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Estimation of initial Maclaurin coefficients of certain subclasses of bounded bi-univalent functions

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Abstract
In this paper, two bounded bi-univalent function subclasses were defined by using Salagean q-differential operator. The functions are defined in the open unit disc of complex plane. The main purpose is to determine some estimations on the initial Maclaurin coefficients for functions in these subclasses. Finally, the Fekete-Szegö inequalities for these are also obtained.

Keywords: Analytic function, Starlike function, Convex function, Subordination, Bi-univalent function, Fekete-Szegö inequalities, Bounded function

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Introduction

Let $\mathcal{A}$ denotes the class of analytic functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \tag{1}$$

normalized by the conditions $f(0) = f'(0) - 1 = 0$, which are defined on the open unit disc $U = \{z \in \mathbb{C} : |z| < 1\}$. Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ consisting of all functions of the form (1) which are univalent in U. In the geometric function theory, there are two important subclasses of $\mathcal{S}$, which are the well-known subclasses of starlike and convex functions, namely, $\mathcal{S}^*$ and $\mathcal{K}$, for which the inequalities $\mathrm{Re}\left\{zf'(z)/f(z)\right\} > 0$ and $\mathrm{Re}\left\{1 + zf''(z)/f'(z)\right\} > 0$ $(z \in U)$ are the sufficient conditions, respectively (see [1], Ch.8). An analytic function f is subordinate to an analytic function g, written $f(z) \prec g(z)$, provided there exist an analytic function w defined on U with $w(0) = 0$ and $|w(z)| < 1$ satisfying $f(z) = g(w(z))$ (see [2]). Ma and Minda [3] gave a unified representation of various subclasses of starlike and convex functions by introducing the classes $\mathcal{S}^*(\varphi)$ and $\mathcal{K}(\varphi)$ of functions $f \in \mathcal{S}$ satisfying $zf'(z)/f(z) \prec \varphi(z)$ and $1 + zf''(z)/f'(z) \prec \varphi(z)$ $(z \in U)$, respectively, where φ is an analytic function with positive real part in the unit disc U, $\varphi(0) = 1$, $\varphi'(0) > 0$, and φ maps U onto a region starlike with respect to 1 and symmetric with respect to the real axis. The classes $\mathcal{S}^*(\varphi)$ and $\mathcal{K}(\varphi)$ include several well-known subclasses as special case. For example, when $\varphi(z) = (1 + Az)/(1 + Bz)(1 \le B < A \le 1)$, the classes $\mathcal{S}^*(\varphi)$ and $\mathcal{K}(\varphi)$ are reduced to the subclasses $\mathcal{S}^*[A, B]$ and $\mathcal{K}[A, B]$, which were introduced by Janowski [4]. For $0 \le \beta < 1$, the classes $\mathcal{S}^*(\beta) = \mathcal{S}^*((1+(1-2\beta)z)/(1-z))$ and

$\mathcal{K}(\beta) = \mathcal{K}((1+(1-2\beta)z)/(1-z))$ are subclasses of starlike and convex functions of order β (see [1], Ch.9), $\mathcal{S}^* := \mathcal{S}^*(0) = \mathcal{S}^*((1+z)/(1-z))$ and $\mathcal{K} := \mathcal{K}(0) = \mathcal{K}((1+z)/(1-z))$. Moreover, the subclasses of strongly starlike and strongly convex functions of order $\alpha(0 \leq \alpha < 1)$ can be obtained by $\mathcal{S}^*_\alpha := \mathcal{S}^*(((1+z)/(1-z))^\alpha)$ and $\mathcal{K}_\alpha := \mathcal{K}(((1+z)/(1-z))^\alpha)$ (see [5]).

The Koebe one quarter theorem ensures that the image of U under every univalent function $f \in \mathcal{S}$ contains a disk of radius $\frac{1}{4}$(see [6]). Thus, every univalent function f has an inverse f^{-1} satisfying

$$f^{-1}(f(z)) = z, (z \in U) \text{ and} f(f^{-1}(\omega)) = \omega(|\omega| < r_0(f), r_0(f) \geq \frac{1}{4}. \tag{2}$$

A function $f \in \mathcal{S}$ is said to be bi-univalent in U if both f and f^{-1} are univalent in U. Let Σ denotes the subclass of $\mathcal{S}$, consisting of all bi-univalent functions defined on the unit disc U. Since $f \in \Sigma$ has the Maclaurin series expansion given by (1), a simple calculation shows that its inverse $g = f^{-1}$ has the series expansion

$$g(\omega) = f^{-1}(\omega) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - .... \tag{3}$$

Examples of functions in the class Σ are

$$\frac{z}{1-z}, -log\,(1-z) \text{ and} \frac{1}{2} log\left(\frac{1+z}{1-z}\right). \tag{4}$$

and so on. However, the familiar Koebe function is not a member of Σ. Other common examples of functions in $\mathcal{S}$ such as

$$z - \frac{z^2}{2} \text{and} \frac{z}{1-z^2} \tag{5}$$

are also not members of Σ (see [7]). Several authors have introduced and investigated subclasses of bi-univalent functions and obtained bounds for the initial coefficients (see [7–11]). The research into Σ was started by Lewin ([9]). It focused on problems connected with coefficients. Many papers concerning bi-univalent functions have been published recently. A function $f \in \Sigma$ is in the class $\mathcal{S}^*_\Sigma(\beta)$ of bi-starlike function of order $\beta(0 \leq \beta < 1)$, or $\mathcal{K}_\Sigma(\beta)$ of bi-convex function of order β if both f and f^{-1} are respectively starlike or convex functions of order β. For $0 \leq \alpha < 1$, the function $f \in \Sigma$ is strongly bi-starlike function of order α if both the functions f and f^{-1} are strongly starlike functions of order α. The class of all such functions is denoted by $\mathcal{S}^*_{\Sigma,\alpha}$. These classes were introduced by Brannan and Taha [8], they obtained estimates on the initial coefficients a_2 and a_3 for functions in these classes. We owe the revival of these topics to Srivastava et al. ([7]). The investigations in this direction have also been carried out, among others, by Ali et al. [12], Frasin and Aouf [13]. Hamidi and Jahangiri (e.g., [14]) have revealed the importance of the Faber polynomials in general studies on the coefficients of bi-univalent functions. In fact, little is known about exact bounds of the initial coefficients of $f \in \Sigma$. For the most general families of functions given by (1), we know that $|a_2| < 1.51$ for bi-univalent functions ([9]), $|a_2| \leq 2$ for bi-starlike functions (Kedzierawski [15]), and $|a_2| < 1$ for bi-convex functions ([8]). Only the last estimate is sharp, equality holds only for $f(z) = z/(1-z)$ and its rotations.

In this study, we are concerned with a different type of classes of bi-univalent functions, which are of the bounded type. A bounded function classes was firstly introduced and discussed by Singh [16]. Singh and Singh [17] introduced a bounded starlike and convex function classes $\mathcal{S}_M^*$ and $\mathcal{K}_M$, respectively, these were followed by the subclasses $\mathcal{S}_M^*(\alpha)$ and $\mathcal{K}_M(\alpha)$ represented a bounded starlike and convex function of order α, respectively.

The q-difference operator, which was introduced by Jackson [18], and may go back to Heine [19], is defined by

$$\partial_q f(z) = \begin{cases} \frac{f(qz)-f(z)}{z(q-1)} & , z \neq 0\,; \\ f^{'}(0) & , z = 0\,; \end{cases}$$

and

$$\partial_q^0 f(z) = f(z), \partial_q^1 f(z) = \partial_q f(z) \text{and} \partial_q^m f(z) = \partial_q \left(\partial_q^{m-1} f(z)\right) (m \in \mathbb{N}).$$

For the function $f(z)$ denoted by (1), we have

$$\partial_q^1 f(z) = 1 + \sum_{k=2}^{\infty} [k]_q\, a_k z^k (z \neq 0),$$

where

$$[\nu]_q = \frac{q^\nu - 1}{q-1} = \sum_{j=0}^{\nu-1} q^j, \nu \in \mathbb{N}.$$

For function $f \in \Sigma$ given by (1) and $n \in \mathbb{N}, 0 \leq q < 1$, Salagean q-differential operator D_q^n, introduced by Govindaraj and Sivasubramanian [20] (see also [21]), defined by

$$D_q^0 f(z) = f(z), D_q^1 f(z) = z\partial_q f(z) \text{and} D_q^m f(z) = D_q \left(D_q^{m-1} f(z)\right) (m \in \mathbb{N}). \tag{6}$$

For the functions $f(z)$ and $g(w)$ denoted by (1) and (3), we have

$$D_q^m f(z) = z + \sum_{k=2}^{\infty} [k]_q^m\, a_k z^k, \tag{7}$$

$$D_q^m g(w) = w - [2]_q^m\, a_2 w^2 + [3]_q^m \left(2a_2^2 - a_3\right) w^3 + \ldots\,. \tag{8}$$

In this present work, we introduce two bounded subclasses of Σ associated with Salagean q-differential operator and obtain the initial Maclaurin coefficients $|a_2|$ and $|a_3|$ for these function classes. Also, we give bounds for the Fekete-Szegö functional $|a_3 - \mu a_2^2|$ for each subclass.

Estimations of $|a_2|$ and $|a_3|$

The main results in this section is to define two bounded subclasses of the class Σ, then some estimations of the first two Maclaurin coefficients of functions belonging to those subclasses were calculated.

Definition 1 *For $0 \leq \lambda < 1, b \in \mathbb{C}^*$ and $M > \frac{1}{2}$, let $\mathcal{S}_{\Sigma,q}^n(\lambda, b, M)$ be the subclass of Σ consisting of functions of the form (1) and satisfying the following condition*

$$\left| \frac{b - 1 + \frac{D_q^{n+1} f(z)}{\lambda D_q^{n+1} f(z) + (1-\lambda) D_q^n f(z)}}{b} - M \right| < M, (z \in U) \tag{9}$$

and

$$\left| \frac{b - 1 + \frac{D_q^{n+1} g(w)}{\lambda D_q^{n+1} g(w) + (1-\lambda) D_q^n g(w)}}{b} - M \right| < M, (w \in U) \tag{10}$$

where $z, w \in U$, and $g = f^{-1} \in \Sigma$ is given by (3). Also, let $\mathcal{C}^n_{\Sigma,q}(\lambda, b, M)$ be the subclass of Σ consisting of functions of the form (1) and satisfying the following condition

$$\left| \frac{b - 1 + \frac{D_q^{n+2} f(z)}{\lambda D_q^{n+2} f(z) + (1-\lambda) D_q^{n+1} f(z)}}{b} - M \right| < M, (z \in U) \tag{11}$$

and

$$\left| \frac{b - 1 + \frac{D_q^{n+2} g(w)}{\lambda D_q^{n+2} g(w) + (1-\lambda) D_q^{n+1} g(w)}}{b} - M \right| < M, (w \in U) \tag{12}$$

where $z, w \in U$, and $g = f^{-1} \in \Sigma$ is given by (3).

It is clear that

$$f(z) \in \mathcal{C}^n_{\Sigma,q}(\lambda, b, M) \Longleftrightarrow z\partial_q f(z) \in \mathcal{S}^n_{\Sigma,q}(\lambda, b, M)$$

Lemma 1 *Let $m = 1 - \frac{1}{M}\left(M > \frac{1}{2}\right)$, f defined by (1) and $g = f^{-1}$, then we have*

$$f(z) \in \mathcal{S}^n_{\Sigma,q}(\lambda, b, M) \Longleftrightarrow \begin{cases} 1 + \frac{1}{b}\left(\frac{D_q^{n+1} f(z)}{\lambda D_q^{n+1} f(z) + (1-\lambda) D_q^n f(z)} - 1\right) \prec \frac{1+z}{1-mz} \\ 1 + \frac{1}{b}\left(\frac{D_q^{n+1} g(w)}{\lambda D_q^{n+1} g(w) + (1-\lambda) D_q^n g(w)} - 1\right) \prec \frac{1+w}{1-mw} \end{cases}, \tag{13}$$

and also,

$$f(z) \in \mathcal{C}^n_{\Sigma,q}(\lambda, b, M) \Longleftrightarrow \begin{cases} 1 + \frac{1}{b}\left(\frac{D_q^{n+2} f(z)}{\lambda D_q^{n+2} f(z) + (1-\lambda) D_q^{n+1} f(z)} - 1\right) \prec \frac{1+z}{1-mz} \\ 1 + \frac{1}{b}\left(\frac{D_q^{n+2} g(w)}{\lambda D_q^{n+2} g(w) + (1-\lambda) D_q^{n+1} g(w)} - 1\right) \prec \frac{1+w}{1-mw} \end{cases}, \tag{14}$$

where

$$(0 \le \lambda < 1, b \in \mathbb{C}^* \text{and} z, w \in U)$$

Lemma 2 *(see [22]) If $h \in \mathcal{P}$, then $|c_n| \le 2$ for each $n \in \mathbb{N}$, where $\mathcal{P}$ is the family of all functions h which is analytic in U for which $\operatorname{Re}\{h(z)\} > 0$, where $h(z) = 1 + c_1 z + c_2 z^2 + \ldots$ for $z \in U$.*

Remark 1 *In Definitions 1, 2 and for special choices of the parameters λ, b, M, also, taking $q \to 1^-$, then we can obtain the following subclasses:*

$$\begin{aligned} \mathcal{S}^0_\Sigma(\lambda, 1-\beta, \infty) &= \mathcal{M}_\Sigma(\beta, \lambda) \\ &= \left\{ f \in \Sigma : \begin{array}{l} \operatorname{Re}\left(\frac{zf'(z)}{(1-\lambda)f(z) + \lambda z f'(z)}\right) > \beta, \\ \operatorname{Re}\left(\frac{wg'(w)}{(1-\lambda)g(w) + \lambda w g'(w)}\right) > \beta \end{array}, 0 < \beta \le 1, z, w \in U \right\} \\ \mathcal{S}^0_\Sigma(\lambda, b, \infty) &= \mathcal{S}_\Sigma(1, \lambda) \\ &= \left\{ f \in \Sigma : \begin{array}{l} \left|\arg\left\{\frac{zf'(z)}{(1-\lambda)f(z) + \lambda z f'(z)}\right\}\right| < \frac{\pi}{2}, \\ \left|\arg\left\{\frac{wg'(w)}{(1-\lambda)g(w) + \lambda w g'(w)}\right\}\right| < \frac{\pi}{2} \end{array}, z, w \in U \right\}, \end{aligned}$$

which were introduced by Murugusundaramoorthy et al. [23].

$$\mathcal{S}^0_\Sigma(\gamma,\tau,\infty) = \mathcal{H}_\Sigma(\tau,0,1,\gamma,0) = \left\{ f\in\Sigma : \begin{array}{l} \operatorname{Re}\left(1+\frac{1}{\tau}\left(\frac{zf'(z)}{(1-\gamma)f(z)+\gamma zf'(z)}-1\right)\right)>0, \\ \operatorname{Re}\left(1+\frac{1}{\tau}\left(\frac{zg'(w)}{(1-\gamma)g(w)+\gamma wg'(w)}-1\right)\right)>0 \end{array}, z,w\in U \right\},$$

which was introduced by Srivastava et al. [24].

$$\mathcal{S}^0_\Sigma(0,1-\beta,\infty) = \mathcal{S}_\Sigma(\beta) = \left\{ f\in\Sigma : \begin{array}{l} \operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right)>\beta, \\ \operatorname{Re}\left(\frac{wg'(w)}{g(w)}\right)>\beta \end{array}, 0\le\beta<1, z,w\in U \right\},$$

$$\mathcal{C}^0_\Sigma(0,1-\beta,\infty) = \mathcal{K}_\Sigma(\beta) = \left\{ f\in\Sigma : \begin{array}{l} \operatorname{Re}\left(1+\frac{zf''(z)}{f'(z)}\right)>\beta, \\ \operatorname{Re}\left(1+\frac{wg''(w)}{g'(w)}\right)>\beta \end{array}, 0\le\beta<1, z,w\in U \right\},$$

which are the classes of bi-starlike and bi-convex functions introduced by Brannan and Taha [8].

Theorem 1 *Let f given by (1) be in the subclass $\mathcal{S}^n_{\Sigma,q}(\lambda,b,M)$. Then*

$$|a_2| \le \sqrt{\frac{|b|(m+1)}{(1-\lambda)\left|[3]^n_q([3]_q-1)-[2]^{2n}_q([2]_q-1)\left(1-\lambda+\lambda[2]_q+\frac{(1-\lambda)(m-1)([2]_q-1)}{b(m+1)}\right)\right|}}.$$

and

$$|a_3| \le \frac{|b|\,(m+1)}{(1-\lambda)}\left\{\frac{1}{[3]^n_q\,([3]_q-1)}+\frac{|b|\,(m+1)}{[2]^{2n}_q\,(1-\lambda)([2]_q-1)^2}\right\},$$

where

$$0\le\lambda<1, b\in\mathbb{C}^*, z\in U \text{and} m=1-\frac{1}{M}(M>\frac{1}{2}),$$

Proof Let $f \in \mathcal{S}^n_{\Sigma,q}(\lambda,b,M)$ and $g = f^{-1}$. Then, it satisfy the conditions (13). By the definition, there exist two analytic functions $u,v : U \to U$ with $u(0) = v(0) = 0$ and $|u(z)| < 1, |v(w)| < 1$ for all $z,w \in U$ satisfying

$$1+\frac{1}{b}\left(\frac{D^{n+1}_q f(z)}{\lambda D^{n+1}_q f(z)+(1-\lambda)D^n_q f(z)}-1\right)=\frac{1+u(z)}{1-mu(z)}, \tag{15}$$

and

$$1+\frac{1}{b}\left(\frac{D^{n+1}_q g(w)}{\lambda D^{n+1}_q g(w)+(1-\lambda)D^n_q g(w)}-1\right)=\frac{1+v(w)}{1-mv(w)}, \tag{16}$$

Now, define the two functions $p(z)$ and $q(z)$ by

$$p(z) := \frac{1+u(z)}{1-u(z)} = 1+p_1z+p_2z^2+...,$$

$$p(z) := \frac{1+v(z)}{1-v(z)} = 1+q_1z+q_2z^2+...$$

It is equivalent to

$$u(z) := \frac{1-p(z)}{1+p(z)} = \frac{1}{2}\left(p_1 z + \left(p_2 - \frac{p_1^2}{2}\right) z^2 + \ldots\right), \tag{17}$$

$$v(z) := \frac{1-q(z)}{1+q(z)} = \frac{1}{2}\left(q_1 z + \left(q_2 - \frac{q_1^2}{2}\right) z^2 + \ldots\right). \tag{18}$$

Then $p(z)$ and $q(z)$ are analytic in U with $p(0) = 1 = q(0)$. In view of Janowski [4], Since $u, v : U \to U$, the functions $p(z), q(z) \in P(M)$ and have a positive real part in U where $P(M)$ is the class of all function $\psi(z) = 1 + \delta_1 z + \delta_2 z^2 + \ldots$ which are analytic in U and satisfy the condition

$$|\psi(z) - \rho| < \rho, (\rho \geq 1, z \in U).$$

Therefore, in view of the Lemma 1, we have

$$\left|p_i\right| \leq 2 \text{and} \left|q_i\right| \leq 2 (n \in \mathbb{N}) \tag{19}$$

By substituting from (7), (8), (17), and (18) into (15)and (16), we obtain

$$\begin{aligned} &1 + [2]_q^n (1-\lambda)([2]_q - 1) a_2 z + (1-\lambda)\Big([3]_q^n ([3]_q - 1) a_3 - [2]_q^{2n} ([2]_q - 1). \\ &(1 - \lambda + \lambda [2]_q) a_2^2\Big) z^2 + \ldots = 1 + \tfrac{b(m+1)p_1}{2} z + \tfrac{b(m+1)}{2}\left(p_2 + \tfrac{p_1^2(m-1)}{2}\right) z^2 + \ldots \end{aligned} \tag{20}$$

$$\begin{aligned} &1 - [2]_q^n (1-\lambda)([2]_q - 1) a_2 w + (1-\lambda)\Big([3]_q^n ([3]_q - 1)(2a_2^2 - a_3) + [2]_q^{2n} ([2]_q - 1). \\ &\quad (1 - \lambda + \lambda [2]_q) a_2^2\Big) w^2 + \ldots = 1 + \tfrac{b(m+1)q_1}{2} w + \tfrac{b(m+1)}{2}\left(q_2 + \tfrac{q_1^2(m-1)}{2}\right) w^2 + \ldots \end{aligned} \tag{21}$$

which yields the following relations

$$[2]_q^n (1-\lambda)([2]_q - 1) a_2 = \frac{b}{2}(m+1) p_1 \tag{22}$$

$$\begin{aligned} &[3]_q^n (1-\lambda)([3]_q - 1) a_3 - [2]_q^{2n} (1-\lambda)([2]_q - 1)(1 - \lambda + \lambda [2]_q) a_2^2 \\ &\qquad = \frac{b}{2}(m+1)\left(p_2 + \frac{p_1^2}{2}(m-1)\right) \end{aligned} \tag{23}$$

$$-[2]_q^n (1-\lambda)([2]_q - 1) a_2 = \frac{b}{2}(m+1) q_1 \tag{24}$$

$$\begin{aligned} &[3]_q^n (1-\lambda)([3]_q - 1)(2a_2^2 - a_3) - [2]_q^{2n} (1-\lambda)([2]_q - 1)(1 - \lambda + \lambda [2]_q) a_2^2 \\ &\qquad = \frac{b}{2}(m+1)\left(q_2 + \frac{q_1^2}{2}(m-1)\right) \end{aligned} \tag{25}$$

From (22) and (24), we obtain

$$p_1 = -q_1, \tag{26}$$

and

$$2 [2]_q^{2n} (1-\lambda)^2 ([2]_q - 1)^2 a_2^2 = \frac{b^2}{4}(m+1)^2 \left(p_1^2 + q_1^2\right). \tag{27}$$

By adding (23) to (25) then use (27), we obtain

$$a_2^2 = \frac{b(m+1)(p_2+q_2)}{4(1-\lambda)\left([3]_q^n([3]_q-1) - [2]_q^{2n}([2]_q-1)(1-\lambda+\lambda[2]_q + \frac{(1-\lambda)(m-1)([2]_q-1)}{b(m+1)})\right)}, \tag{28}$$

applying Lemma 2 to the coefficients p_2 and q_2, we conclude

$$|a_2| \leq \sqrt{\frac{|b|(m+1)}{(1-\lambda)\left|[3]_q^n([3]_q-1)-[2]_q^{2n}([2]_q-1)\left(1-\lambda+\lambda[2]_q+\frac{(1-\lambda)(m-1)([2]_q-1)}{b(m+1)}\right)\right|}}.$$

By subtracting (25) from (23), we have

$$2\,[3]_q^n\,(1-\lambda)([3]_q-1)\left(a_3-a_2^2\right)=\frac{b}{2}(m+1)\left((p_2-q_2)+\frac{(m-1)}{2}\left(p_1^2-q_1^2\right)\right), \quad (29)$$

by substituting from (26) and (27) into (29), we conclude

$$a_3=\frac{b(m+1)\,(p_2-q_2)}{4\,[3]_q^n\,(1-\lambda)([3]_q-1)}+\frac{b^2(m+1)^2\left(p_1^2+q_1^2\right)}{8\,[2]_q^{2n}\,(1-\lambda)^2([2]_q-1)^2}. \quad (30)$$

Finally, by applying Lemma 2 to the coefficients p_1, p_2, q_1 and q_2, we conclude

$$|a_3| \leq \frac{|b|\,(m+1)}{[3]_q^n\,(1-\lambda)([3]_q-1)}+\frac{|b|^2\,(m+1)^2}{[2]_q^{2n}\,(1-\lambda)^2([2]_q-1)^2}$$

The proof is completed. □

For $n=0, b=1-\beta, m=1$, and $q \to 1^-$, we obtain the bounds corresponding to the class $M_\Sigma(\beta,\lambda)$ given by Murugusundaramoorthy et al. [23].

Corollary 1 *Let f given by (1) be a function in the class $M_\Sigma(\beta,\lambda)$, then*

$$|a_2| \leq \frac{\sqrt{2(1-\beta)}}{(1-\lambda)}$$

and

$$|a_3| \leq \frac{4(1-\beta)^2}{(1-\lambda)^2}+\frac{1-\beta}{1-\lambda}$$

Additionally, put $\lambda=0$, we obtain bounds of the class of bi-starlike function of order β donated by $\mathcal{S}_\Sigma(\beta)$.

Corollary 2 *[8] Let f given by (1) be in the class $\mathcal{S}_\Sigma(\beta)$, then*

$$|a_2| \leq \sqrt{2(1-\beta)}$$

and

$$|a_3| \leq (1-\beta)(5-4\beta)$$

Theorem 2 *Let f given by (1) be in the subclass $\mathcal{C}_{\Sigma,q}^n(\lambda,b,M)$. Then*

$$|a_2| \leq \sqrt{\frac{|b|(m+1)}{(1-\lambda)\left|[3]_q^{n+1}([3]_q-1)-[2]_q^{2n+2}([2]_q-1)(1-\lambda+\lambda[2]_q+\frac{(m-1)(1-\lambda)([2]_q-1)}{b(m+1)}\right|}}$$

and

$$|a_3| \leq \frac{|b|\,(m+1)}{(1-\lambda)}\left\{\frac{1}{[3]_q^{n+1}\,([3]_q-1)}+\frac{|b|\,(m+1)}{[2]_q^{2n+2}\,(1-\lambda)([2]_q-1)^2}\right\},$$

where

$$0 \leq \lambda < 1, b \in \mathbb{C}^*, z \in U \text{ and } m=1-\frac{1}{M}\left(M>\frac{1}{2}\right),$$

Proof Let $f \in \mathcal{C}_{\Sigma,q}^n(\lambda,b,M)$ and $g=f^{-1}$. Then, it satisfy the conditions (14). By the definition, there exist two analytic functions $u, v : U \to U$ with $u(0)=v(0)=0$ and

$|u(z)| < 1, |v(w)| < 1$ for all $z, w \in U$ satisfying

$$1 + \frac{1}{b}\left(\frac{D_q^{n+2}f(z)}{\lambda D_q^{n+2}f(z) + (1-\lambda)D_q^{n+1}f(z)} - 1\right) = \frac{1+u(z)}{1-mu(z)}, \tag{31}$$

and

$$1 + \frac{1}{b}\left(\frac{D_q^{n+2}g(w)}{\lambda D_q^{n+2}g(w) + (1-\lambda)D_q^{n+1}g(w)} - 1\right) = \frac{1+v(w)}{1-mv(w)}. \tag{32}$$

Now, define the two functions $r(z)$ and $s(z)$ by

$$r(z) := \frac{1+u(z)}{1-u(z)} = 1 + r_1 z + r_2 z^2 + ...,$$

$$s(z) := \frac{1+v(z)}{1-v(z)} = 1 + s_1 z + s_2 z^2 + ...$$

It is equivalent to

$$u(z) := \frac{1-r(z)}{1+r(z)} = \frac{1}{2}\left(r_1 z + \left(r_2 - \frac{r_1^2}{2}\right) z^2 + ...\right), \tag{33}$$

$$v(z) := \frac{1-s(z)}{1+s(z)} = \frac{1}{2}\left(s_1 z + \left(s_2 - \frac{s_1^2}{2}\right) z^2 + ...\right). \tag{34}$$

Then $r(z)$ and $s(z)$ are analytic in U with $p(0) = 1 = q(0)$. Since $u, v : U \to U$, the functions $r(z)$ and $s(z)$ have a positive real part in U. Therefore, in view of the Lemma 2, we have

$$|r_i| \leq 2 \text{and} |s_i| \leq 2 (n \in \mathbb{N}). \tag{35}$$

By following the same steps in proving Theorem 1, we can complete the proof of this theorem. □

For $n = 0, b = 1 - \beta, m = 1, \lambda = 0$ and $q \to 1^-$, we obtain the bounds corresponding to the class $\mathcal{K}_\Sigma(\beta)$ given by Brannan and Taha [8].

Corollary 3 *Let f given by (1) be in the class* $\mathcal{K}_\Sigma(\beta)$, *then*

$$|a_2| \leq \sqrt{1-\beta}$$

and

$$|a_3| \leq \frac{(4-3\beta)(1-\beta)}{3}$$

Fekete-Szegö inequalities

Fekete and Szegö [25] introduced the generalized functional $|a_3 - \mu a_2^2|$ where μ is some real number. In this section, we obtain the Fekete-Szegö inequality for the functions belonging to the classes $\mathcal{S}_{\Sigma,q}^n(\lambda, b, M)$ and $\mathcal{C}_{\Sigma,q}^n(\lambda, b, M)$. Before establishing our results, we need the following Lemma introduced by Zaprawa [11].

Lemma 3 *Let* $k, l \in \mathbb{R}$ *and* $p_1, p_2 \in \mathbb{C}$. *If* $|p_1|, |p_2| < R$, *then*

$$\left|(k+l)p_1 + (k-l)p_2\right| \leq \begin{cases} 2\,|k|\,R\ ,\ |k| \geq |l| \\ \\ 2\,|l|\,R\ ,\ |k| \leq |l| \end{cases}.$$

Theorem 3 *Let f given by (1) be in the class $\mathcal{S}^{n}_{\Sigma,q}(\lambda, b, M)$ and $\mu \in \mathbb{R}$. Then*

$$\left|a_3 - \mu a_2^2\right| \leq \frac{|b|\,(m+1)}{(1-\lambda)} \begin{cases} \frac{1}{[3]_q^n([3]_q-1)} & , \ |h(\mu)| \leq \frac{1}{[3]_q^n([3]_q-1)} \\ \\ |h(\mu)| & , \ |h(\mu)| \geq \frac{1}{[3]_q^n([3]_q-1)} \end{cases},$$

where

$$h(\mu) = \frac{(1-\mu)}{[3]_q^n\left([3]_q-1\right) - [2]_q^{2n}\left([2]_q-1\right)\left(1-\lambda+\lambda\,[2]_q + \frac{(1-\lambda)(m-1)([2]_q-1)}{b(m+1)}\right)}$$

Proof Using (29), we can write

$$a_3 = a_2^2 + \frac{b(m+1)\,(p_2-q_2)}{4\,[3]_q^n\,(1-\lambda)([3]_q-1)} \tag{36}$$

From (30) and (36), we obtain

$$a_3 - \mu a_2^2 = \frac{b(m+1)}{4(1-\lambda)}\left\{\left(h(\mu) + \frac{1}{[3]_q^n\left([3]_q-1\right)}\right)p_2 + \left(h(\mu) - \frac{1}{[3]_q^n\left([3]_q-1\right)}\right)q_2\right\} \tag{37}$$

where

$$h(\mu) = \frac{(1-\mu)}{[3]_q^n\left([3]_q-1\right) - [2]_q^{2n}\left([2]_q-1\right)\left(1-\lambda+\lambda\,[2]_q + \frac{(1-\lambda)(m-1)([2]_q-1)}{b(m+1)}\right)}$$

Therefore, by applying Lemma 2 to the coefficients p_2 and q_2 which obtain

$$\left|p_2\right| \leq 2\text{and}\left|q_2\right| \leq 2$$

Thus, by applying Lemma 3 into (37), we conclude

$$\left|a_3 - \mu a_2^2\right| \leq \frac{|b|\,(m+1)}{(1-\lambda)} \begin{cases} \frac{1}{[3]_q^n([3]_q-1)} & , \ |h(\mu)| \leq \frac{1}{[3]_q^n([3]_q-1)} \\ \\ |h(\mu)| & , \ |h(\mu)| \geq \frac{1}{[3]_q^n([3]_q-1)} \end{cases},$$

which completes the proof. □

For $n = 0, b = 1-\beta, m = 1$ and $q \to 1^-$, we obtain bounds of the Fekete-Sezgö inequality of the class $M_\Sigma(\beta,\lambda)$ given by Zaprawa [11].

Corollary 4 *Let f given by (1) be in the class $M_\Sigma(\beta,\lambda)$, then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{2(1-\beta)|1-\mu|}{(1-\lambda)^2} & , \ 2|1-\mu| \geq (1-\lambda) \\ \\ \frac{1-\beta}{1-\lambda} & , \ 2|1-\mu| \leq (1-\lambda) \end{cases}$$

Additionally, put $\lambda = 0$, we obtain Fekete-Sezgö inequality of the class $\mathcal{S}_\Sigma(\beta)$.

Corollary 5 *Let f given by (1) be in the class $\mathcal{S}_\Sigma(\beta)$, then*

$$|a_3 - \mu a_2^2| \leq \begin{cases} 2(1-\beta)|1-\mu| & , \ 2|1-\mu| \geq 1 \\ \\ 1-\beta & , \ 2|1-\mu| \leq 1 \end{cases}$$

Theorem 4 *Let f given by (1) be in the class $\mathcal{C}^{n}_{\Sigma,q}(\lambda, b, M)$ and $\mu \in \mathbb{R}$. Then,*

$$\left|a_3 - \mu a_2^2\right| \le \frac{|b|\,(m+1)}{(1-\lambda)} \begin{cases} \frac{1}{[3]_q^{n+1}([3]_q-1)} & , |h(\mu)| \le \frac{1}{[3]_q^{n+1}([3]_q-1)} \\ |h(\mu)| & , |h(\mu)| \ge \frac{1}{[3]_q^{n+1}([3]_q-1)} \end{cases},$$

where

$$h(\mu) = \frac{(1-\mu)}{[3]_q^{n+1}\left([3]_q-1\right) - [2]_q^{2n+2}\left([2]_q-1\right)\left(1-\lambda+\lambda\,[2]_q + \frac{(1-\lambda)(m-1)([2]_q-1)}{b(m+1)}\right)}$$

Proof Just as we derived Theorem 3, we can deduce Theorem 4, so we choose to omit the proof □

Corollary 6 *Let f given by (1) be in the class $\mathcal{K}_{\Sigma}(\beta)$, then*

$$\left|a_3 - \mu a_2^2\right| \le \begin{cases} \frac{1-\beta}{3} & , 3|1-\mu| \le 1 \\ (1-\beta)|1-\mu| & , 3|1-\mu| \ge 1 \end{cases}$$

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Authors' contributions
AHE collected the data regarding the previous articles about subclasses of bi-univalent functions, then choosing the bounded functions to investigate. MAM performed the calculations and was a major contributor in writing the manuscript. Both authors read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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