\text{RCI}(G) = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 1 & 2 & 0 & 3 \\ 0 & 0 & 3 & 0 \\ 1 & 2 & 0 & 3 \end{bmatrix}.$$

From the above matrix, we find the chromatic index $\chi'(G)$ of the graph G is equal to 3.

Example2. Given a graph G, shown in Fig. 2.

Fig. 2 The quartic graph
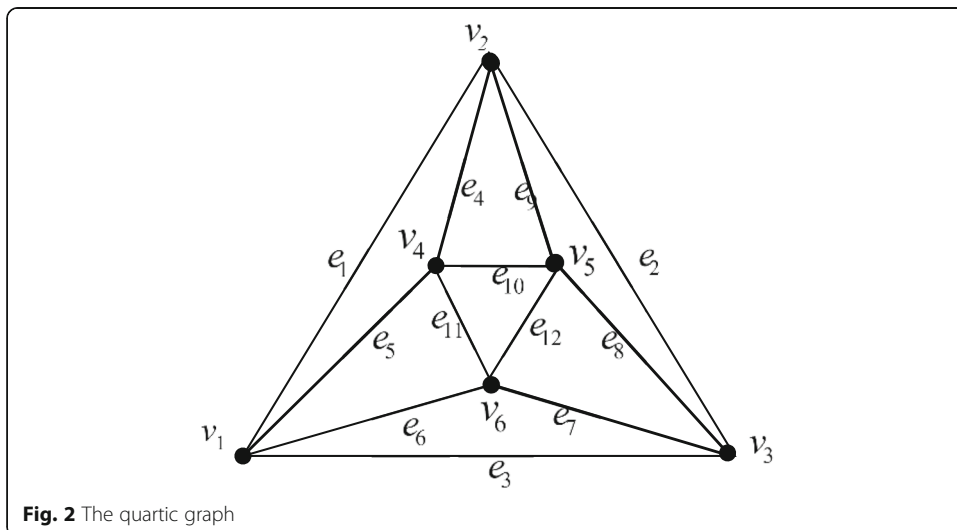
Applying the RF coloring algorithm step by step to evaluate the colors of edges and chromatic index as shown in the sequence of matrices below:

$$I(G) = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{\text{The first row}} \begin{bmatrix} 1 & 0 & 2 & 0 & 3 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & \square & 0 & \square & 0 & 0 & \square & \square & \square & \square & \square & \square \\ 0 & \square & 2 & \square & 0 & 0 & \square & \square & \square & \square & \square & \square \\ 0 & \square & 0 & \square & 3 & 0 & \square & \square & \square & \square & \square & \square \\ 0 & \square & 0 & \square & 0 & 0 & \square & \square & \square & \square & \square & \square \\ 0 & \square & 0 & \square & 0 & 4 & \square & \square & \square & \square & \square & \square \end{bmatrix}$$

$$\xrightarrow{\text{The third row}} \begin{bmatrix} 1 & 0 & 2 & 0 & 3 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 0 & 2 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 3 & 2 & 0 & 0 & 0 & \square & \square & 0 & \square & \square & \square \\ 0 & 0 & 0 & 2 & 3 & 0 & \square & \square & 0 & \square & \square & \square \\ 0 & 0 & 0 & 0 & 0 & 0 & \square & \square & 4 & \square & \square & \square \\ 0 & 0 & 0 & 0 & 0 & 4 & \square & \square & 0 & \square & \square & \square \end{bmatrix} \xleftarrow{\text{The second row}}$$

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 3 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 0 & 2 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 3 & 2 & 0 & 0 & 0 & 1 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 3 & 0 & 0 & 0 & 0 & \square & \square & \square \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 4 & \square & \square & \square \\ 0 & 0 & 0 & 0 & 0 & 4 & 1 & 0 & 0 & \square & \square & \square \end{bmatrix} \xrightarrow{\text{The fourth row}} \begin{bmatrix} 1 & 0 & 2 & 0 & 3 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 0 & 2 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 3 & 2 & 0 & 0 & 0 & 1 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 3 & 0 & 0 & 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 4 & 1 & 0 & \square \\ 0 & 0 & 0 & 0 & 0 & 4 & 1 & 0 & 0 & 0 & 5 & \square \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 2 & 0 & 3 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 0 & 2 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 3 & 2 & 0 & 0 & 1 & 5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 3 & 0 & 0 & 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 4 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 4 & 1 & 0 & 0 & 0 & 5 & 2 \end{bmatrix} \xleftarrow{\text{The fifth row}}$$

Hence, the RF coloring matrix is given by

$$RCI(G) = \begin{bmatrix} 1 & 0 & 2 & 0 & 3 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 3 & 0 & 2 & 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 3 & 2 & 0 & 0 & 0 & 1 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 3 & 0 & 0 & 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 5 & 4 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 4 & 1 & 0 & 0 & 0 & 5 & 2 \end{bmatrix}.$$

It is clear from the RF coloring matrix that the chromatic index of the graph G is 5, i.e., $\chi'(G) = 5$ and the color of e_1, e_7, e_{10} is 1, the color of e_3, e_4, e_{12} is 2, the color of e_2, e_5 is 3, the color of e_6, e_9 is 4, and the color of e_8, e_{11} is 5.

In the following section, we reprove some theorems by using RF coloring algorithm

Theorem 1 *Let S_n be a star graph of order n. Then, $\chi'(S_n) = n - 1$, where $\chi'(S_n)$ is the chromatic index of S_n.*

Proof Let S_n be a star graph of order n as shown in Fig. 3.
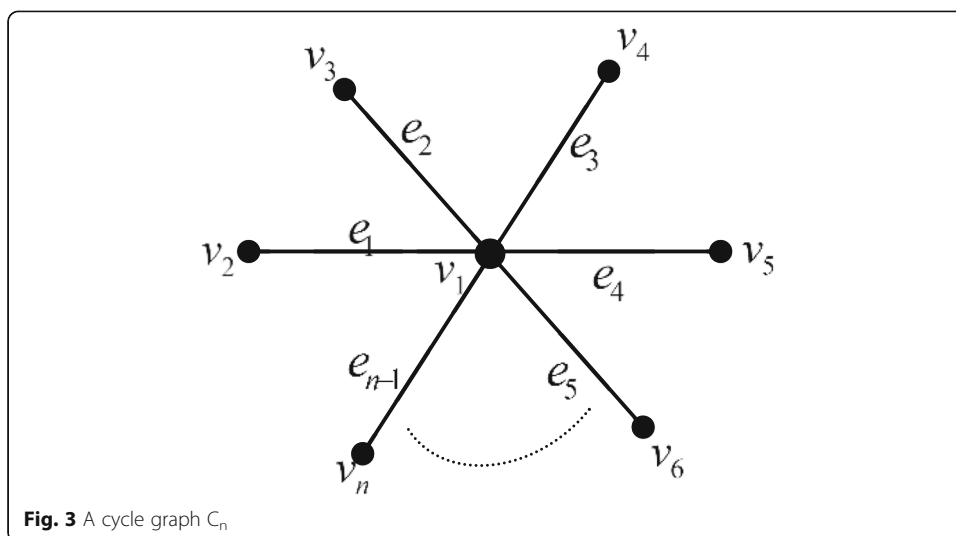
By applying the RF coloring algorithm, we found the RF coloring matrix is given by

$$RCI(S_n) = \begin{bmatrix} 1 & 2 & 3 & \cdots & n-1 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & n-1 \end{bmatrix}$$

The greatest number in the RF coloring matrix is $n - 1$, then the chromatic index of S_n equals $n - 1$.

Theorem 2 *Let C_n be a cycle graph of order n. Then* $\chi'(C_n) = \begin{cases} 2 & \text{if} \quad n \text{ is even} \\ 3 & \text{if} \quad n \text{ is odd} \end{cases},$

where $\chi'(C_n)$ is the chromatic index of C_n.

Fig. 3 A cycle graph C_n

Proof Let C_n be a cycle graph of order n. Applying the RF coloring algorithm, we will stop when the RF coloring matrix will become

$$
\mathrm{RCI}(C_n) = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 & 0 & 2 \\
1 & 2 & 0 & \cdots & 0 & 0 & 0 \\
0 & 2 & 1 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & l & 0 & 0 \\
0 & 0 & 0 & \cdots & l & m & 0 \\
0 & 0 & 0 & \cdots & 0 & m & 2
\end{bmatrix}.
$$

Now we want to evaluate the values of l and m and we have two cases:

Case 1. When n is even then l is the entry in $a^*_{(n-1)(n-2)}$, i.e., $l = 2$. Since $a^*_{(n-1)(n-2)} = 2$ and $a^*_{nm} = 2$ then $m = 1$; hence, the RF coloring matrix is

$$
\mathrm{RCI}(C_n) = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 & 0 & 2 \\
1 & 2 & 0 & \cdots & 0 & 0 & 0 \\
0 & 2 & 1 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 2 & 0 & 0 \\
0 & 0 & 0 & \cdots & 2 & 1 & 0 \\
0 & 0 & 0 & \cdots & 0 & 1 & 2
\end{bmatrix}
$$

and the chromatic index of C_n from the RF coloring matrix is 2, i.e., $\chi'(C_n) = 2$.

Case 2. When n is odd then l is the entry in $a^*_{(n-1)(n-2)}$, i.e., $l = 1$. So, $m = 3$ because $a^*_{(n-1)(n-2)} = 1$ and $a^*_{nm} = 2$; hence, the RF coloring matrix in this case is given by

$$
\mathrm{RCI}(C_n) = \begin{bmatrix}
1 & 0 & 0 & \cdots & 0 & 0 & 2 \\
1 & 2 & 0 & \cdots & 0 & 0 & 0 \\
0 & 2 & 1 & \cdots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 1 & 0 & 0 \\
0 & 0 & 0 & \cdots & 1 & 3 & 0 \\
0 & 0 & 0 & \cdots & 0 & 3 & 2
\end{bmatrix},
$$

and the chromatic index of C_n equals 3, i.e., $\chi'(C_n) = 3$.

Fig. 4 The graph corresponding to the time table problem
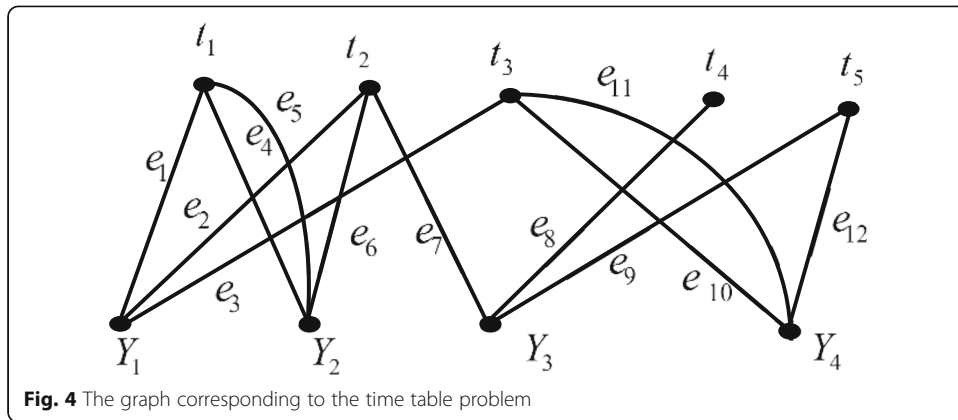
One of the most famous applications on edge coloring of a graph is the timetable; let us give an example and solve it by our new algorithm as follows:

In a particular faculty of science, the Mathematics department has five teachers t_1, t_2, t_3, t_4, t_5. The teaching assignments of the five teachers are given by the array:

	I Year Y_1	II Year Y_2	III Year Y_3	IV Year Y_4
t_1	1	2	—	—
t_2	1	1	1	—
t_3	1	—	—	2
t_4	—	—	1	—
t_5	—	—	1	1

We want to find the minimum period timetable. To solve this problem, first we draw the graph corresponding to this problem, see Fig. 4

By applying the RF coloring algorithm, we find the RF coloring matrix is given by:

$$\begin{bmatrix}
1 & 2 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 2 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 & 2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 \\
1 & 0 & 0 & 2 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 1 & 3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 3
\end{bmatrix}$$

It is clear from the RF coloring matrix above that we have a 3-period timetable given by:

	Period I	Period II	Period III
t_1	Y_1	Y_2	Y_2
t_2	Y_2	Y_3	Y_1
t_3	Y_4	Y_1	Y_4
t_4	—	—	Y_3

The main results *(Continued)*

t_5	Y_3	Y_4	—

Conclusion

In the present manuscript, we have designed a new algorithm to find the chromatic index to color the edges of a graph. It has been found that some classical results are successfully established by our new algorithm. The new introduced algorithm has been successfully applied to a real-life example where a correct result has been obtained. It has a great scope for further research in the field of graph theory, computer programming, and algebraic-specific structures.

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