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ORIGINAL ARTICLE

Investigation of heat transfer in flow of Burgers' fluid during a melting process

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Abstract The heat transfer analysis during melting process in steady flow of an incompressible Burgers' fluid over stretching surface is investigated. The two-dimensional flow equations are modeled and then simplified by employing boundary layer analysis. The solution to the arising nonlinear problem is computed. Interpretation of various emerging parameters is given through graphs for velocity and temperature fields. Furthermore tables are constructed in order to show a comparative study with the previous published results. Comparison shows an excellent agreement with the previous limiting investigations in the field.

MATHEMATICS SUBJECT CLASSIFICATION: 76BXX; 76WXX; 76MXX

1. Introduction

In the recent years, researchers have keen interest in the flow of non-Newtonian fluid due to their practical applications in the field of engineering and technology. For-instance in designing plunge bearings and radial diffusers, in thermal oil recovery, cooling of strips, in traffic engineering where traffic is assumed as continuous fluid etc. The non-Newtonian fluids in view of diverse characteristics cannot be described by a single constitutive relation. Hence researchers have proposed various non-Newtonian fluid models to predict different rheological features. The survey of literature witnesses that there is replete literature on the topic concerning the flows of differential type fluids (a subclass of non-Newtonian fluids) in boundary layer region whereas such flows are in scarce for the rate type fluid models. It is because of the fact that the derivation of governing equations for the rate type fluids in two-dimensional flow analysis is much more complex than those of differential type fluids. However recent researchers have paid considerable attention on rate type fluids. For-instance Jamil et al. [1] examined the helical flows of Oldroyd-B fluids. Constantly accelerated flow between two sided walls perpendicular to the plate has been investigated by Fetecau et al. [2]. Pahlavan

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and Sadeghy [3] examined the homotopy analysis method for solving unsteady MHD flow of Maxwellian fluids above impulsively stretching sheets. Hayat et al. [4] investigated the effects of mass transfer on the stagnation point flow of an upper-convected Maxwell (UCM) fluid. Sajid et al. [5] presented boundary layer flow of an Oldroyd-B fluid in a region of stagnation point over a stretching sheet. Similar solution for the three-dimensional flow of an Oldroyd-B fluid has been presented by Hayat et al. [6]. Recently Hayat et al. [7] investigated stagnation point flow of Burgers' fluid over a stretching surface.

Melting heat transport phenomenon has been introduced recently in view of its relevance to some particular engineering problem. For-instance in the magma solidification, melting of permafrost, preparations of semi-conductor materials, etc. The seminal work by Epstein and Cho [8] incorporated melting effects on heat transport phenomenon to submerged bodies. The work of Epstein has been extended by the various researchers. Cheng and Lin [9] studied melting effect on mixed convective heat transfer with aiding and opposing external flows from the vertical plate in a liquid-saturated porous medium. Ishak et al. [10] studied melting heat transfer in steady laminar flow over a moving surface. Melting heat transfer in boundary layer stagnation-point flow toward a stretching/shrinking sheet has been analyzed by Bachok et al. [11]. Hayat et al. [12] investigated melting heat transfer in the stagnation-point flow of a second grade fluid. Royon and Guiffant [13] analyzed the heat transfer properties of slurry of stabilized paraffin during a melting process.

The purpose of current study is to analyze the characteristics of melting heat transfer on the boundary layer flow of Burgers' fluid [7] over a stretched surface. Nonlinear analysis is formulated. The solutions are derived by homotopy analysis method (HAM) which has been already applied to provide series solutions of various nonlinear problems [14–18]. Graphical results for dimensionless velocity and temperature are displayed and discussed. The numerical values of local Nusselt number have been obtained for various values of embedded parameters. Several tables are constructed in order to make a comparative study with various published articles in the limiting sense. These tables assure the validity of the present investigation.

2. Mathematical analysis

Consider the stagnation point flow of Burgers' fluid. The flow is induced by the stretching of sheet coinciding with the plane $y = 0$ whereas fluid occupies the region $y \geqslant 0$. The x- and y-axes are chosen along and perpendicular to sheet respectively. Velocity of stagnation point flow is taken as $U_e(x) = ax$ and the velocity of stretching sheet is $U_w(x) = cx$ where a and c are the positive constants. Further, the effect of melting heat transfer is taken into account. It is assumed that the temperature of the melting surface is T_m while the temperature in the free stream is T_∞, where $T_\infty > T_m$. The boundary layer equations governing the flow in the absence of viscous dissipation effects are modeled as follows

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{1}$$

$$\begin{aligned} u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \lambda_1\left[u^2\frac{\partial^2 u}{\partial x^2} + v^2\frac{\partial^2 u}{\partial y^2} + 2uv\frac{\partial^2 u}{\partial x\partial y}\right] \\ + \lambda_2\left[u^3\frac{\partial^3 u}{\partial x^3} + v^3\frac{\partial^3 u}{\partial y^3} + u^2\left(\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial x^2} + 2\frac{\partial v}{\partial x}\frac{\partial^2 u}{\partial x\partial y}\right)\right. \\ +3v^2\left(\frac{\partial v}{\partial y}\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x\partial y}\right) + 3uv\left(u\frac{\partial^3 u}{\partial x^2\partial y} + v\frac{\partial^3 u}{\partial x\partial y^2}\right) \\ \left. +2uv\left(2\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial x^2} + \frac{\partial v}{\partial x}\frac{\partial^2 u}{\partial y^2} + \frac{\partial v}{\partial y}\frac{\partial^2 u}{\partial x\partial y}\right)\right] = \nu\frac{\partial^2 u}{\partial y^2} + U_e\frac{dU_e}{dx} \\ +\nu\lambda_3\left[u\frac{\partial^3 u}{\partial x\partial y^2} + v\frac{\partial^3 u}{\partial y^3} - \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial y}\frac{\partial^2 v}{\partial y^2}\right], \end{aligned} \tag{2}$$

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} \tag{3}$$

with the corresponding boundary conditions given by

$$\begin{aligned} &u = U_w(x) = cx, \quad v = 0, \quad T = T_m \quad \text{at } y = 0 \\ &u \to U_e(x) = ax, \quad \frac{\partial u}{\partial y} \to 0, \quad T \to T_\infty \quad \text{as } y \to \infty, \\ &k\left(\frac{\partial T}{\partial y}\right)_{y=0} = \rho[\lambda + c_s(T_m - T_0)]v(x,0), \end{aligned} \tag{4}$$

The boundary conditions for heat transport phenomena state that the heat conducted to the melting surface is equal to the heat of melting plus the sensible heat required to raise the temperature of the solid T_0 to its melting temperature T_m (see Epstein and Cho [11]) Moreover in above equations u and v denote the velocity components in the x- and y-directions respectively, λ_1 and λ_2 the relaxation times respectively, λ_3 the retardation time, ν the kinematic viscosity, T the fluid temperature, T_m the mean fluid temperature, α the thermal diffusivity of the fluid, k the thermal conductivity, λ the latent heat of the fluid, c_s the heat capacity of the solid surface, c_p the specific heat and k the thermal conductivity, The velocity components in terms of stream function ψ and similarity transformations can be expressed as

$$u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}, \tag{5}$$

$$\psi = x\sqrt{c\nu}f(\eta), \quad \theta(\eta) = \frac{T - T_m}{T_\infty - T_m}, \quad \eta = \sqrt{\frac{c}{\nu}}y. \tag{6}$$

Now Eq. (1) is satisfied automatically and Eqs. (2) and (3) become

$$\begin{aligned} f''' - f'^2 + ff'' + \beta_1(2ff'f'' - f^2f''') + \beta_2(f^3f'''' - 2ff'^2f'' - 3f^2f''^2) \\ + \beta_3(f''^2 - ff'''') + A^2 = 0, \end{aligned} \tag{7}$$

$$\theta'' + \Pr f\theta' = 0, \tag{8}$$

$$\begin{aligned} &f'(0) = 1, \quad \Pr f(0) + M_e\theta'(0) = 0, \quad \theta(0) = 0, \\ &f'(\infty) = A, \quad f''(\infty) = 0, \quad \theta(\infty) = 1, \end{aligned} \tag{9}$$

in which β_1 and β_2 denote the Deborah numbers in terms of relaxation times respectively, β_3 the Deborah number in terms of retardation times, A the stagnation point parameter, Pr the Prandtl number and Me the dimensionless melting parameter. These are defined as follows:

$$\beta_1 = \lambda_1 c, \quad \beta_2 = \lambda_2 c^2, \quad \beta_3 = \lambda_3 c, \quad A = \frac{a}{c},$$
$$\Pr = \frac{\nu}{\alpha} = \frac{\mu c_p}{k}, \quad Me = \frac{c_p(T_\infty - T_m)}{\lambda + c_s(T_m - T_0)}. \tag{10}$$

It is noted that $\beta_2 = 0$ corresponds to Oldroyd-B fluid, $\beta_2 = 0 = \beta_3$ gives Maxwell fluid and $\beta_1 = 0 = \beta_2 = \beta_3$ shows Newtonian fluid. Furthermore $Me = 0$ means that melting is absent. Moreover local Nusselt number Nu is given by

$$Nu_x = \frac{xq_w}{k(T_\infty - T_m)}, \tag{11}$$

where q_w denotes the wall heat flux given by

$$q_w = -k\frac{\partial T}{\partial y}\bigg|_{y=0}, \tag{12}$$

The dimensionless variables (6) lead to the following relations

$$Nu_x \mathrm{Re}_x^{-1/2} = -\theta'(0). \tag{13}$$

Eqs. (7) and (8) along with boundary conditions 9 are solved analytically by employing an efficient approach namely the homotopy analysis method (HAM). It is noted that $\hbar_f$ and $\hbar_\theta$ are the HAM parameters which are useful in adjusting and controlling the convergence of the nonlinear differential equations (Eqs. (8) and (9)). Thus Figs. 1 and 2 are plotted to obtain the permissible values of these auxiliary parameters. It is found that range for admissible values of $\hbar_f$ and $\hbar_\theta$ are $-1.5 \leqslant (\hbar_f, \hbar_\theta) \leqslant -0.5$. Further the series solutions converge in the whole region of $\eta(0 < \eta < \infty)$ for $\hbar_f = -0.8 = \hbar_\theta$. In Figs. 3 and 4 the $\hbar$-curves for residual error of f and θ are sketched in order to get the admissible range for $\hbar$. It is obvious from these Figs. that by choosing the values of $\hbar$ from this range we will get the correct result upto 6th decimal place. Table 1 is made just to decide that how much order of approximations are necessary for a convergent solution. It is found that 15th order approximations are sufficient in the present problem.

3. Results and discussion

In order to validate our series solution for convergence, we have constructed Table 1. This table showed that 15th order of approximations are sufficient in the present analysis. Further, we have given a comparative study of present results with the existing results. The results displayed in Tables 2–4 are in an excellent agreement with the existing data. Furthermore Table 5 presents the values of surface mass transfer $\theta'(0)$ for various parameters. It is noted that surface mass

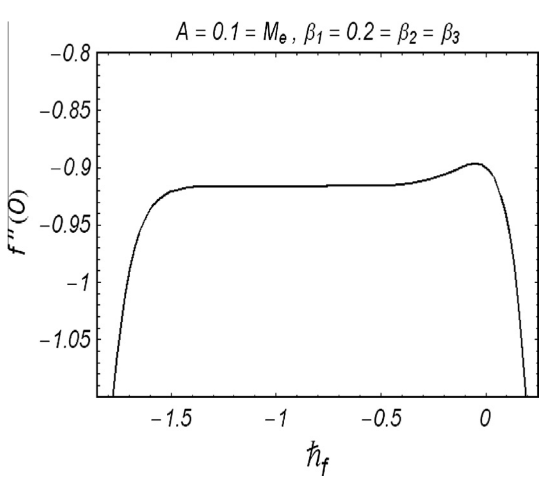

Figure 1 $\hbar$-curve for the $f''(0)$.

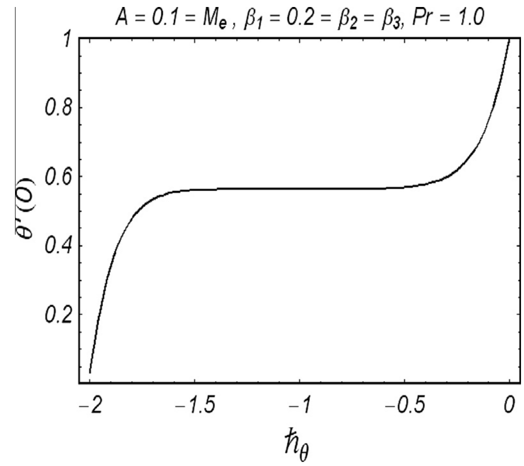

Figure 2 $\hbar$-curve for the $\theta'(0)$.

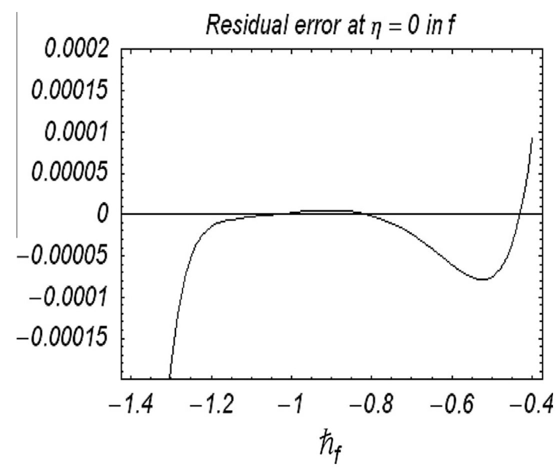

Figure 3 h-curve for error in f.

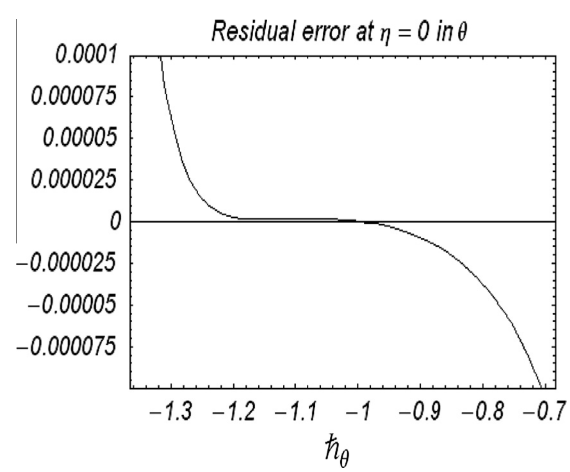

Figure 4 h-curve for error in θ.

Table 1 Convergence of the homotopy solutions for different order of approximation when $A = 0.1 = M_e$, $\beta_1 = \beta_2 = \beta_3 = 0.2$ and $\Pr = 1.0$.

Order of approximation	$-f''(0)$	$\theta'(0)$
1	0.888733	0.700000
5	0.915717	0.567717
10	0.916198	0.562502
15	0.916210	0.562468
25	0.916212	0.562460
35	0.916212	0.562460

Table 2 Comparison of present results of $f''(0)$ with those of [5] for Maxwell and Oldroyd-B fluids when $\beta_2 = 0 = M_e$.

A	Maxwell fluid ($\beta_1 = 0.2, \beta_3 = 0$)		Oldroyd-B fluid ($\beta_1 = 0.2 = \beta_3$)	
	Ref. [5]	Present results	Ref. [5]	Present results
0.01	−1.0499	−1.04991	−0.9583	−0.95815
0.05	−1.0393	−1.03934	−0.9490	−0.94948
0.1	−1.0207	−1.02081	−0.9330	−0.93395
0.5	−0.7078	−0.70782	−0.6549	−0.65785
1.0	0.0000	0.00000	0.0000	0.00000
2.0	2.2225	2.2225	2.2255	2.22571

Table 3 Comparison of present results of $f''(0)$ with those of [19] for different values of A.

A	Present results	Ref. [19] results
0.01	−0.99823	−0.9980
0.1	−0.96954	−0.9694
0.5	−0.66735	−0.6673
2.0	2.01767	2.0175
3.0	4.72964	4.7294

Table 4 Comparison of values of $\theta'(0)$ when $M_e = 0$ with those of [20].

Pr	A	Present results	Ref. [20] results
1.0	0.1	0.602156	0.603
	0.5	0.692460	0.692
1.5	0.1	0.776802	0.777
	0.5	0.864771	0.863

transfer increases with an increase in A and Pr whereas by increasing melting M_e and Deborah number β_2, the magnitude of the local Nusselt number decreases. Figs. 5–12 are prepared to give more physical insight of the problem. These graphs show the influence of various emerging parameters on the velocity and temperature fields for both stretching and shrinking sheet cases.. Fig. 5 shows that an increase in A yields an increase in velocity for shrinking sheet case i.e. $-3 < A < 1$ and the boundary layer thickness decreases. For $A = 1$ it is noted that boundary layer vanishes. Furthermore when the free stream velocity is greater than the velocity of the stretching sheet i.e. $A > 1$, the velocity increases and the boundary layer thickness decreases by increasing A. From the physical

Table 5 Values of local Nusselt number $Re_x^{-1/2}Nu$ for various values of parameters when $A = 0.2$.

A	M_e	β_2	Pr	$Re_x^{-1/2}Nu$
0.0	0.1	0.2	1.0	−0.54323
0.1				−0.56772
0.2				−0.58295
0.3				−0.60294
0.1	0.0			−0.59615
	0.2			−0.53263
	0.5			−0.46264
	0.1	0.0		−0.56882
		0.2		−0.56772
		0.5		−0.55219
		0.2	0.5	−0.36923
			1.0	−0.56772
			1.5	−0.72675

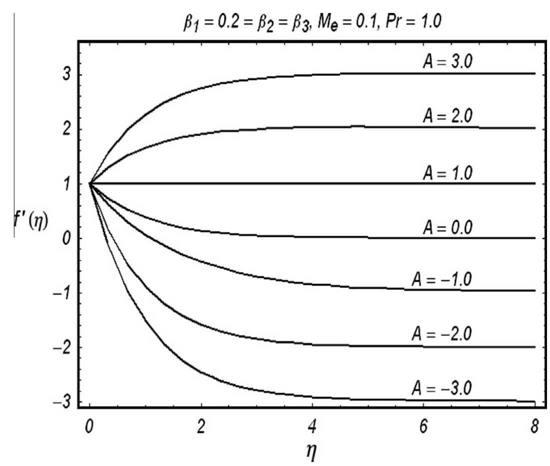

Figure 5 Influence of A on f'.

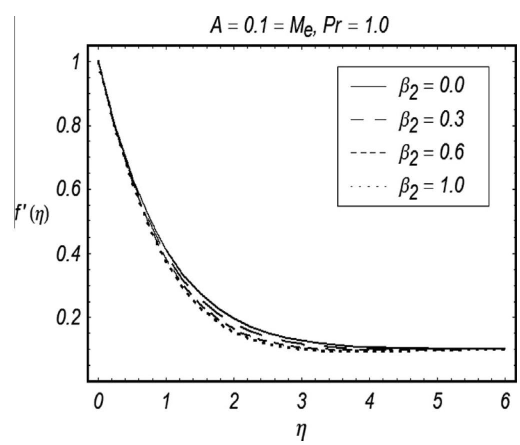

Figure 6 Influence of β_2 on f' when $A > 0$.

point of view, the large values of A accompany with the higher free stream velocity which increases the velocity of fluid. The influences of β_2 for ($A > 0$ and $A < 0$) on f' is shown in Figs. 6 and 7. It is noted that for both stretching and shrinking sheets

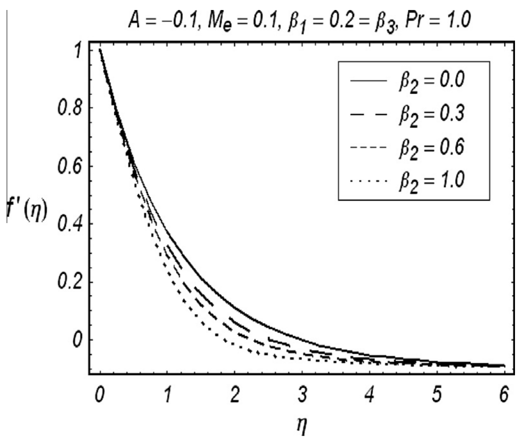

Figure 7 Influence of β_2 on f' when $A < 0$.

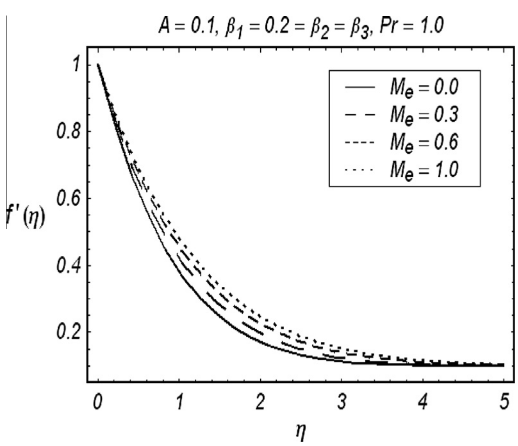

Figure 8 Influence of M_e on f' when $A > 0$.

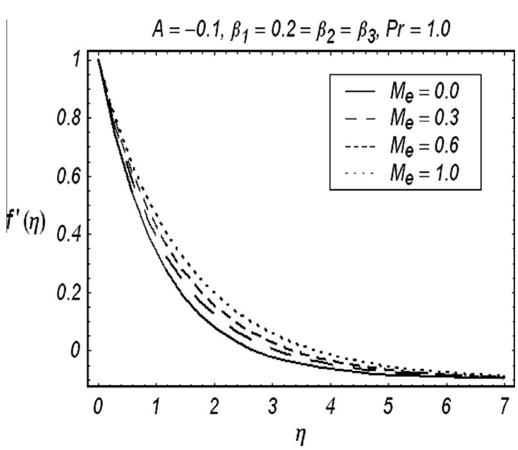

Figure 9 Influence of M_e on f' when $A < 0$.

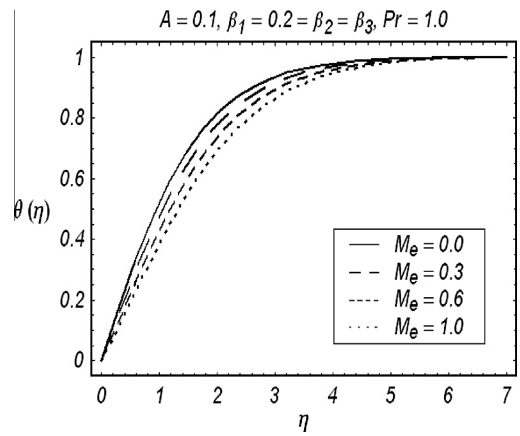

Figure 10 Influence of M_e on θ when $A > 0$.

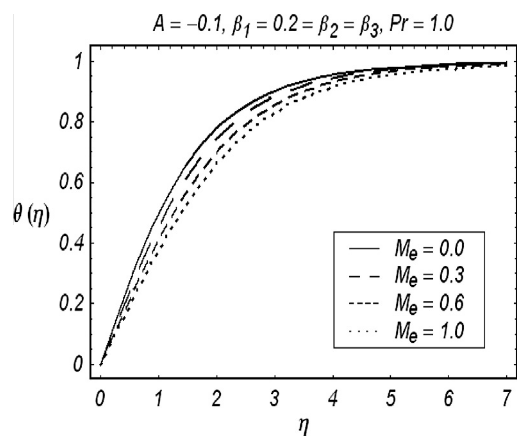

Figure 11 Influence of M_e on θ when $A < 0$.

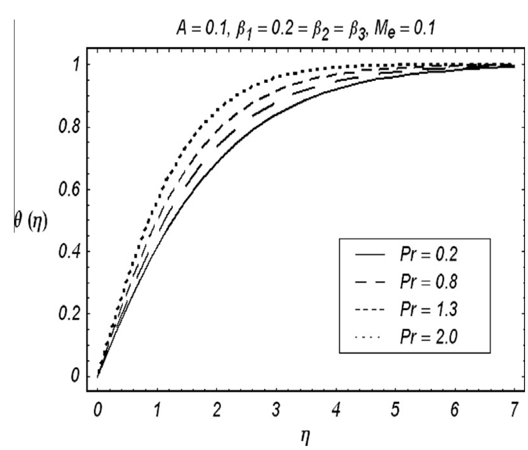

Figure 12 Influence of Pr on θ.

the velocity field and corresponding boundary layer thickness are decreasing functions of β_2. Since β_2 being the Deborah number being dependent on λ_2 has the properties of relaxation phenomenon. It is evident from various previous studies that for the smaller Deborah number, material behaves like fluids whereas for larger Deborah number, material acts like visco-elastic solids and conclusively the material become more densor. Thus the larger values of Deborah number cause a reduction in the fluids velocity and the momentum boundary layer is also reduced due to the lessor molecular movement. Figs. 8 and 9 provide the effects of melting M_e on the velocity field f' for stretching and shrinking cases. These Figs. depicts that M_e causes an increase in the velocity field f' in both cases. It is because of the fact that an increase in melting causes an increase in the molecular motion which enhances the flow.

Figs. 10 and 11 plot the influence of M_e on the temperature profile θ for stretching and shrinking cases. It is noted that temperature field θ decreases with an increase in M_e for both cases. Physically melting process causes an increase in the molecular movement which finally results into dissipation in energy and the reduction in the fluids temperature. The effects of Pr on θ are portrayed in Fig. 12. The larger values of the Prandtl number Pr correspond to the weaker thermal diffusivity, which causes a reduction in the thermal boundary layer.

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