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ORIGINAL ARTICLE

Numerical simulation of nanofluid flow with convective boundary condition

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Abstract In this paper, the heat and mass transfer of an electrically conducting incompressible nanofluid over a heated stretching sheet with convective boundary condition is investigated. The transport model includes the effect of Brownian motion with thermophoresis in the presence of thermal radiation, chemical reaction and magnetic field. Lie group transformations are applied to the governing equations. The transformed ordinary differential equations are solved numerically by employing Runge–Kutta–Fehlberg method with shooting technique. Numerical results for temperature and concentration profiles as well as wall heat and mass flux are elucidated through graphs and tables.

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1. Introduction

In the recent past a new class of fluids, namely nanofluids has attracted the attention of the science and engineering community because of the many possible industrial applications of these fluids. An innovative way of improving the thermal conductivities of heat transfer fluids is to suspend small solid particles in the fluids. Nanofluids are nanometer-sized particles (diameter less than 50 nm) dispersed in a base fluid such as water, ethylene glycol, toluene and oil. Addition of high thermal conductivity metallic nanoparticles (e.g., aluminum, copper, silicon, silver and titanium or their oxides) increases the thermal conductivity of such mixtures; thus enhancing their overall energy transport capability. The enhancement of thermal conductivities by nanofluids was first discussed by Choi [1]. It should be noticed that there have been published several recent papers [2–5] on the mathematical and numerical modeling of convective heat transfer in nanofluids. The boundary layer flow of a nanofluid caused by a stretching surface has drawn the attention of a growing number of researchers [6–10] because of its immense potential to be used as a technological tool in many engineering applications.

The effect of radiation on heat transfer problems has studied by Makinde [11], Ibrahim et al. [12], Hayat et al. [13], Das [14] and Nadeem et al. [15]. Lie group analysis, also

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known as symmetry analysis, is the most powerful, sophisticated, and systematic method for finding similarity solution of non-linear differential equations and is widely used in non-linear dynamical system, especially in the range of deterministic chaos. This technique has been applied by many researchers [16–19] to study different flow phenomena over different geometrics arising in fluid mechanics, chemical engineering and other engineering branches. Hamad and Ferdows [20] considered similarity solution of boundary layer stagnation-point flow toward a heated porous stretching sheet saturated with a nanofluid using Lie group analysis. Recently, heat transfer problems for boundary layer flow concerning with a convective boundary condition were investigated by Ishak [21], Makinde and Aziz [22]. Recently, radiation effects on MHD nanofluid flow toward a stretching surface with convective boundary condition were discussed by Akbar et al. [23].

The aim of the present work was to study the effects of the thermal radiation on the heat and mass transfer of an electrically conducting incompressible nanofluid over a heated stretching sheet with convective boundary conditions. The flow is permeated by a uniform transverse magnetic field in presence of Brownian motion, chemical reaction with thermophoresis.

2. Mathematical analysis

The steady two-dimensional boundary layer flow of an electrically conducting nanofluid over a heated stretching sheet is considered in the region $y > 0$. Keeping the origin fixed, two equal and opposite forces are applied along the x-axis which results in stretching of the sheet and a uniform magnetic field of strength B_0 is imposed along the y-axis. It is assumed that the velocity of the external flow is $U(x) = ax$ and the velocity of the stretching sheet is $u_w(x) = bx$ where a is a positive constant and b is a positive (stretching sheet) constant. The chemical reaction and thermal radiation is taking place in the flow.

Under the above conditions, the governing boundary layer equations are

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{dU}{dx} + v\frac{\partial^2 u}{\partial y^2} - \frac{v}{k}(u-U) - \frac{\sigma B_0^2}{\rho}(u-U), \tag{2}$$

$$\begin{aligned} u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} &= \frac{\kappa}{(\rho c_p)_f}\frac{\partial^2 T}{\partial y^2} + \frac{Q_0}{(\rho c_p)_f}(T - T_\infty) \\ &\quad + \tau\left[D_B\frac{\partial C}{\partial y}\frac{\partial T}{\partial y} + \frac{D_T}{T_\infty}\left(\frac{\partial T}{\partial y}\right)^2\right] - \frac{1}{(\rho c_p)_f}\frac{\partial q_r}{\partial y} \end{aligned} \tag{3}$$

$$u\frac{\partial C}{\partial x} + v\frac{\partial C}{\partial y} = D_B\frac{\partial^2 C}{\partial y^2} + \frac{D_T}{T_\infty}\frac{\partial^2 T}{\partial y^2} - k_1(C - C_\infty) \tag{4}$$

where u, v are the velocity components along the x and y-axis respectively, T is temperature, k is the permeability of the porous medium, v is the kinematic viscosity, σ is the electrical conductivity, C_p is the specific heat at constant pressure, $\tau = (\rho c)_p/(\rho c)_f$ is the ratio of the effective heat capacity of the nanoparticle material and the base fluid, ρ_f is the density of base fluid, ρ_p is the nanoparticle density, D_B is the Brownian diffusion coefficient, D_T is the thermophoretic diffusion coefficient, k_1 is the rate of chemical reaction.

The radiative heat flux term q_r by using the Rosseland approximation is given by

$$q_r = -\frac{4\sigma^*}{3k^*}\frac{\partial T^4}{\partial y} \tag{5}$$

where σ^* is the Stefan–Boltzmann constant and k^* is the mean absorption coefficient. Assuming that the differences in temperature within the flow are such that T^4 can be expressed as a linear combination of the temperature, T^4 may be expanded in Taylor's series about T_∞ and neglecting higher order terms, one may get

$$T^4 = 4T_\infty^3 T - 3T_\infty^4 \tag{6}$$

Thus

$$\frac{\partial q_r}{\partial y} = -\frac{16T_\infty^3\sigma^*}{3k^*}\frac{\partial^2 T}{\partial y^2} \tag{7}$$

The boundary conditions at the plate surface and far into the cold fluid may be written as

$$\left.\begin{array}{ll} u = u_w(x), v = v_w, -\kappa\frac{\partial T}{\partial y} = h_w(T_f - T_w), C = C_w & \text{for } y = 0, \\ u \to U(x), T \to T_\infty, C \to C_\infty & \text{as } y \to \infty \end{array}\right\} \tag{8}$$

where v_w is the wall mass transfer velocity and T_f is the convective fluid temperature.

Introducing the following non-dimensional variables:

$$x' = \frac{x}{\sqrt{vb}}, \quad y' = \frac{y}{\sqrt{vb}}, \quad u' = \frac{u}{\sqrt{vb}}, \quad v' = \frac{v}{\sqrt{vb}}, \\ U' = \frac{U}{\sqrt{vb}}, \quad \theta = \frac{T - T_\infty}{T_f - T_\infty}, \quad \phi = \frac{C - C_\infty}{C_w - C_\infty}, \tag{9}$$

and using classical Lie group approach along the same lines as in Das [10] and Hamad and Ferdows [20], we get

$$\eta = y, \quad \psi = xf(\eta), \quad \theta = \theta(\eta), \quad \phi = \phi(\eta) \tag{10}$$

Substituting (10) into Eqs. (2)–(4) we finally obtain the following system of non-linear ordinary differential equations

$$f''' + ff'' - f'^2 + \frac{a^2}{b^2} - \left(M^2 + \frac{1}{K}\right)\left(f' - \frac{a}{b}\right) = 0, \tag{11}$$

$$(1 + Nr)\frac{1}{Pr}\theta'' + f\theta' + \lambda\theta + Nb\theta'\phi' + Nt\theta'^2 = 0 \tag{12}$$

$$\phi'' + LePrf\phi' + \frac{Nt}{Nb}\theta'' - Kr\phi = 0 \tag{13}$$

The corresponding boundary conditions (8) become

$$\left.\begin{array}{ll} f = S, f' = 1, \theta' = -\gamma(1-\theta), \phi = 1 & \text{at } \eta = 0 \\ f' \to \frac{a}{b}, \theta \to 0, \phi \to 0 & \text{as } \eta \to \infty \end{array}\right\} \tag{14}$$

where $Pr = \frac{v}{\alpha}$ is the Prandtl number, $Le = \frac{\alpha}{D_B}$ is the Lewis number, $Nr = \frac{4T_\infty^3\sigma^*}{3k^*\kappa}$ is the thermal parameter, $Nb = \frac{\tau D_B(C_w - C_\infty)}{v}$ is the Brownian motion parameter, $Nt = \frac{\tau D_T(T_w - T_\infty)}{vT_\infty}$ is the thermophoresis parameter, $S = \frac{v_w}{\sqrt{bv}}$ is the suction/injection parameter, $Kr = \frac{k_1 v}{bD_B}$ is the chemical reaction rate parameter, $K = \frac{bk}{v}$ is the permeability parameter, $M = B_0\sqrt{\frac{\sigma}{b\rho}}$ is the magnetic field parameter and $\gamma = \frac{h_w\sqrt{vb}}{\kappa}$ is the surface convection parameter.

The quantities of physical interest in this problem are the local Nusselt number Nu and the local Sherwood number Su which are defined as

$$Nur = Re_x^{-1/2}Nu = -(1 + Nr)\theta'(0), \tag{15}$$

$$Shr = Re_x^{-1/2}Sh = -\phi'(0) \tag{16}$$

where $Re_x = xu_w/\nu_f$ is the local Reynolds number, *Nur*, the reduced Nusselt number and *Shr*, the reduced Sherwood number.

3. Numerical experiment

The set of highly non-linear ordinary differential Eqs. (11)–(13) with boundary conditions (14) are solved numerically by employing Runge–Kutta–Fehlberg method with shooting technique taking Nr, Nt, Nb, Le, γ as prescribed parameters. A step size of $\Delta\eta = 0.01$ is selected to be satisfactory for a convergence criterion of 10^{-6} in all cases. For numerical computation infinity condition was considered for a large but finite value of η where no considerable variation in temperature, concentration, etc. occurs. Table 1 shows the comparison of the data produced by the present code and that of Das [10] and Hamad and Ferdows [20] in the absence of mass transfer, thermal radiation, magnetic field and convective surface boundary condition. The results show excellent agreement among data. Thus the use of the present numerical code for current model is justified.

4. Results and discussions

The velocity fields, i.e. the momentum equation solutions, have been discussed in Das [10] in detail. This paper focuses on the heat and mass transfer problem with a convective boundary condition at the wall. The solutions for dimensionless temperature and dimensionless concentration are computed for various pertinent parameters.

Table 2 presents the effects for various pertinent parameters on the reduced Nusselt number and the reduced Sherwood number when the stretching sheet is heated convectively. From Table, it can be noticed that the heat transfer rate at the plate increases with increasing values of *Nr* but effect is opposite for *Nb* and *Nt*. This enhancement is due to the nanoparticles of high thermal conductivity being driven away from the hot sheet to the quiescent nanofluid. Further, it is observed from table that an increase in *Nr* and *Nb* leads to increase in the values of the rate of mass transfer while the effect is reverse for *Nt*. It is observed that the heat transfer rate at the plate increases with increase in the values of convection parameter γin the presence of thermal radiation. But the effect is opposite for Sherwood number.

Figs. 1 and 2 show the effects of the Brownian motion *Nb* and the thermophoresis parameter *Nt* on temperature profiles of nanofluid across the boundary layer region in the presence as well as in the absence of thermal radiation. It is found that the temperature increases with the increase in both the values of *Nb* and *Nt*. It may be noted from Fig. 3 that as *Nr* increases,

Table 1 Comparison of results for $f''(0)$ with previously published work.

K	Hamad and Ferdows [20]	Das [10]	Present work
0.0	1.99901	1.99903	1.9990351
0.1	2.01021	2.01016	2.0101278
0.5	2.11021	2.11020	2.1102000
1.0	2.39018	2.39031	2.3903126

Table 2 Effects of various parameters on *Nur* and *Shr*.

Nr	*Nb*	*Nt*	γ	*Nur*	*Shr*
0.0	0.1	0.2	0.2	0.148816	2.52876
0.4				0.197601	2.54675
0.8				0.244394	2.55886
	0.2			0.193821	2.58631
	0.3			0.189919	2.59959
		0.2		0.197601	2.54675
		0.4		0.196306	2.48457
			0.0	0.103318	2.23404
			10	0.611874	2.4208
			∞	0.632462	2.4151

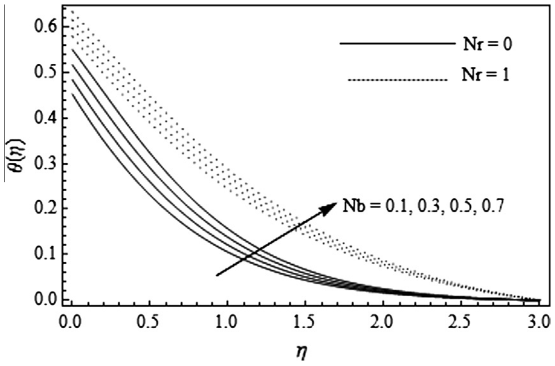

Figure 1 Temperature profiles for various values of *Nb*.

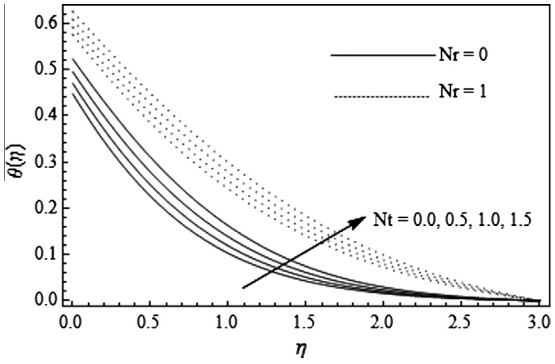

Figure 2 Temperature profiles for various values of *Nt*.

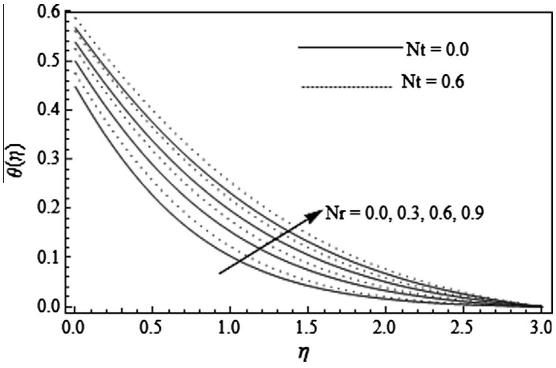

Figure 3 Temperature profiles for various values of *Nr*.

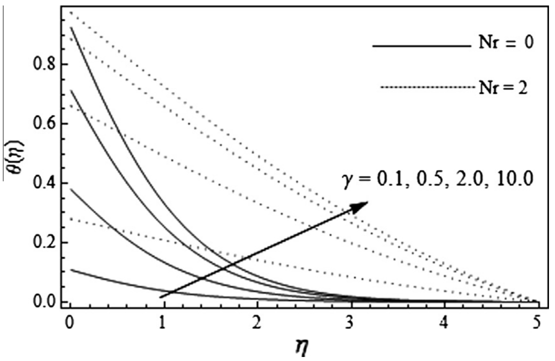

Figure 4 Temperature profiles for various values of γ.

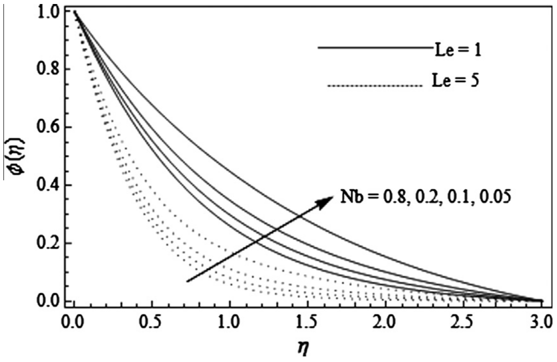

Figure 5 Concentration profiles for various values of Nb.

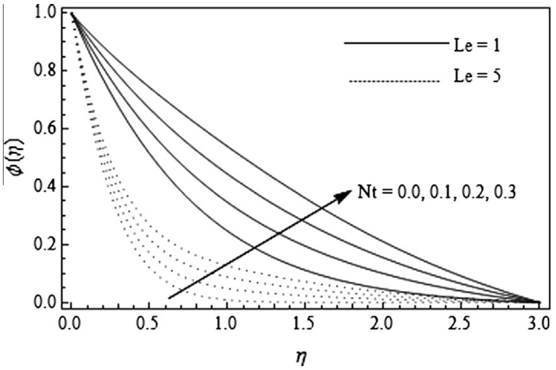

Figure 6 Concentration profiles for various values of Nt.

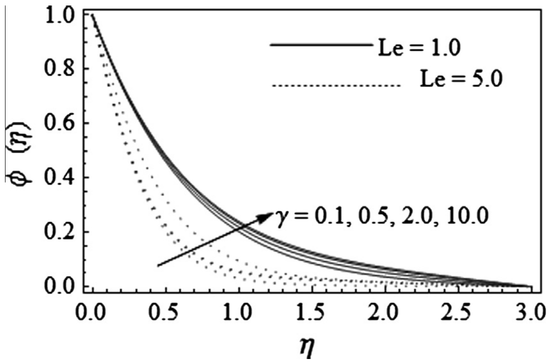

Figure 7 Concentration profiles for various values of γ.

the temperature increases substantially for both $Nt = 0$ and $Nt = 1$. It is observed from the Fig. 4 that temperature increases on increasing γ in the boundary layer region and is maximum at the surface of the plate.

The impact of Brownian motion parameter Nb on the dimensionless concentration is shown in Fig. 5. As the parameter value of Nb increases in the presence as well as in the absence of Lewis number Le, the concentration of nanofluid decreases in the boundary layer region. Fig. 6 shows that concentration of nanofluid increases with the increase in the thermophoretic parameter Nt (for $\eta > 0.3$, not precisely determined) but has no effect near the boundary surface (for $\eta < 0.3$, not precisely determined). Fig. 7 presents the variation in concentration profiles within the boundary layer for various values of surface convection parameter γ. As γ increases concentration of the nanofluid in the boundary layer region increases slightly but effect is significant for large values of Lewis number.

5. Conclusions

In this work, the heat and mass transfer problem for an electrically conducting nanofluid over a convectively heated stretching surface in the presence of thermal radiation, Brownian motion and thermophoresis is investigated. The use of a convective heating boundary condition instead of a constant temperature or a constant heat flux makes this study more general novel. The following conclusion can be drawn from the present investigation:

- An increase in the surface convection parameter, thermal radiation parameter, Brownian motion parameter and thermophoretic parameter lead to an increase in the thermal boundary layer thickness.
- The concentration of nanofluid is an increasing function of each value of the thermophoretic parameters and surface convection parameter.
- The results demonstrate that the surface convection parameter and thermal radiation parameter is able to enhance heat transfer rate at the wall while it decreases for increasing Brownian motion parameter and thermophoretic parameter.
- The rate of mass transfer at the wall decreases with the increase in the surface convection parameter and thermophoretic parameter whereas the effect is reverse for Brownian motion parameter.

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