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ORIGINAL ARTICLE

Solution of the system of fifth order boundary value problem using sextic spline

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Abstract A system of fifth order boundary value problems associated with obstacle, unilateral and contact problems is solved using sextic spline. The results are compared with the method developed by Ghazala and Siddiqi [1] and it has been observed that the method developed in this paper is better than quartic spline method. Two examples are considered for the numerical illustration of the method developed and the results are encouraging.

AMS CLASSIFICATION: 65L10

1. Introduction

Variational inequality theory has become an effective and powerful tool for studying the contact, unilateral, obstacle and equilibrium problems arising in different branches of pure and applied sciences. Variational inequality theory has proved to be immensely useful in the study of many branches of mathematical and engineering sciences. The general variational inequalities can be characterized by a system of differential equations using the penalty function technique, if the obstacle function is known. This technique was used by Lewy and Stampacchia [2] to study the regularity of the solution of variational inequalities. The main advantage of this technique is its simple applicability in solving obstacle and unilateral problems.

Al-Said [3] developed the solution of system of second order boundary value problems using quadratic spline. Gao and Chi [4] solved a system of third-order boundary value problems associated with third-order obstacle problems using the quartic B-splines and the method is claimed to be of second order. Siraj et al. [5] developed the solution of a system of third-order boundary value problems using nonpolynomial spline and the method is claimed to be of second order as well. Siddiqi and Ghazala [6–8] solved the system of fourth order boundary value problem using cubic nonpolynomial spline, cubic spline and nonpolynomial spline.

Ghazala and Siddiqi [1] solved fifth order obstacle problem using quartic spline and it has been observed that the method developed in this paper is better than quartic spline method. Noor et al. [9] also solved fifth order obstacle problem using variation of parameters method, but the exact solution of the problem given in the paper is not correct. Hence the method developed by Noor et al. [9] cannot be compared with the method developed in this paper.

In this paper, sextic spline function is used to develop a technique for the solution of the following system

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$$y^{(5)}(x) = \begin{cases} f(x), & a \leqslant x \leqslant c, \\ f(x) + y(x)g(x) + r, & c \leqslant x \leqslant d, \\ f(x), & d \leqslant x \leqslant b, \end{cases} \tag{1.1}$$

along with the boundary conditions

$$\left.\begin{array}{ll} y(a) = y(b) = \alpha_0, & y^{(1)}(a) = y^{(1)}(b) = \alpha_1, \\ y(c) = y(d) = \alpha_2, & y^{(1)}(c) = y^{(1)}(d) = \alpha_3, \\ y^{(2)}(a) = \alpha_4, & y^{(2)}(c) = y^{(2)}(d) = \alpha_5, \end{array}\right\} \tag{1.2}$$

where r and α_i, $i = 0, 1, \ldots, 5$ are finite real constants and the functions $f(x)$ and $g(x)$ are continuous on $[a, b]$ and $[c, d]$, respectively. Such type of systems arise in connection with contact, obstacle and unilateral problems.

The sextic spline method and the corresponding end conditions are derived in Section 2. Section 3, is devoted to the application of to a system of fifth order boundary value problems. In Section 4, two examples are considered for the usefulness of the method developed.

2. Sextic spline method

To develop the sextic spline approximation S to the problem (1.1), the interval $[a, b]$ is divided into k equal subintervals (s.t k is divisible by 4), using the grid points $x_i = a + ih$; $i = 0, 1, \ldots, k$, where $h = (b - a)/k$.

The restriction S_i of S to each subinterval $[x_i, x_{i+1}]$, $i = 0, 1, \ldots, k - 1$, is defined as

$$S_i(x) = a_i(x - x_i)^6 + b_i(x - x_i)^5 + c_i(x - x_i)^4 + d_i(x - x_i)^3 + e_i(x - x_i)^2 + f_i(x - x_i) + g_i. \tag{2.1}$$

For

$$\left.\begin{array}{ll} S_i(x_i) = y_i, & S_i^{(1)}(x_i) = m_i, \\ S_i^{(5)}(x_i) = t_i, & S_i^{(3)}(x_i) = n_i, \end{array}\right\} \quad i = 0, 1, \ldots, k \tag{2.2}$$

and assuming $y(x)$ to be the exact solution of the system (1.1) and y_i be an approximation to $y(x_i)$, obtained by the spline $S(x_i)$.

Applying the second, third and fourth derivative continuities at the knots, i.e. $S_{i-1}^{(\mu)}(x_i) = S_i^{(\mu)}(x_i)$ for $\mu = 2, 3, 4$, Siddiqi and Ghazala [10] derived the following consistency relation which is necessary to find the solution of problem (1.1)

$$\begin{aligned} & t_{i-3} + 57t_{i-2} + 302t_{i-1} + 302t_i + 57t_{i+1} + t_{i+2} \\ & \quad = \frac{-720}{h^5}[y_{i-3} - 5y_{i-2} + 10y_{i-1} - 10y_i + 5y_{i+1} - y_{i+2}]; \\ & \quad i = 3, 4, \ldots, k - 2, \end{aligned} \tag{2.3}$$

The end conditions corresponding to the system (1.1) with (1.2) are determined as under

$$\left.\begin{array}{l} (i)\ \displaystyle\sum_{k=i-1}^{i+3} b_k y_k + c_0 h y_{i-1}^{(1)} + d_0 h^5 y_{i-1}^{(5)} + h^5 \sum_{k=i-1}^{i+3} d_k t_k = 0, \\ \qquad i = 1, n+1, 3n+1, \\ (ii)\ \displaystyle\sum_{k=i-1}^{i+3} e_k y_k + c_1 h y_{i-2}^{(1)} + d_1 h^2 y_{i-2}^{(2)} + h^5 \sum_{k=i-1}^{i+3} l_k t_k = 0, \\ \qquad i = 2, n+2, 3n+2, \\ (iii)\ \displaystyle\sum_{k=i-3}^{i+1} m_k y_k + c_2 h y_{i+1}^{(1)} + d_2 h^5 y_{i+1}^{(5)} + h^5 \sum_{k=i-3}^{i+1} n_k t_k = 0, \\ \qquad i = n-1, 3n-1, 4n-1, \end{array}\right\} \tag{2.4}$$

where $b_k, d_k, e_k, l_k, m_k, n_k, c_i, d_i, i = 0, 1, 2$ are arbitrary parameters to be determined. For the fourth order end conditions,

$$\begin{aligned} & (b_{i-1}, b_i, b_{i+1}, b_{i+2}, b_{i+3}) = (25, -48, 36, -16, 3), \\ & (d_{i-1}, d_i, d_{i+1}, d_{i+2}, d_{i+3}) = (-1, 0, 0, 0, -1), \\ & (e_{i-3}, e_{i-2}, e_{i-1}, e_i, e_{i+1}) \\ & \quad = \left(\frac{198,160}{12,019}, \frac{-459,650}{12,019}, \frac{417,960}{12,019}, \frac{-192,670}{12,019}, \frac{36,200}{12,019}\right), \\ & (l_{i-3}, l_{i-2}, l_{i-1}, l_i, l_{i+1}) = (-1, 0, 0, 0, -1), \\ & (m_{i-3}, m_{i-2}, m_{i-1}, m_i, m_{i+1}) = (-3, 16, -36, 48, -25), \\ & (n_{i-3}, n_{i-2}, n_{i-1}, n_i, n_{i+1}) = (-1, 0, 0, 0, -1), \\ & (c_0, c_1, c_2) = \left(12, \frac{56,940}{12,019}, 12\right), \\ & (d_0, d_1, d_2) = \left(\frac{-2}{5}, \frac{28,260}{12,019}, \frac{-2}{5}\right). \end{aligned}$$

3. Applications

To illustrate the implementation of the method developed, the following fifth order obstacle boundary value problem can be considered as

$$\left.\begin{array}{l} -y^{(5)}(x) \geqslant f(x), \\ y(x) \geqslant \psi(x), \\ (y^{(5)}(x) - f(x))\,(y(x) - \psi(x)) = 0, \\ y(-1) = y(1) = y^{(1)}(-1) = y^{(1)}(1) = y^{(2)}(-1) = 0, \quad \text{on } \Omega = [-1, 1], \end{array}\right\} \tag{3.1}$$

where f is a given force acting on string and $\psi(x)$ is the elastic obstacle. The problem (3.1) arise in several branches of pure and applied sciences including transportation, equilibrium, optimization, mechanics, structural analysis, fluid flow through porous media and image processing in the medical sciences. Using the ideas and technique of Lewy and Stampacchia [2], the obstacle problem (3.1) can be characterized by the following system of variational inequality problem

$$-y^{(5)} + \mu(y - \psi)(y - \psi) = f(x), \quad -1 < x < 1, \tag{3.2}$$

$$y(-1) = y(1) = 0, \quad y^{(1)}(-1) = y^{(1)}(1) = y^{(2)}(-1) = \epsilon, \tag{3.3}$$

where ϵ is a small positive constant, ψ is the obstacle function and $\mu(t)$ is the penalty function defined by

$$\mu(t) = \begin{cases} 1, & t \geqslant 0, \\ 0, & t < 0. \end{cases} \tag{3.4}$$

Since the obstacle function ψ is known, it is possible to find the exact solution of the problem in the interval $-1/2 \leqslant x \leqslant 1/2$.

Assuming that the obstacle function ψ is defined by

$$\psi(x) = \begin{cases} -1, & -1 \leqslant x \leqslant -1/2, \quad 1/2 \leqslant x \leqslant 1, \\ 1, & -1/2 \leqslant x \leqslant 1/2. \end{cases} \tag{3.5}$$

From Eqs. (3.2)–(3.5), the following system of equations can be obtained as

$$y^{(5)} = \begin{cases} f, & -1 \leqslant x \leqslant -1/2, \quad 1/2 \leqslant x \leqslant 1, \\ -1 + y + f, & -1/2 \leqslant x \leqslant 1/2, \end{cases} \tag{3.6}$$

with the boundary conditions

$$y(-1) = y(-1/2) = y(1/2) = y(1) = 0, \tag{3.7}$$

$$y^{(1)}(-1) = y^{(1)}(-1/2) = y^{(1)}(1/2) = y^{(1)}(1) = y^{(2)}(-1) = y^{(2)}(-1/2) = y^{(2)}(1/2) = \epsilon, \tag{3.8}$$

and the conditions of continuity for $y, y^{(1)}$ and $y^{(2)}$ at $x = -1/2$ and $1/2$.

If $\mu(t)$ and $\psi(x)$ are taken as

$$\mu(t) = \begin{cases} 4, & t \geqslant 0, \\ 0, & t < 0, \end{cases} \tag{3.9}$$

$$\psi(x) = \begin{cases} -1/4, & -1 \leqslant x \leqslant -1/2, \quad 1/2 \leqslant x \leqslant 1, \\ 1/4, & -1/2 \leqslant x \leqslant 1/2, \end{cases} \tag{3.10}$$

then the following system of equations can be obtained as

$$y^{(5)} = \begin{cases} f, & -1 \leqslant x \leqslant -1/2, \quad 1/2 \leqslant x \leqslant 1, \\ 1 - 4y + f, & -1/2 \leqslant x \leqslant 1/2. \end{cases} \tag{3.11}$$

It is to be mentioned that the system of equations associated with the obstacle problem (3.1) is a special case of the system of fifth order boundary value problem (1.1).

4. Numerical examples

Example 1. The following problem is considered, as

$$y^{(5)}(x) = \begin{cases} 1, & -1 \leqslant x \leqslant -1/2, \quad 1/2 \leqslant x \leqslant 1, \\ 2 - 4y(x), & -1/2 \leqslant x \leqslant 1/2, \end{cases} \tag{4.1}$$

together with the boundary conditions at $x = -1, x = 1$ and the conditions of continuity for $y, y^{(1)}$ and $y^{(2)}$ at $x = -1/2$ and $x = 1/2$. The analytic solution of the system (4.1) is

$$y(x) = \begin{cases} 1/480(1 + 7x + 19x^2 + 25x^3 + 16x^4 + 4x^5), \\ -1 \leqslant x \leqslant -1/2, \\ y(-1) = y(-1/2) = 0, \\ y^{(1)}(-1) = y^{(1)}(-1/2) = y^{(2)}(-1) = 0, \\ \\ 1/2 - \beta_1 exp(-2^{2/5}x) - \beta_2 exp(\alpha_1 x)cos(\alpha_2 x) \\ -\beta_3 exp(\alpha_3 x)cos(\alpha_4 x) + \beta_4 exp(\alpha_3 x)sin(\alpha_4 x) \\ +\beta_5 exp(\alpha_1 x)sin(\alpha_2 x), \\ -1/2 \leqslant x \leqslant 1/2, \\ y(-1/2) = y(1/2) = 0, \\ y^{(1)}(-1/2) = y^{(1)}(1/2) = y^{(2)}(-1/2) = 0, \\ \\ 1/960(-1 + 8x - 25x^2 + 38x^3 - 28x^4 + 8x^5), \\ 1/2 \leqslant x \leqslant 1, \\ y(1/2) = y(1) = 0, \\ y^{(1)}(1/2) = y^{(1)}(1) = y^{(2)}(1/2) = 0, \end{cases} \tag{4.2}$$

where

$$\beta_1 = 0.083472402636811683486236902194475814000555110022391306503,$$

$$\beta_2 = 0.197226655875097793394145018248435611353617476770476389696,$$

$$\beta_3 = 0.218780476045802273706722229445094829786402746435753855139,$$

$$\beta_4 = 0.005598628986902192748619920098904556943240511107281886036,$$

$$\beta_5 = 0.031624084344408001357285362723625687421220160348823034071,$$

$$\alpha_1 = \left(\frac{1}{22^{3/5}} - \frac{\sqrt{5}}{22^{3/5}}\right), \quad \alpha_2 = \frac{\sqrt{5+\sqrt{5}}}{22^{1/10}},$$

$$\alpha_3 = \left(\frac{1}{22^{3/5}} + \frac{\sqrt{5}}{22^{3/5}}\right), \quad \alpha_4 = \frac{\sqrt{5-\sqrt{5}}}{22^{1/10}}.$$

The observed maximum errors (in absolute values) are summarized in Table 1 and it is evident from the Table 1 that the results presented by the method developed in this paper are better than those given by Ghazala and Siddiqi [1]. It is confirmed from the Table 1 that if h is reduced by factor 1/2, then $\|E\|$ is reduced by a factor 1/4, which indicates that the present method gives second order results.

Example 2. The following problem is considered, as

Table 1 Maximum absolute errors for problem (4.1).

h	The method presented in the paper	Ghazala and Siddiqi [1]
1/14	3.10×10^{-7}	2.10×10^{-4}
1/28	6.15×10^{-8}	2.93×10^{-5}
1/56	9.20×10^{-9}	3.85×10^{-6}
1/112	1.24×10^{-9}	7.93×10^{-7}
1/224	1.62×10^{-10}	6.23×10^{-8}

Table 2 Maximum absolute errors for problem (4.3).

h	The method presented in the paper	Ghazala and Siddiqi [1]
1/14	9.29×10^{-7}	4.39×10^{-4}
1/28	1.84×10^{-7}	6.12×10^{-5}
1/56	2.76×10^{-8}	8.05×10^{-6}
1/112	3.73×10^{-9}	1.03×10^{-6}
1/224	4.84×10^{-10}	1.30×10^{-7}

$$y^{(5)}(x) = \begin{cases} 2, & -1 \leqslant x \leqslant -1/2, \quad 1/2 \leqslant x \leqslant 1, \\ -1 + y(x), & -1/2 \leqslant x \leqslant 1/2. \end{cases} \tag{4.3}$$

together with the boundary conditions at $x = -1,\ x = 1$ and the conditions of continuity for $y, y^{(1)}$ and $y^{(2)}$ at $x = -1/2$ and $x = 1/2$. The analytic solution of the system (4.3) is

$$y(x) = \begin{cases} 1/240(1 + 7x + 19x^2 + 25x^3 + 16x^4 + 4x^5), \\ -1 \leqslant x \leqslant -1/2, \\ y(-1) = y(-1/2) = 0, \\ y^{(1)}(-1) = y^{(1)}(-1/2) = y^{(2)}(-1) = 0, \\ \\ 1 - \beta_1 exp(\alpha_1 x)cos(\alpha_2 x) - \beta_2 exp(\alpha_3 x)cos(\alpha_4 x) \\ -\beta_3 exp(x) + \beta_4 exp(\alpha_3 x)sin(\alpha_4 x) + \beta_5 exp(\alpha_1 x)sin(\alpha_2 x), \\ -1/2 \leqslant x \leqslant 1/2, \\ y(-1/2) = y(1/2) = 0, \\ y^{(1)}(-1/2) = y^{(1)}(1/2) = y^{(2)}(-1/2) = 0, \\ \\ 1/480(-1 + 8x - 25x^2 + 38x^3 - 28x^4 + 8x^5), \\ 1/2 \leqslant x \leqslant 1, \\ y(1/2) = y(1) = 0, \\ y^{(1)}(1/2) = y^{(1)}(1) = y^{(2)}(1/2) = 0, \end{cases} \tag{4.4}$$

where

$$\beta_1 = 0.421971253377808180784844197586481579304275360089514777474,$$

$$\beta_2 = 0.363965259605979866780921476737942599547341650816790531421,$$

$$\beta_3 = 0.214323949709503239977271742174791615518760110013465674423,$$

$$\beta_4 = 0.031318976821447235311388883995038835884137184947278662482,$$

$$\beta_5 = 0.032949018430124054163299461434266313081077383697306730612,$$

$$\alpha_1 = \left(\frac{-1}{4} + \frac{\sqrt{5}}{4}\right), \quad \alpha_2 = \frac{1}{2}\sqrt{\frac{1}{2}\left(5 + \sqrt{5}\right)},$$
$$\alpha_3 = \left(\frac{-1}{4} - \frac{\sqrt{5}}{4}\right), \quad \alpha_4 = \frac{1}{2}\sqrt{\frac{1}{2}\left(5 - \sqrt{5}\right)}.$$

To analyze the usefulness of the method, the results are summarized in Table 2 and it is evident from the Table 2 that the results presented by the method developed in this paper are better than those given by Ghazala and Siddiqi [1]. It is confirmed from the Table 2 that $\|E\|$ is reduced by a factor 1/4, if h is reduced by factor 1/2, which shows that the method is of second order.

5. Conclusion

Sextic spline method is developed for the approximate solutions of system of fifth order BVPs. It is noted that the method developed is better than quartic spline method [1] and sextic spline method is a powerful mathematical tool for the solution of system of fifth order BVPs. Numerical examples also illustrate the accuracy of the method.

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