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On spacelike equiform-Bishop Smarandache curves on S_1^2

E. M. Solouma[1,2] and W. M. Mahmoud[3*]

*Correspondence:
wageeda76@yahoo.com
[3]Department of Mathematics, Faculty of Science, Aswan University, Aswan, Egypt
Full list of author information is available at the end of the article

Abstract

In this paper, we introduce the equiform-Bishop frame of a spacelike curve *r* lying fully on S_1^2 in Minkowski 3-space $\mathbb{R}_1^3$. By using this frame, we investigate the equiform-Bishop Frenet invariants of special spacelike equiform-Bishop Smarandache curves of a spacelike base curve in $\mathbb{R}_1^3$. Furthermore, we study the geometric properties of these curves when the spacelike base curve *r* is specially contained in a plane. Finally, we give a computational example to illustrate these curves.

Mathematical Subject Classification (2010): 53B30, 53C40, 53C50

Keywords: Smarandache curve, Equiform Frenet frame, Bishop frame, Minkowski 3-space

Introduction

In the theory of curves in the Euclidean and Minkowski spaces, a regular curve whose position vector is composed by Frenet frame vectors on another regular curve is called a Smarandache curve [1].

Smarandache geometries were proposed by Smarandache in [2] which are generalization of classical geometries, i.e., these Euclid, Lobachevshy-Bolyai-Gauss, and Riemann geometries may be united altogether in the same space, by some Smarandache geometries under the combinatorial procedure. These geometries can be either partially Euclidean and partially non-Euclidean or only non-Euclidean.

An axiom is said to be Smarandachely denied if the axiom behaves in at least two different ways within the same space, i.e., validated and invalided, or only invalided but in multiple distinct ways [3–6].

A Smarandache geometry is a geometry which has at least one Smarandachely denied axiom (1969). Recently, special Smarandache curves have been studied by some authors [7–11].

In this work, we introduce the equiform-Bishop frame of a spacelike curve *r* lying fully on S_1^2 in Minkowski 3-space $\mathbb{R}_1^3$. Also, we introduce a special spacelike equiform-Bishop Smarandache curves according to these frame of a spacelike curve *r* in $\mathbb{R}_1^3$. In the "Basic concepts" section, we give the basic conceptions of Minkowski 3-space $\mathbb{R}_1^3$, the Bishop frame, and the equiform-Bishop frame that will be used during this work. In the "Main results" section, we investigate the special spacelike euiform-Bishop TB_1, TB_2,

B_1B_2, and TB_1B_2-Smarandache curves in terms of the equiform-Bishop curvature functions $K_1(\sigma)$, and $K_2(\sigma)$ of the spacelike curve r in $\mathbb{R}^3_1$. Furthermore, we obtain some properties on these curves when the spacelike base curve r is contained in a plane. In the "Example" section, we give a computational example to clarify these curves. We hope these results will be helpful to mathematicians who are specialized on mathematical modeling.

Basic concepts

The Minkowski 3-space $\mathbb{R}^3_1$ is the Euclidean 3-space $\mathbb{R}^3$ provided with the Lorentzian inner product

$$\mathcal{D} = -d\varsigma_1^2 + d\varsigma_2^2 + d\varsigma_3^2,$$

where $(\varsigma_1, \varsigma_2, \varsigma_3)$ is a rectangular coordinate system of $\mathbb{R}^3_1$. The arbitrary vector $\upsilon \in \mathbb{R}^3_1$ can have one of three Lorentzian clause depicts; it can be spacelike if $\mathcal{D}(\upsilon,\upsilon) > 0$ or $\upsilon = 0$, timelike if $\mathcal{D}(\upsilon,\upsilon) < 0$, and lightlike if $\mathcal{D}(\upsilon,\upsilon) = 0$ and $\upsilon \neq 0$. Similarly, a curve r parametrized by $r = r(s) : I \subset \mathbb{R} \to \mathbb{R}^3_1$ can be spacelike, timelike, or lightlike if all of its velocity vectors $r'(s)$ are spacelike , timelike, or lightlike, respectively [12, 13].

Denote by $\{t, n, b\}$ the moving Frenet frame along the regular spacelike curve r with arc-length parameter s in $\mathbb{R}^3_1$. The Frenet trihedron consists of the tangent vector t, the principal normal vector n, and the binormal vector b. Then, the Frenet frame has the following properties [12]:

$$\begin{pmatrix} \dot{t}(s) \\ \dot{n}(s) \\ \dot{b}(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\varepsilon\kappa(s) & 0 & \tau(s) \\ 0 & \tau(s) & 0 \end{pmatrix} \begin{pmatrix} t(s) \\ n(s) \\ b(s) \end{pmatrix}, \tag{1}$$

where $\left(\cdot = \frac{d}{ds}\right)$, $\varepsilon = \pm 1$, $\mathcal{D}(t,t) = 1$, $\mathcal{D}(n,n) = \varepsilon$, $\mathcal{D}(b,b) = -\varepsilon$, and $\mathcal{D}(t,n) = \mathcal{D}(t,b) = \mathcal{D}(n,b) = 0$. If $\varepsilon = 1$, then $r = r(s)$ is a spacelike curve with spacelike principal normal n and timelike binormal b. Also, if $\varepsilon = -1$, then $r = r(s)$ is a spacelike curve with timelike principal normal n and spacelike binormal b.

Let $r = r(s)$ be a regular curve in $\mathbb{R}^3_1$. If the tangent vector field of this curve forms a constant angle with a constant vector field U, then this curve is called a general helix or an inclined curve [14].

Definition 1 *A regular curve in Minkowski 3-space, whose position vector is composed by Frenet frame vectors on another curve, is called a Smarandache curve* [15].

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative [16, 17].

Let us consider the Bishop frame $\{t, b_1, b_2\}$ of the spacelike curve $r(s)$ with a spacelike or timelike normal $b_1 (\varepsilon = 1$ or $\varepsilon = -1)$. The Bishop frame $\{t, b_1, b_2\}$ is expressed as [17, 18].

$$\begin{pmatrix} \dot{t}(s) \\ \dot{b}_1(s) \\ \dot{b}_2(s) \end{pmatrix} = \begin{pmatrix} 0 & k_1(s) & -k_2(s) \\ -\varepsilon k_1(s) & 0 & 0 \\ -\varepsilon k_2(s) & 0 & 0 \end{pmatrix} \begin{pmatrix} t(s) \\ b_1(s) \\ b_2(s) \end{pmatrix}, \tag{2}$$

where $\mathcal{D}(t,t) = 1$, $\mathcal{D}(b_1,b_1) = \varepsilon$, $\mathcal{D}(b_2,b_2) = -\varepsilon$ and $\mathcal{D}(t,b_1) = \mathcal{D}(t,b_2) = \mathcal{D}(b_1,b_2) = 0$. Here, we shall call $k_1(s)$ and $k_2(s)$ as Bishop curvatures. The relation matrix may be expressed as

$$\begin{pmatrix} t(s) \\ b_1(s) \\ b_2(s) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cosh\theta(s) & \sinh\theta(s) \\ 0 & \sinh\theta(s) & \cosh\theta(s) \end{pmatrix} \begin{pmatrix} t(s) \\ n(s) \\ b(s) \end{pmatrix}, \tag{3}$$

where

$$\begin{cases} \theta(s) = arg\tanh\left(\dfrac{k_2}{k_1}\right), \ k_1 \neq 0 \\ \tau(s) = -\varepsilon\dfrac{d\theta(s)}{ds}, \\ \kappa(s) = \sqrt{|k_1^2(s) - k_2^2(s)|}, \end{cases} \tag{4}$$

and

$$\begin{cases} k_1(s) = \kappa(s)\cosh\theta(s), \\ k_2(s) = \kappa(s)\sinh\theta(s). \end{cases}$$

Let $r : I \subset \mathbb{R} \to \mathbb{R}^3_1$ be a spacelike curve in Minkowski space $\mathbb{R}^3_1$. We define the equiform-Bishop parameter of r by $\sigma = \int k_1 ds$. Then, we have $\rho = \dfrac{ds}{d\sigma}$, where $\rho = \dfrac{1}{k_1}$ is the radius of curvature of the curve r. We recall $\{T, B_1, B_2,\}$ be the moving equiform-Bishop frame where $T(\sigma) = \rho\, t(s)$, $B_1(\sigma) = \rho\, b_1(s)$, and $B_2(\sigma) = \rho\, b_2(s)$ are the equiform-Bishop tangent vector, equiform-Bishop principal normal vector, and equiform-Bishop binormal vector respectively. Additionally, the first and second equiform-Bishop curvatures of the curve $r = r(\sigma)$ are defined by $K_1(\sigma) = \dot{\rho} = \dfrac{d\rho}{ds}$ and $K_2(\sigma) = \dfrac{k_2}{k_1}$. So, the moving equiform-Bishop frame of $r = r(\sigma)$ is given as [19]:

$$\begin{pmatrix} T'(\sigma) \\ B_1'(\sigma) \\ B_2'(\sigma) \end{pmatrix} = \begin{pmatrix} K_1(\sigma) & 1 & -K_2(\sigma) \\ -\varepsilon & K_1(\sigma) & 0 \\ -\varepsilon K_2(\sigma) & 0 & K_1(\sigma) \end{pmatrix} \begin{pmatrix} T(\sigma) \\ B_1(\sigma) \\ B_2(\sigma) \end{pmatrix}, \tag{5}$$

where $\left({}' = \dfrac{d}{d\sigma}\right)$, $\mathcal{D}(T,T) = \rho^2$, $\mathcal{D}(B_1,B_1) = \varepsilon\rho^2$, $\mathcal{D}(B_2,B_2) = -\varepsilon\rho^2$, and $\mathcal{D}(T,B_1) = \mathcal{D}(T,B_2) = \mathcal{D}(B_1,B_2) = 0$.

The pseudo-Riemannian sphere of unit radius and with center in the origin in the space $\mathbb{R}^3_1$ is defined by

$$S^2_1 = \{p \in \mathbb{R}^3_1 : \mathcal{D}(p,p) = 1\}.$$

Main results

In this section, we introduce a special spacelike equiform-Bishop Smarandache curves according to the equiform-Bishop frame in Minkowski 3-space $\mathbb{R}^3_1$. Furthermore, we obtain the natural curvature functions of these curves and studying some properties on it when the spacelike base curve $r = r(s)$ specially is contained in a plane. Let $r = r(\sigma)$ be a regular unit speed spacelike curve with spacelike equiform-Bishop principal normal and timelike equiform-Bishop binormal.

Definition 2 *Let $r : I \subset \mathbb{R} \to S_1^2$ be a regular unit speed spacelike curve lying fully on S_1^2. The spacelike equiform-Bishop TB_1-Smarandache curve $\varphi : I \subset \mathbb{R} \to S_1^2$ of r defined by*

$$\varphi = \varphi(\sigma^*) = \frac{1}{\sqrt{2}\,\rho}\left(a\,T(\sigma) + b\,B_1(\sigma)\right), \quad a^2 + b^2 = 2. \tag{6}$$

Theorem 1 *Let $r : I \subset \mathbb{R} \to S_1^2$ be a regular unit speed spacelike curve lying fully on S_1^2 with the moving equiform-Bishop frame $\{T, B_1, B_2\}$. If $\varphi : I \subset \mathbb{R} \to S_1^2$ is the spacelike equiform-Bishop TB_1-Smarandache curve of $r = r(\sigma)$ with non-zero natural curvature functions, then its Frenet frame $\{T_\varphi, N_\varphi, B_\varphi\}$ is given by*

$$\begin{pmatrix} T_\varphi \\ N_\varphi \\ B_\varphi \end{pmatrix} = \begin{pmatrix} \frac{-b}{\rho\sqrt{b^2+a^2(1-K_2^2)}} & \frac{a}{\rho\sqrt{b^2+a^2(1-K_2^2)}} & \frac{-aK_2}{\rho\sqrt{b^2+a^2(1-K_2^2)}} \\ \frac{\omega_1}{\rho\sqrt{\omega_1^2+\omega_2^2-\omega_3^2}} & \frac{\omega_2}{\rho\sqrt{\omega_1^2+\omega_2^2-\omega_3^2}} & \frac{\omega_3}{\rho\sqrt{\omega_1^2+\omega_2^2-\omega_3^2}} \\ \frac{-a(\omega_3+\omega_2K_2)}{\Delta_1} & \frac{a\omega_1K_2-b\omega_3}{\Delta_1} & \frac{-(a\omega_1+b\omega_2)}{\Delta_1} \end{pmatrix} \begin{pmatrix} T \\ B_1 \\ B_2 \end{pmatrix}, \tag{7}$$

where

$$\begin{aligned} \omega_1 &= a\left(K_2^2 - 1\right)\left[b^2 + a^2\left(1 - K_2^2\right)\right] - 2a^2bK_2K_2', \\ \omega_2 &= 2a^3K_2K_2' - b\left[b^2 + a^2\left(1 - K_2^2\right)\right], \\ \omega_3 &= \left(bK_2 - aK_2'\right)\left[b^2 + a^2\left(1 - K_2^2\right)\right] - 2a^3K_2^2K_2', \\ \Delta_1 &= \rho^2\sqrt{\omega_1^2 + \omega_2^2 - \omega_3^2}\sqrt{b^2 + a^2\left(1 - K_2^2\right)}. \end{aligned} \tag{8}$$

Proof Differentiationg Eq. (6) with respect to σ and using Eq. (5), we get

$$\varphi'(\sigma^*) = \frac{d\varphi}{d\sigma^*}\frac{d\sigma^*}{d\sigma} = \frac{1}{\sqrt{2}\,\rho}\left(b\,T(\sigma) + a\,B_1(\sigma) - aK_2B_2(\sigma)\right), \tag{9}$$

hence

$$T_\varphi(\sigma^*) = \frac{1}{\rho\sqrt{b^2 + a^2\left(1 - K_2^2\right)}}\left(b\,T(\sigma) + a\,B_1(\sigma) - aK_2B_2(\sigma)\right), \tag{10}$$

with the parameterization

$$\frac{d\sigma^*}{d\sigma} = \frac{\rho\sqrt{b^2 + a^2\left(1 - K_2^2\right)}}{\sqrt{2}}. \tag{11}$$

Again differentiating Eq. (10) with respect to σ, we have

$$T_\varphi'(\sigma^*) = \frac{\sqrt{2}}{\rho\left[b^2 + a^2\left(1 - K_2^2\right)\right]^2}\left(\omega_1T(\sigma) + \omega_2B_1(\sigma) + \omega_3B_2(\sigma)\right).$$

where

$$\begin{aligned} \omega_1 &= a\left(K_2^2 - 1\right)\left[b^2 + a^2\left(1 - K_2^2\right)\right] - 2a^2bK_2K_2', \\ \omega_2 &= 2a^3K_2K_2' - b\left[b^2 + a^2\left(1 - K_2^2\right)\right], \\ \omega_3 &= \left(bK_2 - aK_2'\right)\left[b^2 + a^2\left(1 - K_2^2\right)\right] - 2a^3K_2^2K_2'. \end{aligned}$$

The curvature and the principal normal of φ are given as follows

$$\kappa_\varphi(\sigma^*) = \left\|T_\varphi'(\sigma^*)\right\| = \frac{\sqrt{2}\sqrt{\omega_1^2 + \omega_2^2 - \omega_3^2}}{\left[b^2 + a^2\left(1 - K_2^2\right)\right]^2},$$

and

$$N_\varphi(\sigma^*) = \frac{\omega_1 T(\sigma) + \omega_2 B_1(\sigma) + \omega_3 B_2(\sigma)}{\rho\sqrt{\omega_1^2 + \omega_2^2 - \omega_3^2}}.$$

On the other hand, we can express

$$B_\varphi(\sigma^*) = \frac{1}{\Delta_1}\left\{-a(\omega_3 + \omega_2 K_2)T(\sigma) + a\omega_1 K_2 - b\omega_3 B_1(\sigma)\right.$$
$$\left. -(a\omega_1 + b\omega_2)B_2(\sigma)\right\},$$

where

$$\Delta_1 = \rho^2\sqrt{\omega_1^2 + \omega_2^2 - \omega_3^2}\sqrt{b^2 + a^2\left(1 - K_2^2\right)}.$$

Now, from Eq. (9), we have

$$\varphi''(\sigma^*) = \frac{1}{\sqrt{2}\,\rho}\left\{a\left(K_2^2 - 1\right)T(\sigma) - b\,B_1(\sigma) + \left(bK_2 - aK_2'\right)B_2(\sigma)\right\},$$

similarly

$$\varphi'''(\sigma^*) = \frac{1}{\sqrt{2}\,\rho}\left(\mu_1 T(\sigma) + \mu_2 B_1(\sigma) + \mu_3 B_2(\sigma)\right),$$

where

$$\mu_1 = aK_2'(1 + k_2) + K_1(aK_2 + bK_1 - a) + 2bK_1',$$
$$\mu_2 = (aK_2 + bK_1 - a) - b\left(K_1K_1' + K_1''\right),$$
$$\mu_3 = -K_2(aK_2 + bK_1 - a) - a\left(K_1K_1' + K_2''\right).$$

As a consequence with the above computation, the torsion of φ is obtained as

$$\tau_\varphi = \frac{\sqrt{2}}{\rho}\left\{\frac{[aK_2 + bK_1 - a]\left[\mu_3(a - bK_1) - a\mu_2K_2\right] - a\mu_1\left[K_2'(bK_1 - a) - bK_1'K_2\right]}{\left[a^2K_2' - 2abK_2\right]^2 + \left[b\left(aK_2' - bK_2\right) - a^2K_2\left(K_2^2 - 1\right)\right]^2 - \left[b^2 - a^2\left(K_2^2 - 1\right)\right]^2}\right\}.$$

□

Corollary 1 *Let $r : I \subset \mathbb{R} \to S_1^2$ be a regular unit speed spacelike curve lying fully on S_1^2 with the moving equiform-Bishop frame $\{T, B_1, B_2\}$. If the base curve $r = r(s)$ is contained in a plane, then the spacelike equiform-Bishop TB_1-Smarandache curve is a circular helix if $K_2 \neq \pm\frac{\sqrt{2}}{a}$ and $K_2 \neq \pm 1$. Moreover, its natural curvature functions are dependent only on the second equiform-Bishop curvature and given by*

$$\kappa_\varphi(\sigma^*) = \frac{\sqrt{2}\sqrt{a^2\left(1 - K_2^2\right)^2 + b^2\left(1 - K_2^2\right)}}{b^2 + a^2(1 - K_2^2)},$$
$$\tau_\varphi(\sigma^*) = \left\{\frac{\sqrt{2}}{\rho}\right\}\left\{\frac{\sqrt{2}(1 + a)(1 - K_2)}{4a^2b^2K_2^2 + \left(K_2^2 - 1\right)\left[b^2 - a^2\left(K_2^2 - 1\right)\right]^2}\right\}. \tag{12}$$

Definition 3 *Let $r : I \subset \mathbb{R} \to S_1^2$ be a regular unit speed spacelike curve lying fully on S_1^2. The spacelike equiform-Bishop TB_2-Smarandache curve $\varphi : I \subset \mathbb{R} \to S_1^2$ of r defined by*

$$\varphi = \varphi(\sigma^*) = \frac{1}{\sqrt{2}\,\rho}\left(a\,T(\sigma) + b\,B_2(\sigma)\right), \quad a^2 - b^2 = 2. \tag{13}$$

Theorem 2 *Let $r : I \subset \mathbb{R} \to S_1^2$ be a regular unit speed spacelike curve lying fully on S_1^2 with the moving equiform-Bishop frame $\{T, B_1, B_2\}$. If $\varphi : I \subset \mathbb{R} \to S_1^2$ is the spacelike equiform-Bishop TB_2-Smarandache curve of $r = r(\sigma)$ with non-zero natural curvature functions, then its Frenet frame $\{T_\varphi, N_\varphi, B_\varphi\}$ is given by*

$$\begin{pmatrix} T_\varphi \\ N_\varphi \\ B_\varphi \end{pmatrix} = \begin{pmatrix} \frac{-bK_2}{\rho\sqrt{b^2K_2^2+a^2(1-K_2^2)}} & \frac{a}{\rho\sqrt{b^2K_2^2+a^2(1-K_2^2)}} & \frac{aK_2}{\rho\sqrt{b^2K_2^2+a^2(1-K_2^2)}} \\ \frac{\xi_1}{\rho\sqrt{\xi_1^2+\xi_2^2-\xi_3^2}} & \frac{\xi_2}{\rho\sqrt{\xi_1^2+\xi_2^2-\xi_3^2}} & \frac{\xi_3}{\rho\sqrt{\xi_1^2+\xi_2^2-\xi_3^2}} \\ \frac{a(\xi_2K_2-\xi_3)}{\Delta_2} & \frac{a\xi_1K_2-b\xi_3K_2}{\Delta_2} & \frac{-(a\xi_1+b\xi_2K_2)}{\Delta_2} \end{pmatrix} \begin{pmatrix} T \\ B_1 \\ B_2 \end{pmatrix}, \tag{14}$$

where

$$\begin{aligned} \xi_1 &= [bK_1 + a(K_2-1)]\left[(a-bK_2)^2 - a^2K_2^2\right], \\ \xi_2 &= -bK_1'\left[(a-bK_2)^2 - a^2K_2^2\right] + (a-bK_1)\left[bK_2'(a-bK_2) + a^2K_2K_2'\right], \\ \xi_3 &= -2(aK_1K_2 + K_2')\left[(a-bK_2)^2 - a^2K_2^2\right] + aK_2\left[bK_2'(a-bK_2) - a^2K_2K_2'\right], \\ \Delta_2 &= \rho^2\sqrt{\xi_1^2+\xi_2^2-\xi_3^2}\sqrt{b^2K_2^2+a^2(1-K_2^2)}: \quad K_2 \neq \frac{\pm a}{\sqrt{a^2-b^2}}. \end{aligned} \tag{15}$$

Proof Differentiationg Eq. (13) with respect to σ and using Eq. (5), we have

$$\varphi'(\sigma^*) = \frac{d\varphi}{d\sigma^*}\frac{d\sigma^*}{d\sigma} = \frac{1}{\sqrt{2}\,\rho}\left(bK_2\,T(\sigma) + a\,B_1(\sigma) + aK_2\,B_2(\sigma)\right), \tag{16}$$

then, we have

$$T_\varphi(\sigma^*) = \frac{1}{\rho\sqrt{b^2K_2^2 + a^2\left(1-K_2^2\right)}}\left(bK_2\,T(\sigma) + a\,B_1(\sigma) + aK_2\,B_2(\sigma)\right), \tag{17}$$

where

$$\frac{d\sigma^*}{d\sigma} = \frac{\sqrt{b^2K_2^2 + a^2\left(1-K_2^2\right)}}{\sqrt{2}}. \tag{18}$$

Then

$$T_\varphi'(\sigma^*) = \frac{\sqrt{2}}{\rho\left[b^2K_2^2 + a^2\left(1-K_2^2\right)\right]^2}\left(\xi_1 T(\sigma) + \xi_2 B_1(\sigma) + \xi_3 B_2(\sigma)\right),$$

where

$$\begin{aligned} \xi_1 &= [bK_1 + a(K_2-1)]\left[(a-bK_2)^2 - a^2K_2^2\right], \\ \xi_2 &= -bK_1'\left[(a-bK_2)^2 - a^2K_2^2\right] + (a-bK_1)\left[bK_2'(a-bK_2) + a^2K_2K_2'\right], \\ \xi_3 &= -2\left(aK_1K_2 + K_2'\right)\left[(a-bK_2)^2 - a^2K_2^2\right] + aK_2\left[bK_2'(a-bK_2) - a^2K_2K_2'\right]. \end{aligned}$$

Therefore, the natural curvature functions κ_φ, τ_φ can be expressed as follows:

$$\kappa_\varphi(\sigma^*) = \frac{\sqrt{2}\sqrt{\xi_1^2+\xi_2^2-\xi_3^2}}{\left[b^2K_2^2 + a^2\left(1-K_2^2\right)\right]^2},$$

and

$$N_\varphi(\sigma^*) = \frac{\xi_1 T(\sigma) + \xi_2 B_1(\sigma) + \xi_3 B_2(\sigma)}{\rho\sqrt{\xi_1^2+\xi_2^2-\xi_3^2}}.$$

Also, the binormal vector of φ is

$$B_\varphi(\sigma^*) = \frac{1}{\Delta_2}\left\{a(\xi_2K_2 - \xi_3)T(\sigma) + (a\xi_1K_2 - b\xi_3K_2)B_1(\sigma) - (a\xi_1 + b\xi_2K_2)B_2(\sigma)\right\},$$

where

$$\Delta_2 = \rho^2 \sqrt{\xi_1^2 + \xi_2^2 - \xi_3^2}\sqrt{b^2K_2^2 + a^2\left(1 - K_2^2\right)}.$$

Differentiating Eq. (16) with respect to σ, we get

$$\varphi''(\sigma^*) = \frac{1}{\sqrt{2}\,\rho}\left\{-\varepsilon[\,a + b(K_1K_2 + K_1')]\,T(\sigma)\right.$$
$$\left. -\varepsilon bK_2\,B_1(\sigma) + \left[\varepsilon bK_2^2 - aK_1'\right]B_2(\sigma)\right\},$$

and

$$\varphi'''(\sigma^*) = \frac{1}{\sqrt{2}\,\rho}\left(\alpha_1 T(\sigma) + \alpha_2 B_1(\sigma) + \alpha_3 B_2(\sigma)\right),$$

where

$$\alpha_1 = bK_2 + \left[aK_1' - bK_2^2 - K_1'' - (K_1K_2)'\right],$$
$$\alpha_2 = b(2K_2K_2' - 2K_1' - K_1K_2) - aK_1'',$$
$$\alpha_3 = -K_2\left[a + b(K_1K_2 + K_1')\right].$$

Then

$$\tau_\varphi = \frac{\sqrt{2}}{\rho}\left\{\frac{\begin{array}{c}[\,bK_2^2 - aK_1']\,[\,b\alpha_2K_2 - a\alpha_1] + [\,b^2\alpha_3 - ab\alpha_1]\,K_2^2 \\ + a(\alpha_3 + \alpha_2K_2)[\,a + b(K_1K_2 + K_1')]\end{array}}{\begin{array}{c}a^4K_1'^4 + \left[aK_2\left(a + b(K_1K_2 + K_1')\right)\right]^2 \\ - \left[b^2K_2^2 + a\left(a + b(K_1K_2 + K_1')\right)\right]^2\end{array}}\right\}.$$

□

Corollary 2 *Let $r : I \subset \mathbb{R} \to S_1^2$ be a regular unit speed spacelike curve lying fully on S_1^2 with the moving equiform-Bishop frame $\{T, B_1, B_2\}$. If the base curve $r = r(s)$ is contained in a plane, then the spacelike equiform-Bishop TB_2-Smarandache curve is a circular helix if $K_2 \neq \pm\frac{a}{\sqrt{2}}$ and $K_2 \neq \pm 1$ and its natural curvature functions are dependent only on the second equiform-Bishop curvature and given by*

$$\kappa_\varphi(\sigma^*) = \frac{\sqrt{2}\,bK_2\sqrt{1 - K_2^2}}{b^2K_2^2 + a^2\left(1 - K_2^2\right)},$$
$$\tau_\varphi(\sigma^*) = \left\{\frac{\sqrt{2}}{\rho}\right\}\left\{\frac{K_2\left(3b^2K_2^2 - a^2\right)}{a^3\left(1 - K_2^2\right)}\right\}. \tag{19}$$

Definition 4 *Let $r : I \subset \mathbb{R} \to S_1^2$ be a regular unit speed spacelike curve lying fully on S_1^2. The spacelike equiform-Bishop B_1B_2-Smarandache curve $\varphi : I \subset \mathbb{R} \to S_1^2$ of r defined by*

$$\varphi = \varphi(\sigma^*) = \frac{1}{\sqrt{2}\,\rho}\left(a\,B_1(\sigma) + b\,B_2(\sigma)\right), \quad a^2 - b^2 = 2. \tag{20}$$

Theorem 3 *Let $r : I \subset \mathbb{R} \to S_1^2$ be a regular unit speed spacelike curve lying fully on S_1^2 with the moving equiform-Bishop frame $\{T, B_1, B_2\}$. If $\varphi : I \subset \mathbb{R} \to S_1^2$ is the spacelike equiform-Bishop B_1B_2-Smarandache curve of $r = r(\sigma)$ with non-zero natural curvature* *functions, then its Frenet frame* $\{T_\varphi, N_\varphi, B_\varphi\}$ *is given by*

$$\begin{pmatrix} T_\varphi \\ N_\varphi \\ B_\varphi \end{pmatrix} = \begin{pmatrix} \frac{-1}{\rho} & 0 & 0 \\ 0 & \frac{-1}{\rho\sqrt{1-K_2^2}} & \frac{K_2}{\rho\sqrt{1-K_2^2}} \\ 0 & \frac{-K_2}{\rho^2\sqrt{1-K_2^2}} & \frac{1}{\rho^2\sqrt{1-K_2^2}} \end{pmatrix} \begin{pmatrix} T \\ B_1 \\ B_2 \end{pmatrix}, \quad K_2 \neq \pm 1. \tag{21}$$

Proof Differentiationg Eq. (20) with respect to σ and using Eq. (5), we get

$$\varphi'(\sigma^*) = \frac{d\varphi}{d\sigma^*}\frac{d\sigma^*}{d\sigma} = \frac{-(a+bK_2)T(\sigma)}{\sqrt{2}\,\rho}, \tag{22}$$

hence

$$T_\varphi(\sigma^*) = \frac{-\,T(\sigma)}{\rho}, \tag{23}$$

with the parameterization

$$\frac{d\sigma^*}{d\sigma} = \frac{a+bK_2}{\sqrt{2}}. \tag{24}$$

Differentiating Eq. (23) with respect to σ, we have

$$T'_\varphi(\sigma^*) = \frac{-\sqrt{2}}{\rho(a+bK_2)}\left(B_1(\sigma) - K_2 B_2(\sigma)\right).$$

The curvature of φ is given by

$$\kappa_\varphi(\sigma^*) = \frac{\sqrt{2}\sqrt{1-K_2^2}}{a+bK_2}, \quad K_2 \neq \frac{-a}{b}.$$

Furthermore, the principal normal and binormal vectors of φ are defined as follows:

$$N_\varphi(\sigma^*) = \frac{-1}{\rho\sqrt{1-K_2^2}}\left(B_1(\sigma) - K_2 B_2(\sigma)\right),$$

$$B_\varphi(\sigma^*) = \frac{1}{\rho^2\sqrt{1-\varepsilon K_2^2}}\left(-K_2 B_1(\sigma) + B_2(\sigma)\right).$$

From Eq. (22), we get

$$\varphi''(\sigma^*) = \frac{-1}{\sqrt{2}\,\rho}\left\{bK_2'\,T(\sigma) + (a+bK_2)B_1(\sigma) \right.$$
$$\left. -K_2'(a+bK_2)B_2(\sigma)\right\},$$

similarly

$$\varphi'''(\sigma^*) = \frac{-1}{\sqrt{2}\,\rho}\left(\left[bK_2'' + (a+bK_2)(K_2K_2' - 1)\right]T(\sigma) + 2bK_2'\,B_1(\sigma)\right.$$
$$\left. -\left[(a+bK_2)K_2'' + K_2'(aK_2 + bK_2')\right]B_2(\sigma)\right).$$

Then, we obtain the torsion of φ as follows. Then

$$\tau_\varphi = \frac{\sqrt{2}}{\rho}\left\{\frac{(a+bK_2)K_2'' + K_2'(aK_2 + 3bK_2')}{({K_2'}^2 - 1)(a+bK_2)^2}\right\}.$$

□

Corollary 3 *Let* $r : I \subset \mathbb{R} \to S_1^2$ *be a regular unit speed spacelike curve lying fully on* S_1^2 *with the moving equiform-Bishop frame* $\{T, B_1, B_2\}$*. If the base curve* $r = r(s)$ *is contained* *in a plane, then the spacelike equiform-Bishop B_1B_2-Smarandache curve is also contained in a plane.*

Definition 5 *Let $r : I \subset \mathbb{R} \to S_1^2$ be a regular unit speed spacelike curve lying fully on S_1^2. The spacelike equiform-Bishop TB_1B_2-Smarandache curve $\varphi : I \subset \mathbb{R} \to S_1^2$ of r defined by*

$$\begin{aligned} \varphi = \varphi(\sigma^*) = \frac{1}{\sqrt{3}\,\rho}\left(a\,T(\sigma) + b\,B_1(\sigma) + +c\,B_2(\sigma)\right), \\ a^2 + b^2 - c^2 = 3. \end{aligned} \tag{25}$$

Theorem 4 *Let $r : I \subset \mathbb{R} \to S_1^2$ be a regular unit speed spacelike curve lying fully on S_1^2 with the moving equiform-Bishop frame $\{T, B_1, B_2\}$. If $\varphi : I \subset \mathbb{R} \to S_1^2$ is the spacelike equiform-Bishop TB_1B_2-Smarandache curve of $r = r(\sigma)$ with non-zero natural curvature functions, then its Frenet frame $\{T_\varphi, N_\varphi, B_\varphi\}$ is given by*

$$\begin{pmatrix} T_\varphi \\ N_\varphi \\ B_\varphi \end{pmatrix} = \begin{pmatrix} \frac{-(b+cK_2)}{\rho\sqrt{(b+cK_2)^2+(a^2-c^2K_2^2)}} & \frac{a}{\rho\sqrt{(b+cK_2)^2+(a^2-c^2K_2^2)}} & \frac{-cK_1}{\rho\sqrt{(b+cK_2)^2+(a^2-c^2K_2^2)}} \\ \frac{\ell_1}{\rho\sqrt{\ell_1^2+\ell_2^2-\ell_3^2}} & \frac{\ell_2}{\rho\sqrt{\ell_1^2+\ell_2^2-\ell_3^2}} & \frac{\ell_3}{\rho\sqrt{\ell_1^2+\ell_2^2-\ell_3^2}} \\ \frac{-(a\,\ell_3+c\,\ell_2K_1)}{\Delta_3} & \frac{c\,\ell_1K_1-\ell_3(b+cK_2)}{\Delta_3} & \frac{-[a\,\ell_1+\ell_2(b+cK_2)]}{\Delta_3} \end{pmatrix} \begin{pmatrix} T \\ B_1 \\ B_2 \end{pmatrix}, \tag{26}$$

where

$$\begin{aligned} \ell_1 &= (b+cK_2)\left[a^2K_2K_2' - c(b+cK_2)K_2'\right] - \left[(b+cK_2)^2 + a^2(1-K_2^2)\right]\left[cK_2' + a(1-K_2^2)\right], \\ \ell_2 &= (b+cK_2)\left[(b+cK_2)^2 + a^2(1-K_2^2)\right] - a\left[a^2K_2K_2' - c(b+cK_2)K_2'\right], \\ \ell_3 &= \left[b+cK_2 - aK_2'\right]\left[(b+cK_2)^2 + a^2(1-K_2^2)\right] + aK_1K_2\left[a^2K_2K_2' - c(b+cK_2)K_2'\right], \\ \Delta_3 &= \rho^2\sqrt{\ell_1^2+\ell_2^2-\ell_3^2}\sqrt{(b+cK_2)^2 + (a^2 - c^2K_2^2)}. \end{aligned} \tag{27}$$

Proof Differentiationg Eq. (25) with respect to σ and using Eq. (5), this leads to

$$\varphi'(\sigma^*) = \frac{d\varphi}{d\sigma^*}\frac{d\sigma^*}{d\sigma} = \frac{1}{\sqrt{3}\,\rho}\left(-(b+cK_2)\,T(\sigma) + a\,B_1(\sigma) - cK_1\,B_2(\sigma)\right), \tag{28}$$

then

$$T_\varphi(\sigma^*) = \frac{-(b+cK_2)\,T(\sigma) + a\,B_1(\sigma) - cK_1\,B_2(\sigma)}{\rho\sqrt{(b+cK_2)^2 + \left(a^2 - c^2K_2^2\right)}}, \tag{29}$$

where

$$\frac{d\sigma^*}{d\sigma} = \frac{\sqrt{(b+cK_2)^2 + \left(a^2 - c^2K_2^2\right)}}{\sqrt{3}}. \tag{30}$$

Then, from Eq. (29) , we get

$$T_\varphi'(\sigma^*) = \frac{\sqrt{3}\left(\ell_1 T(\sigma) + \ell_2 B_1(\sigma) + \ell_3 B_2(\sigma)\right)}{\rho\left[(b+cK_2)^2 + \left(a^2 - c^2K_2^2\right)\right]^2},$$

where

$$\begin{aligned} \ell_1 &= (b+cK_2)\left[a^2K_2K_2' - c(b+cK_2)K_2'\right] - \left[(b+cK_2)^2 + a^2(1-K_2^2)\right]\left[cK_2' + a(1-K_2^2)\right], \\ \ell_2 &= (b+cK_2)\left[(b+cK_2)^2 + a^2(1-K_2^2)\right] - a\left[a^2K_2K_2' - c(b+cK_2)K_2'\right], \\ \ell_3 &= \left[b+cK_2 - aK_2'\right]\left[(b+cK_2)^2 + a^2(1-K_2^2)\right] + aK_1K_2\left[a^2K_2K_2' - c(b+cK_2)K_2'\right]. \end{aligned}$$

Then, the curvature and the principal normal vector of φ are respectively

$$\kappa_\varphi(\sigma^*) = \frac{\sqrt{3}\sqrt{\ell_1^2+\ell_2^2-\ell_3^2}}{\left[(b+cK_2)^2+\left(a^2-c^2K_2^2\right)\right]^2},$$

and

$$N_\varphi(\sigma^*) = \frac{\ell_1 T(\sigma)+\ell_2 B_1(\sigma)+\ell_3 B_2(\sigma)}{\rho\sqrt{\ell_1^2+\ell_2^2-\ell_3^2}}.$$

Besides, the binormal vector of φ is given by

$$\begin{aligned} B_\varphi(\sigma^*) =& \frac{1}{\Delta_3}\left\{-(a\,\ell_3+c\,\ell_2K_1)T(\sigma)+\left[c\,\ell_1K_1-\ell_3(b+cK_2)\right]B_1(\sigma)\right. \\ & \left.-\left[a\,\ell_1+\ell_2(b+cK_2)\right]B_2(\sigma)\right\}, \end{aligned}$$

where

$$\Delta_3 = \rho^2\sqrt{\ell_1^2+\ell_2^2-\ell_3^2}\sqrt{(b+cK_2)^2+\left(a^2-c^2K_2^2\right)}.$$

The derivatives φ'' and φ''' of φ are

$$\begin{aligned} \varphi''(\sigma^*) =& \frac{1}{\sqrt{3}\,\rho}\left\{-\left[a+c\left(K_2'-K_1K_2\right)\right]T(\sigma)-\left[b+cK_2\right]B_1(\sigma)\right. \\ & \left.+\left[(b+cK_2)K_2-cK_1'\right]B_2(\sigma)\right\}, \end{aligned}$$

and

$$\varphi'''(\sigma^*) = \frac{1}{\sqrt{3}\,\rho}\left(\gamma_1 T(\sigma)+\gamma_2 B_1(\sigma)+\gamma_3 B_2(\sigma)\right),$$

where

$$\begin{aligned} \gamma_1 &= c\left(K_2-K_2''\right)-K_2\left[b+cK_2-3aK_2'\right], \\ \gamma_2 &= -\left[2cK_2'+a\left(1-K_2^2\right)\right], \\ \gamma_3 &= cK_2'-aK_2''+K_2\left[2cK_2'+a\left(1-K_2^2\right)\right]. \end{aligned}$$

Then

$$\tau_\varphi = \frac{\sqrt{3}}{\rho}\left\{\frac{\begin{array}{c} a^2K_2'+\left[b+cK_2\right]\left[(\gamma_2-\gamma_3)(b+cK_2)+a\,\gamma_1(1-K_1)-a\,\gamma_3K_2'\right] \\ +a\left(\gamma_3+\gamma_2K_1\right)\left[cK_2'-a\left(1-K_2^2\right)\right] \end{array}}{\begin{array}{c} \left[a^2K_2'-a(b+cK_2)(1-K_1)\right]^2+\left[aK_1\left[cK_2'+a\left(1-K_2^2\right)\right]\right. \\ \left.+(b+cK_2)\left[b+cK_2+aK_2'\right]\right]^2-\left[(b+cK_2)^2+a\left[cK_2'+a\left(1-K_2^2\right)\right]\right]^2 \end{array}}\right\}.$$

□

Corollary 4 *Let $r: I\subset\mathbb{R}\to S_1^2$ be a regular unit speed spacelike curve lying fully on S_1^2 with the moving equiform-Bishop frame $\{T, B_1, B_2\}$. If the base curve $r=r(s)$ is contained in a plane, then the spacelike equiform-Bishop TB_1B_2-Smarandache curve is a circular helix if $K_2\neq\pm\frac{a^2+b^2}{2}$ and $K_2\neq\pm1$. Also, its natural curvature functions are dependent* *only on the second equiform-Bishop curvature and given by*

$$\kappa_\varphi(\sigma^*) = \frac{a\sqrt{3\left(1-K_2^2\right)}}{(b+cK_2)^2+\left(a^2-c^2K_2^2\right)},$$
$$\tau_\varphi(\sigma^*) = \left\{\frac{\sqrt{3}}{\rho}\right\}\left\{\frac{\begin{array}{l} abK_2(b+cK_2)(K_1+K_2)-a^3K_1\left(1-K_2^2\right)^2 \\ +a(b+cK_2)\left(1-K_2^2\right)[a+c(1+K_1K_2)+2(b+cK_2)] \end{array}}{\begin{array}{l} a^4(1-K_1)^2\left(1-K_2^2\right)^2+a(1-K_1)(b+cK_2)\,[b+cK_2 \\ -2(1-K_2^2)] \end{array}}\right\}. \tag{31}$$

Example

In this section, we construct a computational examples of the spacelike equiform-Bishop Smarandache curves in $\mathbb{R}_1^3$ with the moving equiform-Bishop frame $\{T, B_1, B_2\}$ of the spacelike equiform-Bishop curve $r = r(\sigma)$. Let $r(s) = (s, s\sin(\ln s), s\cos(\ln s))$ be a unit speed spacelike curve parametrized by arc-length s with spacelike principal normal vector in $\mathbb{R}_1^3$ (see Fig. 1). Then, it is easy to show that

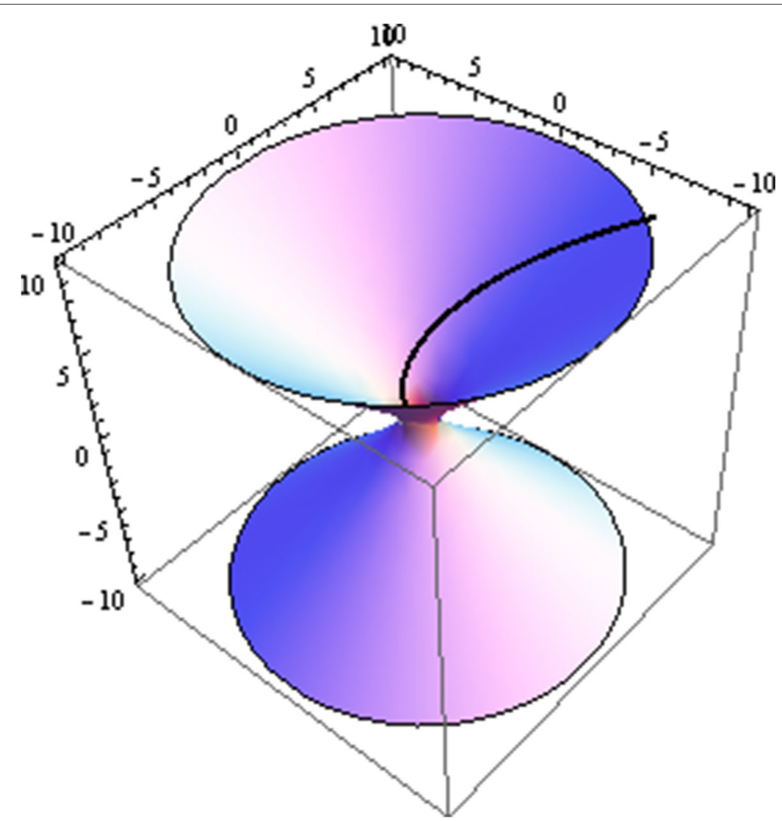

Fig. 1 Spacelike curve $r = r(s)$ on S_1^2

$$t(s) = (1, \sin(\ln s) + \cos(\ln s), \cos(\ln s) - \sin(\ln s)).$$

This vector is spacelike and future-directed, we have $\kappa = \frac{\sqrt{2}}{s}$. Hence,

$$\begin{cases} n(s) = \frac{1}{\sqrt{2}}(0, \cos(\ln s) - \sin(\ln s), -\sin(\ln s) - \cos(\ln s)), \\ b(s) = \frac{1}{\sqrt{2}}(2, \sin(\ln s) + \cos(\ln s), \cos(\ln s) - \sin(\ln s)). \end{cases}$$

The torsion is $\tau = \frac{1}{s}$ and $\theta(s) = \int \left(\frac{1}{s}\right) ds = \ln s + c$. Here, we can take $c = 0$. From Eq. (4), we get $k_1(s) = \left(\frac{\sqrt{2}}{s}\right)\cosh(\ln s)$, $k_2(s) = \left(\frac{\sqrt{2}}{s}\right)\sinh(\ln s)$. Also from Eq. (2), we get $b_1(s) = -\int k_1(s)t(s)ds$ and $b_2(s) = -\int k_2(s)t(s)ds$, then we have

$$b_1(s) = \frac{1}{\sqrt{2}\,s}\left(s^2 - 1, s^2\sin(\ln s) - \cos(\ln s), \sin(\ln s) + s^2\cos(\ln s)\right),$$

$$b_2(s) = \frac{1}{\sqrt{2}\,s}\left(s^2 + 1, s^2\sin(\ln s) + \cos(\ln s), -\sin(\ln s) + s^2\cos(\ln s)\right).$$

Now, the equiform-Bishop parameter is $\sigma = \int k_1\,ds = \sqrt{2}\sinh(\ln s) + c$. In this case, we take $c = 0$, then we have $s = \left(\frac{\sigma + \sqrt{\sigma^2+2}}{\sqrt{2}}\right)$ and $\rho = \left(\frac{\sigma + \sqrt{\sigma^2+2}}{\sqrt{2}\sqrt{\sigma^2+2}}\right)$. Furthermore, the equiform-Bishop curvatures are given by

$$\begin{cases} K_1(\sigma) = \dfrac{1 - \sigma\sqrt{\sigma^2+2}}{\sqrt{2}(\sigma^2+2)}, \\ K_2(\sigma) = \dfrac{\sigma}{\sqrt{\sigma^2+2}}. \end{cases}$$

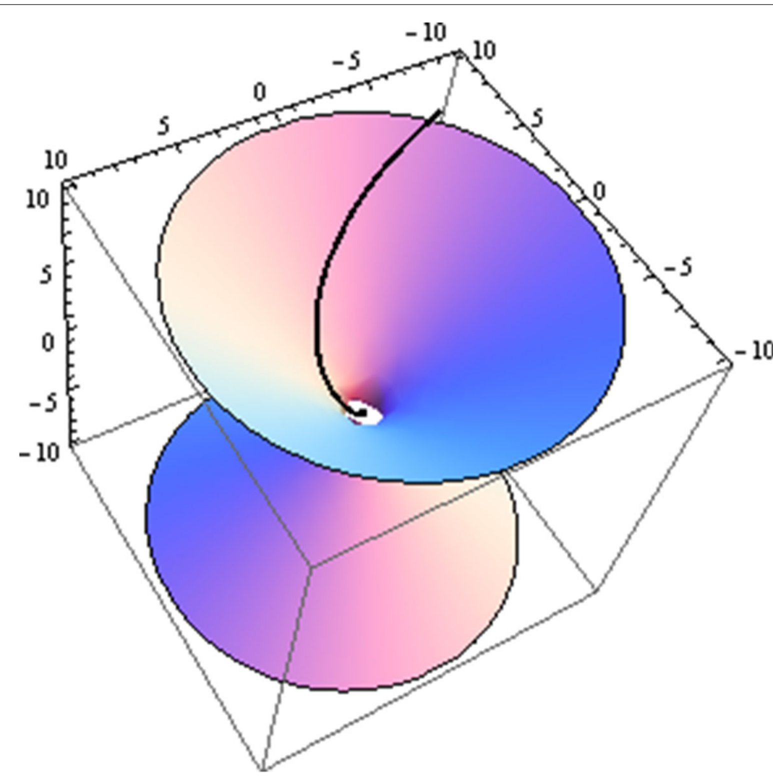

Fig. 2 Spacelike equiform-Bishop curve $r = r(\sigma)$ on S_1^2

So the spacelike equiform-Bishop curve $r = r(\sigma)$ is defined as (see Fig. 2

$$r(\sigma) = \left(\frac{\sigma+\sqrt{\sigma^2+2}}{\sqrt{2}}\right)\left(1, \sin\left(\ln\left(\frac{\sigma+\sqrt{\sigma^2+4}}{\sqrt{2}}\right)\right), \cos\left(\ln\left(\frac{\sigma+\sqrt{\sigma^2+4}}{\sqrt{2}}\right)\right)\right).$$

It easy to show that

$$T(\sigma) = \left(\frac{\sigma+\sqrt{\sigma^2+2}}{\sqrt{2}\sqrt{\sigma^2+2}}\right)\left(1, \sin\left(\ln\left(\frac{\sigma+\sqrt{\sigma^2+4}}{\sqrt{2}}\right)\right) + \cos\left(\ln\left(\frac{\sigma+\sqrt{\sigma^2+4}}{\sqrt{2}}\right)\right)\right.$$
$$\left., \cos\left(\ln\left(\frac{\sigma+\sqrt{\sigma^2+4}}{\sqrt{2}}\right)\right) - \sin\left(\ln\left(\frac{\sigma+\sqrt{\sigma^2+4}}{\sqrt{2}}\right)\right)\right).$$

It is easy to show that T is an equiform-Bishop spacelike vector. Also

$$B_1(\sigma) = \left(\frac{\sigma+\sqrt{\sigma^2+2}}{2\sqrt{\sigma^2+2}}\right)\left(0, \cos\left(\ln\left(\frac{\sigma+\sqrt{\sigma^2+4}}{\sqrt{2}}\right)\right) - \sin\left(\ln\left(\frac{\sigma+\sqrt{\sigma^2+4}}{\sqrt{2}}\right)\right)\right.$$
$$\left., -\sin\left(\ln\left(\frac{\sigma+\sqrt{\sigma^2+4}}{\sqrt{2}}\right)\right) - \cos\left(\ln\left(\frac{\sigma+\sqrt{\sigma^2+4}}{\sqrt{2}}\right)\right)\right).$$

$$B_2(\sigma) = \left(\frac{\sigma+\sqrt{\sigma^2+2}}{2\sqrt{\sigma^2+2}}\right)\left(2, \sin\left(\ln\left(\frac{\sigma+\sqrt{\sigma^2+4}}{\sqrt{2}}\right)\right) + \cos\left(\ln\left(\frac{\sigma+\sqrt{\sigma^2+4}}{\sqrt{2}}\right)\right)\right.$$
$$\left., \cos\left(\ln\left(\frac{\sigma+\sqrt{\sigma^2+4}}{\sqrt{2}}\right)\right) - \sin\left(\ln\left(\frac{\sigma+\sqrt{\sigma^2+4}}{\sqrt{2}}\right)\right)\right).$$

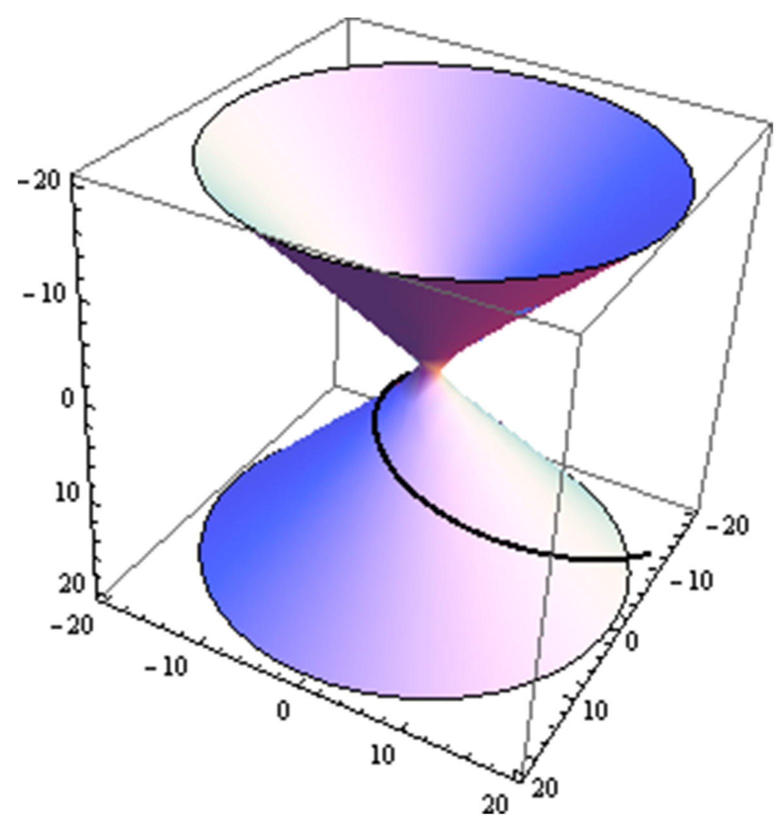

Fig. 3 The spacelike equiform-Bishop TB_1-Smarandache curve $\varphi(\sigma^*)$ on S_1^2

It is clear that B_1 is an equiform-Bishop spacelike vector and B_2 is an equiform-Bishop timelike vector. Moreover, if we take $a = b = 1$, the spacelike equiform-Bishop TB_1-Smarandache curve $\varphi(\sigma^*)$ of the curve $r(\sigma)$ is given by (see Fig. 3)

$$\begin{aligned}\varphi(\sigma^*) = &\left(\frac{\sigma+\sqrt{\sigma^2+2}}{2\sqrt{\sigma^2+2}}\right)\left(\sqrt{2}, \left(\sqrt{2}-1\right)\sin\left(\ln\left(\frac{\sigma+\sqrt{\sigma^2+4}}{\sqrt{2}}\right)\right)\right.\\ &+\left(\sqrt{2}+1\right)\cos\left(\ln\left(\frac{\sigma+\sqrt{\sigma^2+4}}{\sqrt{2}}\right)\right), \left(\sqrt{2}-1\right)\cos\left(\ln\left(\frac{\sigma+\sqrt{\sigma^2+4}}{\sqrt{2}}\right)\right)\\ &\left.-\left(\sqrt{2}+1\right)\sin\left(\ln\left(\frac{\sigma+\sqrt{\sigma^2+4}}{\sqrt{2}}\right)\right)\right).\end{aligned}$$

If we take $a = 3$ and $b = \sqrt{7}$, the spacelike equiform-Bishop TB_2-Smarandache curve $\varphi(\sigma^*)$ of the curve $r(\sigma)$ is given by (see Fig. 4)

$$\begin{aligned}\varphi(\sigma^*) = &\left(3\sqrt{2}+\sqrt{7}\right)\left(\frac{\sigma+\sqrt{\sigma^2+2}}{2\sqrt{2}\sqrt{\sigma^2+2}}\right)\left(1, \sin\left(\ln\left(\frac{\sigma+\sqrt{\sigma^2+4}}{\sqrt{2}}\right)\right)\right.\\ &+\cos\left(\ln\left(\frac{\sigma+\sqrt{\sigma^2+4}}{\sqrt{2}}\right)\right), \cos\left(\ln\left(\frac{\sigma+\sqrt{\sigma^2+4}}{\sqrt{2}}\right)\right)\\ &\left.-\sin\left(\ln\left(\frac{\sigma+\sqrt{\sigma^2+4}}{\sqrt{2}}\right)\right)\right).\end{aligned}$$

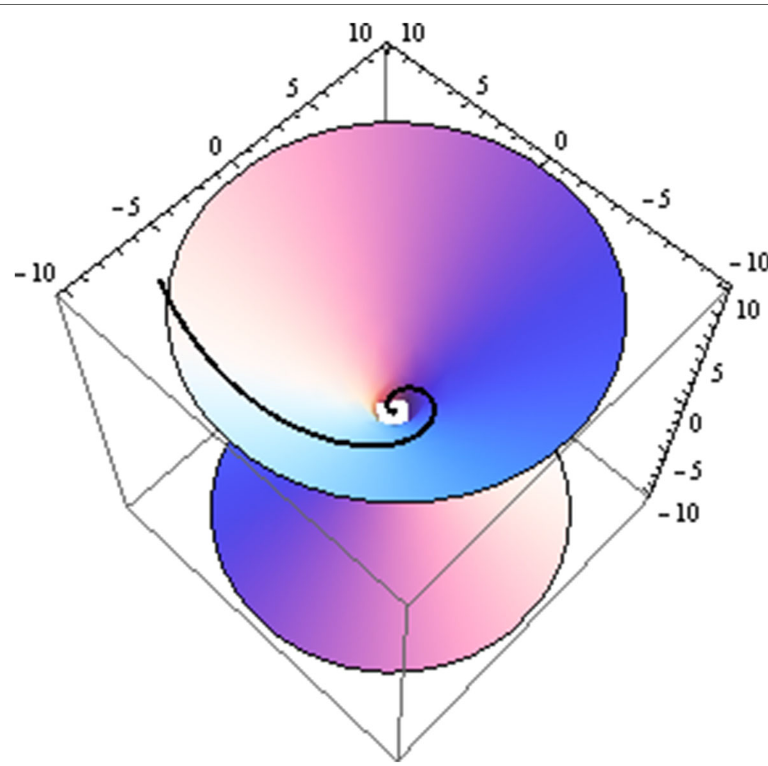

Fig. 4 The spacelike equiform-Bishop TB_2-Smarandache curve $\varphi(\sigma^*)$ on S_1^2

If we take $a = 2$ and $b = \sqrt{2}$, the spacelike equiform-Bishop B_1B_2-Smarandache curve $\varphi(\sigma^*)$ of the curve $r(\sigma)$ is given by (see Fig. 5)

$$\varphi(\sigma^*) = \left(\frac{\sigma + \sqrt{\sigma^2+2}}{2\sqrt{2}\sqrt{\sigma^2+2}}\right)\left(2\sqrt{2}, (2+\sqrt{2})\cos\left(\ln\left(\frac{\sigma+\sqrt{\sigma^2+4}}{\sqrt{2}}\right)\right)\right.$$
$$+(\sqrt{2}-2)\sin\left(\ln\left(\frac{\sigma+\sqrt{\sigma^2+4}}{\sqrt{2}}\right)\right), (\sqrt{2}-2)\cos\left(\ln\left(\frac{\sigma+\sqrt{\sigma^2+4}}{\sqrt{2}}\right)\right)$$
$$\left.-(2+\sqrt{2})\sin\left(\ln\left(\frac{\sigma+\sqrt{\sigma^2+4}}{\sqrt{2}}\right)\right)\right).$$

If we take $a = 2$, $b = 1$, and $c = \sqrt{2}$, the spacelike equiform-Bishop TB_1B_2-Smarandache curve $\varphi(\sigma^*)$ of the curve $r(\sigma)$ is given by (see Fig. 6)

$$\varphi(\sigma^*) = \left(\frac{\sigma + \sqrt{\sigma^2+2}}{2\sqrt{3}\sqrt{\sigma^2+2}}\right)\left(4\sqrt{2}, \left(4\sqrt{2}-1\right)\sin\left(\ln\left(\frac{\sigma+\sqrt{\sigma^2+4}}{\sqrt{2}}\right)\right)\right.$$
$$+\left(4\sqrt{2}+1\right)\cos\left(\ln\left(\frac{\sigma+\sqrt{\sigma^2+4}}{\sqrt{2}}\right)\right), \left(4\sqrt{2}+1\right)\cos\left(\ln\left(\frac{\sigma+\sqrt{\sigma^2+4}}{\sqrt{2}}\right)\right)$$
$$\left.+\left(4\sqrt{2}-1\right)\sin\left(\ln\left(\frac{\sigma+\sqrt{\sigma^2+4}}{\sqrt{2}}\right)\right)\right).$$

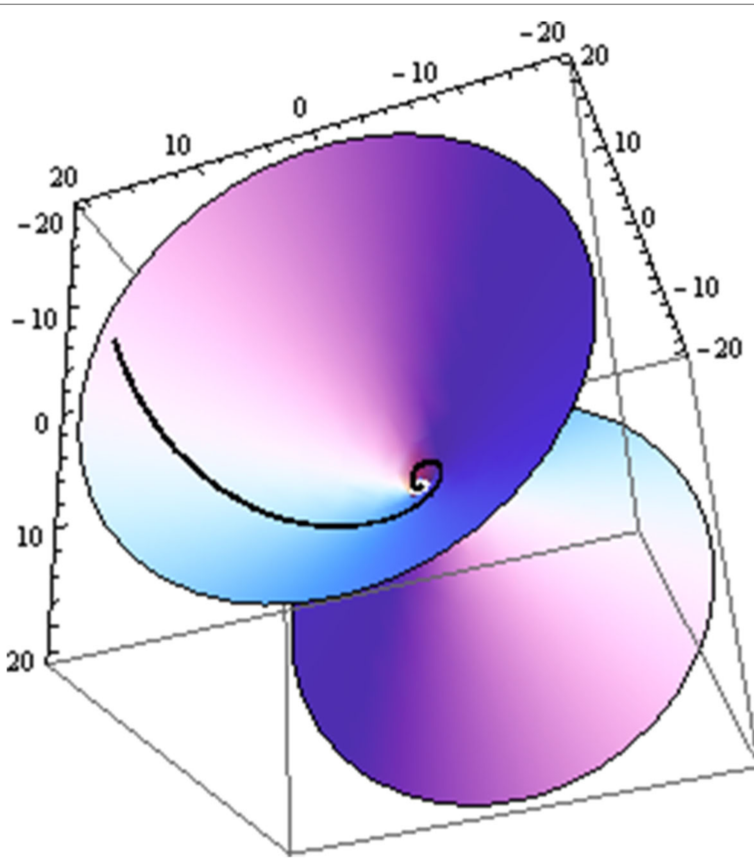

Fig. 5 The spacelike equiform-Bishop B_1B_2-Smarandache curve $\varphi(\sigma^*)$ on S_1^2

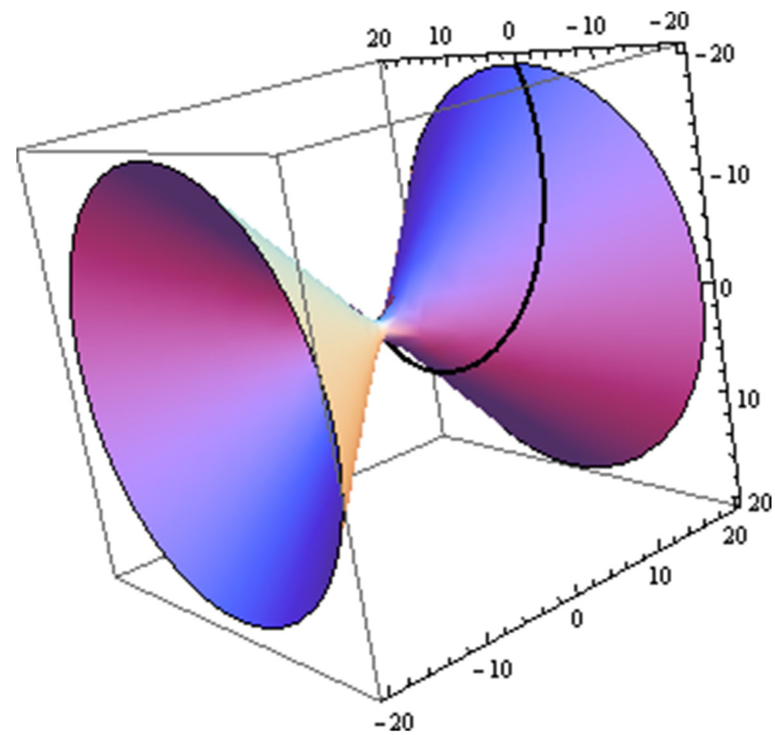

Fig. 6 The spacelike equiform-Bishop TB_1B_2-Smarandache curve $\varphi(\sigma^*)$ on S_1^2

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Author details
[1]Department of Mathematics and Statistics, College of Science, Al Imam Mohammad Ibn Saud Islamic University, Riyadh, Kingdom of Saudi Arabia. [2]Department of Mathematics, Faculty of Science, Beni-Suef University, Beni Suef, Egypt. [3]Department of Mathematics, Faculty of Science, Aswan University, Aswan, Egypt.

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