Journal of the Egyptian Mathematical Society (2015) 23, 416–423

Egyptian Mathematical Society

Journal of the Egyptian Mathematical Society

www.etms-eg.org
www.elsevier.com/locate/joems

ORIGINAL ARTICLE

Modified alternative (G'/G)-expansion method to general Sawada–Kotera equation of fifth-order

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Received 24 August 2013; revised 6 June 2014; accepted 18 June 2014
Available online 11 February 2015

KEYWORDS

(G'/G)-expansion method;
Fifth-order KdV equations;
General SK equation;
Traveling wave solutions;
Solitons

Abstract The alternative (G'/G)-expansion method has been further modified by introducing the generalized Riccati equation to construct new exact solutions. In order to illustrate the novelty and advantages of this approach, the general Sawada–Kotera (GSK) equation is considered and abundant new exact traveling wave solutions are obtained in a uniform way. These solutions may be imperative and significant for the explanation of some practical physical phenomena. It is shown that the modified alternative (G'/G)-expansion method is an efficient and advance mathematical tool for solving nonlinear partial differential equations in mathematical physics.

MATHEMATICS SUBJECT CLASSIFICATION: C02; C30; C32

1. Introduction

The rapid development of nonlinear sciences witnesses a wide range of reliable and efficient techniques (see for example [1–43]) which are of great help in tackling physical problems even of highly complex nature. After the observation of soliton phenomena by John Scott Russell in 1834 [1,25] and since the KdV equation was solved by Gardner et al. [2] by inverse scattering method, finding exact solutions of nonlinear evolution equations (NLEEs) has turned out to be one of the most exciting and particularly active areas of research. The appearance of solitary wave solutions in nature is quite common. Bell-shaped sech-solutions and kink-shaped tanh-solutions model wave phenomena in elastic media, plasmas, solid state physics, condensed matter physics, electrical circuits, optical fibers, chemical kinematics, fluids, bio-genetics, etc. The traveling wave solutions of the KdV equation and the Boussinesq equation which describe water waves are well-known examples. Apart from their physical relevance, the closed-form solutions of NLEEs if available facilitate the numerical solvers in comparison, and aids in the stability analysis. In soliton theory, there are several techniques to deal with the problems of solitary wave solutions for NLEEs, such as, Hirota's bilinear transformation [3], Backlund transformation [4], improved homotopy perturbation [5], Darboux transformation [6], Tanh-function [7], homogeneous balance [8], Jacobi elliptic function [9,10], F-expansion [11] and Exp-function [12–15].

Recently, Wang et al. [16] established a widely used direct and concise method called the (G'/G)-expansion method for obtaining the exact travelling wave solutions of NLEEs, where $G(\xi)$ satisfies the second order linear ordinary differential equation (ODE) $G'' + \lambda G' + \mu G = 0$, where λ and μ are arbitrary constants. Applications of the (G'/G)-expansion method can be found in the articles [17–26] for better understanding.

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Peer review under responsibility of Egyptian Mathematical Society.

Production and hosting by Elsevier

http://dx.doi.org/10.1016/j.joems.2014.06.013
1110-256X

In order to establish the effectiveness and reliability of the (G'/G)-expansion method and to expand the possibility of its application, further research has been carried out by several researchers. For instance, Zhang et al. [27] presented an improved (G'/G)-expansion method to seek more general traveling wave solutions. Zayed [28] presented a new approach of the (G'/G)-expansion method where $G(\xi)$ satisfies the Jacobi elliptic equation $[G'(\xi)]^2 = e_2G^4(\xi) + e_1G^2(\xi) + e_0, e_2, e_1, e_0$ are arbitrary constants, and obtained new exact solutions. Zayed [29] again presented an alternative approach of this method in which $G(\xi)$ satisfies the Riccatiequation $G'(\xi) = A + BG^2(\xi)$, where A and B are arbitrary constants. Still, substantial work has to be done in order for the (G'/G)-expansion method to be well established, since every nonlinear equation has its own physically significant rich structure. For finding the new exact solutions of NLEEs, it is important to present various method and ansatz, but it seems to be more important how to obtain more new exact solutions to NLEEs under the known method and ansatz. In the present article, we further modify the alternative (G'/G)-expansion method (presented by Zayed [29]) by introducing the generalized Riccati equation mapping, its twenty seven solutions and constructed abundant new traveling wave solutions of the general Sawada– Kotera equation which is a special case of the fifth-order KdV equation (fKdV)

$$u_t + u_{xxxxx} + \gamma uu_{xxx} + \beta u_xu_{xx} + \alpha u^2u_x = 0, \tag{1.1}$$

where α, β and γ are arbitrary nonzero and real parameters, and $u = u(x,t)$ is a differentiable function. The fKdV Eq. (1.1) is an important mathematical model with wide applications in quantum mechanics and nonlinear optics, furthermore describes motions of long waves in shallow water under gravity and in a one-dimensional nonlinear lattice. The general SK equation is characterized by the values

$$\beta = \gamma, \quad \alpha = \frac{1}{5}\gamma^2. \tag{1.2}$$

With (1.2), (1.1) reduces to general SK equation:

$$u_t + u_{xxxxx} + \gamma uu_{xxx} + \gamma u_xu_{xx} + \frac{1}{5}\gamma^2u^2u_x = 0. \tag{1.3}$$

The article is arranged as follows: In Section 2, the modified alternative (G'/G)-expansion method is discussed. In Section 3, we apply this method to the nonlinear fifth-order KdV equation (fKdV) pointed out above; in Section 4, graphical representation and in Section 5 conclusions are given.

2. Methodology

Consider the general nonlinear PDE of the type

$$P(u, u_t, u_x, u_{tt}, u_{xx}, u_{xt}, \ldots) = 0, \tag{2.1}$$

where $u = u(x,t)$ is an unknown function, P is a polynomial in $u(x,t)$ and its partial derivatives in which the highest order partial derivatives and the nonlinear terms are involved. The main steps of the modified alternative (G'/G)-expansion method combined with the generalized Riccati equation mapping are as follows:

Step 1: The travelling wave variable ansatz

$$u(x,t) = u(\xi), \quad \xi = x - Vt, \tag{2.2}$$

where V is the speed of the traveling wave, permits us to transform Eq. (2.1) into an ODE:

$$Q(u, u', u'', u''', \ldots) = 0, \tag{2.3}$$

where the superscripts stands for the ordinary derivatives with respect to ξ.

Step 2: If possible, integrate Eq. (2.3) term by term one or more times. This yields constant(s) of integration.

Step 3: Suppose the traveling wave solution of Eq. (2.3) can be expressed by a polynomial in (G'/G) as follows:

$$u(\xi) = \sum_{n=0}^{m} a_n\left(\frac{G'}{G}\right)^n, \quad a_m \neq 0 \tag{2.4}$$

where $G = G(\xi)$ satisfies the generalized Riccati equation,

$$G' = r + pG + qG^2, \tag{2.5}$$

where a_n $(n = 0, 1, 2, \ldots, m)$, r, p and q are arbitrary constants to be determined later. The generalized Riccati Eq. (2.5) has twenty seven solutions [41] as follows:

Family 1. When $p^2 - 4qr < 0$ and $pq \neq 0$ $(or\ rq \neq 0)$, the solutions of Eq. (2.5) are,

$$G_1 = \frac{1}{2q}\left[-p + \sqrt{4qr - p^2}\,tan\left(\frac{1}{2}\sqrt{4qr - p^2}\,\xi\right)\right],$$

$$G_2 = -\frac{1}{2q}\left[p + \sqrt{4qr - p^2}\,cot\left(\frac{1}{2}\sqrt{4qr - p^2}\,\xi\right)\right],$$

$$G_3 = \frac{1}{2q}\left[-p + \sqrt{4qr - p^2}\left(tan\left(\sqrt{4qr - p^2}\,\xi\right) \pm sec\left(\sqrt{4qr - p^2}\,\xi\right)\right)\right],$$

$$G_4 = -\frac{1}{2q}\left[p + \sqrt{4qr - p^2}\left(cot\left(\sqrt{4qr - p^2}\,\xi\right) \pm csc\left(\sqrt{4qr - p^2}\,\xi\right)\right)\right],$$

$$G_5 = \frac{1}{4q}\left[-2p + \sqrt{4qr - p^2}\left(tan\left(\frac{1}{4}\sqrt{4qr - p^2}\,\xi\right) - cot\left(\frac{1}{4}\sqrt{4qr - p^2}\,\xi\right)\right)\right],$$

$$G_6 = \frac{1}{2q}\left[-p + \frac{\sqrt{(A^2 - B^2)(4qr - p^2)} - A\sqrt{4qr - p^2}\,cos\left(\sqrt{4qr - p^2}\,\xi\right)}{Asin\left(\sqrt{4qr - p^2}\,\xi\right) + B}\right],$$

$$G_7 = \frac{1}{2q}\left[-p + \frac{\sqrt{(A^2 - B^2)(4qr - p^2)} + A\sqrt{4qr - p^2}\,cos\left(\sqrt{4qr - p^2}\,\xi\right)}{Asin\left(\sqrt{4qr - p^2}\,\xi\right) + B}\right],$$

where A and B are two non-zero real constants and satisfy the condition $A^2 - B^2 > 0$.

$$G_8 = \frac{-2rcos\left(\frac{1}{2}\sqrt{4qr - p^2}\,\xi\right)}{\sqrt{4qr - p^2}\,sin\left(\frac{1}{2}\sqrt{4qr - p^2}\,\xi\right) + pcos\left(\frac{1}{2}\sqrt{4qr - p^2}\,\xi\right)},$$

$$G_9 = \frac{2rsin\left(\frac{1}{2}\sqrt{4qr - p^2}\,\xi\right)}{-psin\left(\frac{1}{2}\sqrt{4qr - p^2}\,\xi\right) + \sqrt{(4qr - p^2)}\,cos\left(\frac{1}{2}\sqrt{4qr - p^2}\,\xi\right)},$$

$$G_{10} = \frac{-2rcos\left(\sqrt{4qr - p^2}\,\xi\right)}{\sqrt{(4qr - p^2)}\,sin\left(\sqrt{4qr - p^2}\,\xi\right) + pcos\left(\sqrt{4qr - p^2}\,\xi\right) \pm \sqrt{(4qr - p^2)}},$$

$$G_{11} = \frac{2r\sin\left(\sqrt{4qr-p^2}\xi\right)}{-p\sin\left(\sqrt{4qr-p^2}\xi\right)+\sqrt{(4qr-p^2)}\cos\left(\sqrt{4qr-p^2}\xi\right)\pm\sqrt{(4qr-p^2)}},$$

$$G_{12} = \frac{4r\sin\left(\frac{1}{4}\sqrt{4qr-p^2}\xi\right)\cos\left(\frac{1}{4}\sqrt{4qr-p^2}\xi\right)}{-2p\sin\left(\frac{1}{4}\sqrt{4qr-p^2}\xi\right)\cos\left(\frac{1}{4}\sqrt{4qr-p^2}\xi\right)+2\sqrt{(4qr-p^2)}\cos^2\left(\frac{1}{4}\sqrt{4qr-p^2}\xi\right)-\sqrt{(4qr-p^2)}}.$$

Family 2. When $p^2-4qr>0$ and $pq\neq 0$ $(or\ rq\neq 0)$, the solutions of Eq. (2.5) are,

$$G_{13} = -\frac{1}{2q}\left[p+\sqrt{p^2-4qr}\tanh\left(\frac{1}{2}\sqrt{p^2-4qr}\xi\right)\right],$$

$$G_{14} = -\frac{1}{2q}\left[p+\sqrt{p^2-4qr}\coth\left(\frac{1}{2}\sqrt{p^2-4qr}\xi\right)\right],$$

$$G_{15} = -\frac{1}{2q}\left[p+\sqrt{p^2-4qr}\left(\tanh\left(\sqrt{p^2-4qr}\xi\right)\pm i\,\mathrm{sech}\left(\sqrt{p^2-4qr}\xi\right)\right)\right],$$

$$G_{16} = -\frac{1}{2q}\left[p+\sqrt{p^2-4qr}\left(\coth\left(\sqrt{p^2-4qr}\xi\right)\pm \mathrm{csch}\left(\sqrt{p^2-4qr}\xi\right)\right)\right],$$

$$G_{17} = -\frac{1}{4q}\left[2p+\sqrt{p^2-4qr}\left(\tanh\left(\frac{1}{4}\sqrt{p^2-4qr}\xi\right)+\coth\left(\frac{1}{4}\sqrt{p^2-4qr}\xi\right)\right)\right],$$

$$G_{18} = \frac{1}{2q}\left[-p+\frac{\sqrt{(A^2+B^2)(p^2-4qr)}-A\sqrt{p^2-4qr}\cosh\left(\sqrt{p^2-4qr}\xi\right)}{A\sinh\left(\sqrt{p^2-4qr}\xi\right)+B}\right],$$

$$G_{19} = \frac{1}{2q}\left[-p-\frac{\sqrt{(B^2-A^2)(p^2-4qr)}+A\sqrt{p^2-4qr}\cosh\left(\sqrt{p^2-4qr}\xi\right)}{A\sinh\left(\sqrt{p^2-4qr}\xi\right)+B}\right],$$

where A and B are two non-zero real constants and satisfy the condition $B^2-A^2>0$.

$$G_{20} = \frac{2r\cosh\left(\frac{1}{2}\sqrt{p^2-4qr}\xi\right)}{\sqrt{p^2-4qr}\sinh\left(\frac{1}{2}\sqrt{p^2-4qr}\xi\right)-p\cosh\left(\frac{1}{2}\sqrt{p^2-4qr}\xi\right)},$$

$$G_{21} = \frac{2r\sinh\left(\frac{1}{2}\sqrt{p^2-4qr}\xi\right)}{\sqrt{p^2-4qr}\cosh\left(\frac{1}{2}\sqrt{p^2-4qr}\xi\right)-p\sinh\left(\frac{1}{2}\sqrt{p^2-4qr}\xi\right)},$$

$$G_{22} = \frac{2r\cosh\left(\sqrt{p^2-4qr}\xi\right)}{\sqrt{p^2-4qr}\sinh\left(\sqrt{p^2-4qr}\xi\right)-p\cosh\left(\sqrt{p^2-4qr}\xi\right)\pm i\sqrt{p^2-4qr}},$$

$$G_{23} = \frac{2r\sinh\left(\sqrt{p^2-4qr}\xi\right)}{-p\sinh\left(\sqrt{p^2-4qr}\xi\right)+\sqrt{p^2-4qr}\cosh\left(\sqrt{p^2-4qr}\xi\right)\pm\sqrt{p^2-4qr}},$$

$$G_{24} = \frac{4r\sinh\left(\frac{1}{4}\sqrt{p^2-4qr}\xi\right)\cosh\left(\frac{1}{4}\sqrt{p^2-4qr}\xi\right)}{-2p\sinh\left(\frac{1}{4}\sqrt{p^2-4qr}\xi\right)\cosh\left(\frac{1}{4}\sqrt{p^2-4qr}\xi\right)+2\sqrt{p^2-4qr}\cosh^2\left(\frac{1}{4}\sqrt{p^2-4qr}\xi\right)-\sqrt{p^2-4qr}}.$$

Family 3. When $r=0$ and $pq\neq 0$, the solutions of Eq. (2.5) are,

$$G_{25} = \frac{-pd}{q[d+\cosh(p\xi)-\sinh(p\xi)]},$$

$$G_{26} = -\frac{p[\cosh(p\xi)+\sinh(p\xi)]}{q[d+\cosh(p\xi)+\sinh(p\xi)]},$$

where d is an arbitrary constant.

Family 4. When $q\neq 0$ and $r=p=0$, the solution of Eq. (2.5) is,

$$G_{27} = -\frac{1}{q\xi+c_1},$$

where c_1 is an arbitrary constant.

Step 4: To determine the positive integer m, substitute Eq. (2.4) along with Eq. (2.5) into Eq. (2.3) and then consider homogeneous balance between the highest order derivatives and the nonlinear terms appearing in Eq. (2.3).

Step 5: Substituting Eq. (2.4) along with Eq. (2.5) into Eq. (2.3) together with the value of m obtained in step 4, we obtain polynomials in G^i and G^{-i} $(i=0,1,2,3,\ldots)$ and vanishing each coefficient of the resulted polynomial to zero, yields a set of algebraic equations for a_n, r, p, q and V.

Step 6: Suppose the value of the constants a_n, r, p, q and V can be determined by solving the set of algebraic equations obtained in step 4. Since the general solutions of Eq. (2.5) are known, substituting a_n, r, p, q and V into Eq. (2.4), we obtain new exact traveling wave solutions of the nonlinear evolution Eq. (2.1).

3. Some new traveling wave solutions of GSK equation

In this section, the modified alternative (G'/G)-expansion method is employed to construct some new traveling wave solutions of the general Sawada–Kotera (GSK) equation which is a very important nonlinear fifth-order KdV equation (fKdV) in mathematical physics and engineering. Let us consider the general Sawada–Kotera equation:

$$u_t + u_{xxxxx} + \gamma u u_{xxx} + \gamma u_x u_{xx} + \frac{1}{5}\gamma^2 u^2 u_x = 0. \tag{3.1}$$

Now, we use the wave transformation Eq. (2.2) into Eq. (3.1), which yield:

$$-Vu' + u^{(v)} + \gamma u u''' + \gamma u' u'' + \frac{1}{5}\gamma^2 u^2 u' = 0. \tag{3.2}$$

Integrating Eq. (3.2) with respect to ξ, we get

$$C - Vu + u^{(iv)} + \gamma u u'' + \frac{1}{15}\gamma^2 u^3 = 0. \tag{3.3}$$

According to step 4, the solution of Eq. (3.3) can be expressed by a polynomial in (G'/G) as follows:

$$u(\xi) = a_0 + a_1\left(\frac{G'}{G}\right) + a_2\left(\frac{G'}{G}\right)^2 + \cdots + a_m\left(\frac{G'}{G}\right)^m, \quad a_m \neq 0 \tag{3.4}$$

where a_i, $(i = 0, 1, 2, \ldots, m)$ all are constants to be determined and $G = G(\xi)$ satisfies the generalized Riccati Eq. (2.5). Considering the homogeneous balance between the highest order derivatives and the nonlinear terms in Eq. (3.3), we obtain $m = 2$.

Therefore, solution of Eq. (3.4) takes the form

$$u(\xi) = a_0 + a_1\left(\frac{G'}{G}\right) + a_2\left(\frac{G'}{G}\right)^2, \quad a_2 \neq 0. \tag{3.5}$$

By means of Eqs. (2.5), (3.5) can be rewritten as,

$$u(\xi) = a_0 + a_1(p + rG^{-1} + qG) + a_2(p + rG^{-1} + qG)^2. \tag{3.6}$$

Substituting Eq. (3.6) into Eq. (3.3), the left hand side of this equation is converted into polynomials in G^i *and* G^{-i} $(i = 0, 1, 2, 3, \ldots)$. Setting each coefficient of these polynomials to zero, we obtain a set of simultaneous algebraic equations for a_0, a_1, a_2, p, q, r and V as follows:

$$C_0\left(\frac{G'}{G}\right)^0 + C_1\left(\frac{G'}{G}\right)^1 + C_2\left(\frac{G'}{G}\right)^2 + \cdots + C_{12}\left(\frac{G'}{G}\right)^{12} = 0.$$

Solving the over-determined set of algebraic equations by using the symbolic computation software, such as, Maple, we obtain

$$a_0 = -\frac{5(p^2 - 4qr)}{\gamma}, \quad a_1 = \frac{30p}{\gamma}, \quad a_2 = -\frac{30}{\gamma},$$
$$V = p^4 - 8p^2 qr + 16q^2 r^2,$$
$$C = \frac{10}{3}\,\frac{p^6 - 12p^4 qr + 48p^2 q^2 r^2 - 64q^3 r^3}{\gamma}, \tag{3.7}$$

where p, q and r are arbitrary constants.Now on the basis of the solutions of Eq. (2.5), we obtain some new types of solutions of Eq. (3.1).

Family 1. When $p^2 - 4qr < 0$ and $pq \neq 0$ (or $rq \neq 0$), the periodic form solutions of Eq. (3.1) are:

$$u_1 = \frac{20\Delta^2}{\gamma} + \frac{30p}{\gamma}\left(\frac{2\Delta^2 sec^2(\Delta\xi)}{-p + 2\Delta tan(\Delta\xi)}\right) - \frac{30}{\gamma}\left(\frac{2\Delta^2 sec^2(\Delta\xi)}{-p + 2\Delta tan(\Delta\xi)}\right)^2,$$

where $\Delta = \frac{1}{2}\sqrt{4qr - p^2}$, $\quad \xi = x - 16\Delta^4 t$ and p, q and r are arbitrary constants.

$$u_2 = \frac{20\Delta^2}{\gamma} - \frac{30p}{\gamma}\left(\frac{2\Delta^2 csc^2(\Delta\xi)}{p + 2\Delta cot(\Delta\xi)}\right) - \frac{30}{\gamma}\left(\frac{2\Delta^2 csc^2(\Delta\xi)}{p + 2\Delta cot(\Delta\xi)}\right)^2,$$

$$u_3 = \frac{20\Delta^2}{\gamma} + \frac{30p}{\gamma}\left(\frac{4\Delta^2 sec(2\Delta\xi)(1 \pm sin(2\Delta\xi))}{-pcos(2\Delta\xi) + 2\Delta sin(2\Delta\xi) \pm 2\Delta}\right) - \frac{30}{\gamma}\left(\frac{4\Delta^2 sec(2\Delta\xi)(1 \pm sin(2\Delta\xi))}{-pcos(2\Delta\xi) + 2\Delta sin(2\Delta\xi) \pm 2\Delta}\right)^2,$$

$$u_4 = \frac{20\Delta^2}{\gamma} - \frac{30p}{\gamma}\left(\frac{4\Delta^2 csc(2\Delta\xi)(1 \pm cos(2\Delta\xi))}{psin(2\Delta\xi) + 2\Delta cos(2\Delta\xi) \pm 2\Delta}\right) - \frac{30}{\gamma}\left(\frac{4\Delta^2 csc(2\Delta\xi)(1 \pm cos(2\Delta\xi))}{psin(2\Delta\xi) + 2\Delta cos(2\Delta\xi) \pm 2\Delta}\right)^2,$$

$$u_5 = \frac{20\Delta^2}{\gamma} - \frac{30p}{\gamma}\left(\frac{2\Delta^2 csc(\Delta\xi)}{psin(\Delta\xi) + 2\Delta cos(\Delta\xi)}\right) - \frac{30}{\gamma}\left(\frac{2\Delta^2 csc(\Delta\xi)}{psin(\Delta\xi) + 2\Delta cos(\Delta\xi)}\right)^2,$$

$$u_6 = \frac{20\Delta^2}{\gamma} - \frac{30p}{\gamma}\left(\frac{4A\Delta^2\left\{\sqrt{A^2 - B^2}cos(2\Delta\xi) - Bsin(2\Delta\xi) - A\right\}\{Asin(2\Delta\xi) + B\}}{\{A^2cos^2(2\Delta\xi) - A^2 - B^2 - 2ABsin(2\Delta\xi)\}\left\{pAsin(2\Delta\xi) + 2A\Delta cos(2\Delta\xi) + pB - 2\Delta\sqrt{A^2 - B^2}\right\}}\right) - \frac{30}{\gamma}\left(\frac{4A\Delta^2\left\{\sqrt{A^2 - B^2}cos(2\Delta\xi) - Bsin(2\Delta\xi) - A\right\}\{Asin(2\Delta\xi) + B\}}{\{A^2cos^2(2\Delta\xi) - A^2 - B^2 - 2ABsin(2\Delta\xi)\}\left\{pAsin(2\Delta\xi) + 2A\Delta cos(2\Delta\xi) + pB - 2\Delta\sqrt{A^2 - B^2}\right\}}\right)^2,$$

$$u_7 = \frac{20\Delta^2}{\gamma} - \frac{30p}{\gamma}\left(\frac{4A\Delta^2\left\{\sqrt{A^2 - B^2}cos(2\Delta\xi) + Bsin(2\Delta\xi) + A\right\}\{Asin(2\Delta\xi) + B\}}{\{A^2cos^2(2\Delta\xi) - A^2 - B^2 - 2ABsin(2\Delta\xi)\}\left\{pAsin(2\Delta\xi) - 2A\Delta cos(2\Delta\xi) + pB - 2\Delta\sqrt{A^2 - B^2}\right\}}\right) - \frac{30}{\gamma}\left(\frac{4A\Delta^2\left\{\sqrt{A^2 - B^2}cos(2\Delta\xi) + Bsin(2\Delta\xi) + A\right\}\{Asin(2\Delta\xi) + B\}}{\{A^2cos^2(2\Delta\xi) - A^2 - B^2 - 2ABsin(2\Delta\xi)\}\left\{pAsin(2\Delta\xi) - 2A\Delta cos(2\Delta\xi) + pB - 2\Delta\sqrt{A^2 - B^2}\right\}}\right)^2,$$

where A and B are two non-zero real constants and satisfy the condition $A^2 - B^2 > 0$.

$$u_8 = \frac{20\Delta^2}{\gamma} - \frac{30p}{\gamma}\left(\frac{2\Delta^2 sec(\Delta\xi)\{pcos(\Delta\xi) + 2\Delta sin(\Delta\xi)\}}{2(p^2 - 2qr)cos^2(\Delta\xi) + 4\Delta psin(\Delta\xi)cos(\Delta\xi) + 4\Delta^2}\right) - \frac{30}{\gamma}\left(\frac{2\Delta^2 sec(\Delta\xi)\{pcos(\Delta\xi) + 2\Delta sin(\Delta\xi)\}}{2(p^2 - 2qr)cos^2(\Delta\xi) + 4\Delta psin(\Delta\xi)cos(\Delta\xi) + 4\Delta^2}\right)^2,$$

$$u_9 = \frac{20\Delta^2}{\gamma} + \frac{30p}{\gamma}\left(\frac{2\Delta^2 csc(\Delta\xi)\{psin(\Delta\xi) - 2\Delta cos(\Delta\xi)\}}{2(p^2 - 2qr)cos^2(\Delta\xi) + 4\Delta psin(\Delta\xi)cos(\Delta\xi) - p^2}\right) - \frac{30}{\gamma}\left(\frac{2\Delta^2 csc(\Delta\xi)\{psin(\Delta\xi) - 2\Delta cos(\Delta\xi)\}}{2(p^2 - 2qr)cos^2(\Delta\xi) + 4\Delta psin(\Delta\xi)cos(\Delta\xi) - p^2}\right)^2,$$

$$u_{10} = \frac{20\Delta^2}{\gamma} - \frac{30p}{\gamma}\left(\frac{2\Delta^2 sec(2\Delta\xi)\{1 \pm sin(2\Delta\xi)\}\{pcos(2\Delta\xi) + 2\Delta sin(2\Delta\xi) \pm 2\Delta\}}{2(p^2 - 2qr)cos^2(2\Delta\xi) + 2\Delta\{1 \pm sin(2\Delta\xi)\}\{2\Delta \pm pcos(2\Delta\xi)\}}\right) - \frac{30}{\gamma}\left(\frac{2\Delta^2 sec(2\Delta\xi)\{1 \pm sin(2\Delta\xi)\}\{pcos(2\Delta\xi) + 2\Delta sin(2\Delta\xi) \pm 2\Delta\}}{2(p^2 - 2qr)cos^2(2\Delta\xi) + 2\Delta\{1 \pm sin(2\Delta\xi)\}\{2\Delta \pm pcos(2\Delta\xi)\}}\right)^2,$$

$$u_{11} = \frac{20\Delta^2}{\gamma} \pm \frac{30p}{\gamma}\left(\frac{2\Delta^2 csc(2\Delta\xi)\{-psin(2\Delta\xi) + 2\Delta cos(2\Delta\xi) \pm 2\Delta\}}{(2qr - p^2)cos(2\Delta\xi) - 2p\Delta sin(2\Delta\xi) \pm 2\mathrm{qr}}\right) - \frac{30}{\gamma}\left(\frac{2\Delta^2 csc(2\Delta\xi)\{-psin(2\Delta\xi) + 2\Delta cos(2\Delta\xi) \pm 2\Delta\}}{(2qr - p^2)cos(2\Delta\xi) - 2p\Delta sin(2\Delta\xi) \pm 2\mathrm{qr}}\right)^2,$$

$$u_{12} = \frac{20\Delta^2}{\gamma} + \frac{30p}{\gamma}\left(\frac{2\Delta^2 csc(\Delta\xi)\{psin(\Delta\xi) - 2\Delta cos(\Delta\xi)\}}{2(p^2 - 2qr)cos^2(2\Delta\xi) + 4p\Delta sin(\Delta\xi)cos(\Delta\xi) - p^2}\right) - \frac{30}{\gamma}\left(\frac{2\Delta^2 csc(\Delta\xi)\{psin(\Delta\xi) - 2\Delta cos(\Delta\xi)\}}{2(p^2 - 2qr)cos^2(2\Delta\xi) + 4p\Delta sin(\Delta\xi)cos(\Delta\xi) - p^2}\right)^2.$$

Family 2. When $p^2 - 4qr > 0$ and $pq \neq 0$ (or $rq \neq 0$), the soliton and soliton-like solutions of Eq. (3.1) are:

$$u_{13} = -\frac{20\Omega^2}{\gamma} + \frac{30p}{\gamma}\left(\frac{2\Omega^2 sech^2(\Omega\xi)}{p + 2\Omega tanh(\Omega\xi)}\right) - \frac{30}{\gamma}\left(\frac{2\Omega^2 sech^2(\Omega\xi)}{p + 2\Omega tanh(\Omega\xi)}\right)^2,$$

where $\Omega = \frac{1}{2}\sqrt{p^2 - 4qr}$, $\xi = x - 16\Omega^4 t$ and p, q and r are arbitrary constants.

$$u_{14} = -\frac{20\Omega^2}{\gamma} - \frac{30p}{\gamma}\left(\frac{2\Omega^2 \mathrm{csch}^2(\Omega\xi)}{p + 2\Omega coth(\Omega\xi)}\right) - \frac{30}{\gamma}\left(\frac{2\Omega^2 csch^2(\Omega\xi)}{p + 2\Omega coth(\Omega\xi)}\right)^2,$$

$$u_{15} = -\frac{20\Omega^2}{\gamma} + \frac{30p}{\gamma}\left(\frac{4\Omega^2 sech(2\Omega\xi)(1 \mp isinh(2\Omega\xi))}{pcosh(2\Omega\xi) + 2\Omega sinh(2\Omega\xi) \pm i2\Omega}\right) - \frac{30}{\gamma}\left(\frac{4\Omega^2 sech(2\Omega\xi)(1 \mp isinh(2\Omega\xi))}{pcosh(2\Omega\xi) + 2\Omega sinh(2\Omega\xi) \pm i2\Omega}\right)^2,$$

$$u_{16} = -\frac{20\Omega^2}{\gamma} - \frac{30p}{\gamma}\left(\frac{4\Omega^2 csch(2\Omega\xi)(1 \pm cosh(2\Omega\xi))}{psinh(2\Omega\xi) + 2\Omega cosh(2\Omega\xi) \pm 2\Omega}\right) - \frac{30}{\gamma}\left(\frac{4\Omega^2 csch(2\Omega\xi)(1 \pm cosh(2\Omega\xi))}{psinh(2\Omega\xi) + 2\Omega cosh(2\Omega\xi) \pm 2\Omega}\right)^2,$$

$$u_{17} = -\frac{20\Omega^2}{\gamma} - \frac{30p}{\gamma}\left(\frac{\Omega^2 sech^2(\Omega\xi/2)}{2\{cosh^2(\Omega\xi/2) - 1\}\{p + \Omega(tanh(\Omega\xi/2) + coth(\Omega\xi/2))\}}\right) - \frac{30}{\gamma}\left(\frac{\Omega^2 sech^2(\Omega\xi/2)}{2\{cosh^2(\Omega\xi/2) - 1\}\{p + \Omega(tanh(\Omega\xi/2) + coth(\Omega\xi/2))\}}\right)^2,$$

$$u_{18} = -\frac{30p}{\gamma}\left(\frac{4A\Omega^2\left\{A - Bsinh(2\Omega\xi) - \sqrt{A^2 + B^2}cosh(2\Omega\xi)\right\}}{\{Asinh(2\Omega\xi) + B\}\left\{pAsinh(2\Omega\xi) + 2A\Omega cosh(2\Omega\xi) + pB - 2\Omega\sqrt{A^2 + B^2}\right\}}\right) - \frac{30}{\gamma}\left(\frac{4A\Omega^2\left\{A - Bsinh(2\Omega\xi) - \sqrt{A^2 + B^2}cosh(2\Omega\xi)\right\}}{\{Asinh(2\Omega\xi) + B\}\left\{pAsinh(2\Omega\xi) + 2A\Omega cosh(2\Omega\xi) + pB - 2\Omega\sqrt{A^2 + B^2}\right\}}\right)^2 - \frac{20\Omega^2}{\gamma},$$

$$u_{19} = -\frac{30p}{\gamma}\left(\frac{4A\Omega^2\left\{A - Bsinh(2\Omega\xi) + \sqrt{A^2 + B^2}cosh(2\Omega\xi)\right\}}{\{Asinh(2\Omega\xi) + B\}\left\{pAsinh(2\Omega\xi) + 2A\Omega cosh(2\Omega\xi) + pB + 2\Omega\sqrt{A^2 + B^2}\right\}}\right) - \frac{30}{\gamma}\left(\frac{4A\Omega^2\left\{A - Bsinh(2\Omega\xi) + \sqrt{A^2 + B^2}cosh(2\Omega\xi)\right\}}{\{Asinh(2\Omega\xi) + B\}\left\{pAsinh(2\Omega\xi) + 2A\Omega cosh(2\Omega\xi) + pB + 2\Omega\sqrt{A^2 + B^2}\right\}}\right)^2 - \frac{20\Omega^2}{\gamma},$$

where A and B are two non-zero real constants and satisfy the condition $B^2 - A^2 > 0$.

$$u_{20} = -\frac{20\Omega^2}{\gamma} - \frac{30p}{\gamma}\left(\frac{2\Omega^2 sech(\Omega\xi)}{2\Omega sinh(\Omega\xi) - pcosh(\Omega\xi)}\right) - \frac{30}{\gamma}\left(\frac{2\Omega^2 sech(\Omega\xi)}{2\Omega sinh(\Omega\xi) - pcosh(\Omega\xi)}\right)^2,$$

$$u_{21} = -\frac{20\Omega^2}{\gamma} + \frac{30p}{\gamma}\left(\frac{2\Omega^2 csch(\Omega\xi)}{2\Omega cosh(\Omega\xi) - psinh(\Omega\xi)}\right) - \frac{30}{\gamma}\left(\frac{2\Omega^2 csch(\Omega\xi)}{2\Omega cosh(\Omega\xi) - psinh(\Omega\xi)}\right)^2,$$

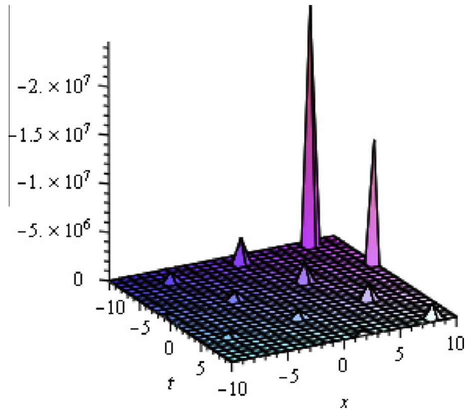

Fig. 1 Soliton corresponding to solution u_1 for $p = q = 2$, $r = 3$ and $\gamma = 5$.

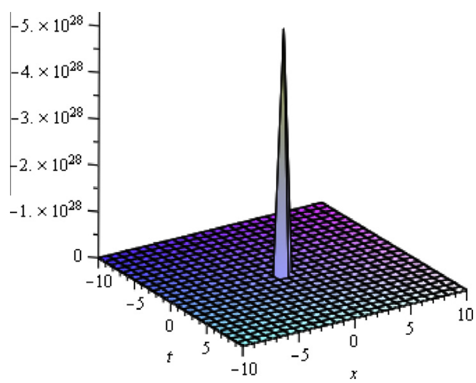

Fig. 2 Soliton corresponding to solution u_5 for $p = q = 1$, $r = 2$ and $\gamma = 5$.

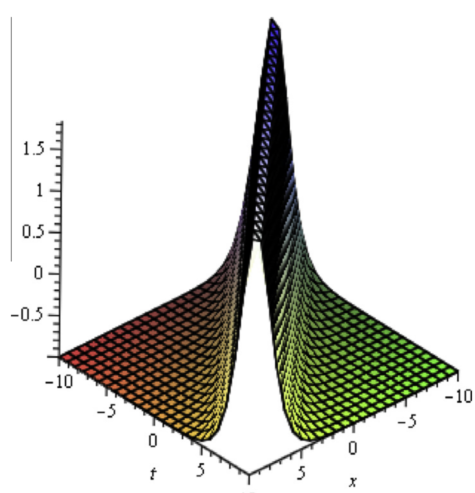

Fig. 3 Soliton corresponding to solution u_{13} for $p = 3,\ q = 2$, $r = 1$ and $\gamma = 5$.

$$u_{22} = -\frac{20\Omega^2}{\gamma} + \frac{30p}{\gamma}\left(\frac{4\Omega^2 sech(2\Omega\xi)\{1 \mp isinh(2\Omega\xi)\}}{pcosh(2\Omega\xi) - 2\Omega sinh(2\Omega\xi) \mp i2\Omega}\right) - \frac{30}{\gamma}\left(\frac{4\Omega^2 sech(2\Omega\xi)\{1 \mp isinh(2\Omega\xi)\}}{pcosh(2\Omega\xi) - 2\Omega sinh(2\Omega\xi) \mp i2\Omega}\right)^2,$$

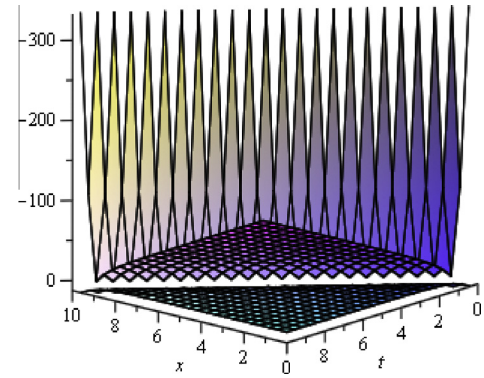

Fig. 4 Soliton corresponding to solution u_{21} for $p = 3,\ q = 2$, $r = 1$ and $\gamma = 5$.

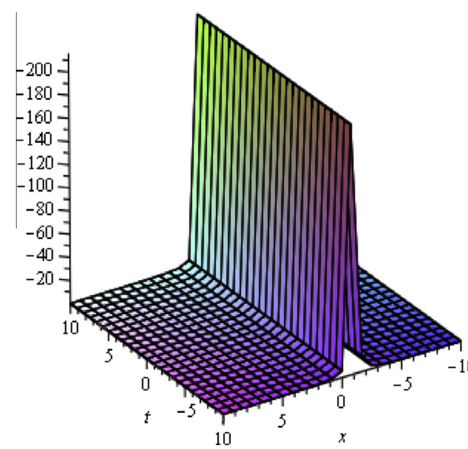

Fig. 5 Soliton corresponding to solution u_{27} for $p = 0,\ q = 1$, $r = 0,\ \gamma = 5$ and $c_1 = 1$.

$$u_{23} = -\frac{20\Omega^2}{\gamma} + \frac{30p}{\gamma}\left(\frac{4\Omega^2 csch(2\Omega\xi)\{1 \pm cosh(2\Omega\xi)\}}{2\Omega cosh(2\Omega\xi) - psinh(2\Omega\xi) \pm 2\Omega}\right) - \frac{30}{\gamma}\left(\frac{4\Omega^2 csch(2\Omega\xi)\{1 \pm cosh(2\Omega\xi)\}}{2\Omega cosh(2\Omega\xi) - psinh(2\Omega\xi) \pm 2\Omega}\right)^2,$$

$$u_{24} = -\frac{20\Omega^2}{\gamma} + \frac{30p}{\gamma}\left(\frac{2\Omega^2 csch(\Omega\xi)}{2\Omega cosh(\Omega\xi) - psinh(\Omega\xi)}\right) - \frac{30}{\gamma}\left(\frac{2\Omega^2 csch(\Omega\xi)}{2\Omega cosh(\Omega\xi) - psinh(\Omega\xi)}\right)^2.$$

Family 3. When $r = 0$ and $pq \neq 0$, the solutions of Eq. (3.1) are,

$$u_{25} = -\frac{20\Omega^2}{\gamma} + \frac{30p}{\gamma}\left(\frac{p(cosh(p\xi) - sinh(p\xi))}{d + cosh(p\xi) - sinh(p\xi)}\right) - \frac{30}{\gamma}\left(\frac{p(cosh(p\xi) - sinh(p\xi))}{d + cosh(p\xi) - sinh(p\xi)}\right)^2,$$

$$u_{26} = -\frac{20\Omega^2}{\gamma} + \frac{30p}{\gamma}\left(\frac{pd}{d + cosh(p\xi) + sinh(p\xi)}\right) - \frac{30}{\gamma}\left(\frac{pd}{d + cosh(p\xi) + sinh(p\xi)}\right)^2.$$

Family 4. When $q \neq 0$ and $r = p = 0$, the solution of Eq. (3.1) is,

$$u_{27} = -\frac{20\Omega^2}{\gamma} - \frac{30p}{\gamma}\left(\frac{q}{q\xi + c_1}\right) - \frac{30}{\gamma}\left(\frac{q}{q\xi + c_1}\right)^2,$$

where c_1 is an arbitrary constant.

4. Graphical presentation

Graph is a powerful tool for communication and describes lucidly the solutions of the problems. Therefore, some graphs of the solutions are given below. The graphs readily have shown the solitary wave form of the solutions (see Figs. 1–5).

5. Conclusion

Alternative (G'/G)-expansion method has been modified by introducing the generalized Riccati equation mapping and abundant exact traveling wave solutions of the general Sawada–Kotera equation with the help of symbolic computation are obtained. It is important to point out that the obtained solutions have not been reported in the previous literature. The new type of traveling wave solutions found in this article might have significant impact on future research. We assured the correctness of our solutions by putting them back into the original Eq. (3.1). This article is only an imploring work and we look forward the modified alternative (G'/G)-expansion method may be applicable to other kinds of NLEEs in mathematical physics.

Acknowledgment

The authors are highly grateful for the valuable comments made by the unknown referees' and editor which really improved the quality of presented work.

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