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ORIGINAL ARTICLE

On using third and fourth kinds Chebyshev polynomials for solving the integrated forms of high odd-order linear boundary value problems

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Abstract This article presents some spectral Petrov–Galerkin numerical algorithms based on using Chebyshev polynomials of third and fourth kinds for solving the integrated forms of high odd-order two point boundary value problems governed by homogeneous and nonhomogeneous boundary conditions. The principle idea behind obtaining the proposed numerical algorithms is based on constructing trial and test functions as compact combinations of shifted Chebyshev polynomials of third and fourth kinds. The algorithms lead to linear systems with specially structured matrices that can be efficiently inverted. Some numerical examples are illustrated for the sake of demonstrating the validity and the applicability of the proposed algorithms. The presented numerical results indicate that the proposed algorithms are reliable and very efficient.

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1. Introduction

Chebyshev polynomials have become increasingly crucial in numerical analysis, from both theoretical and practical points of view. It is well-known that there are four kinds of Chebyshev polynomials, and all of them are special cases of the more widest class of Jacobi polynomials. The first and second kinds are special cases of the symmetric Jacobi polynomials (i.e., ultraspherical polynomials), while the third and fourth kinds are special cases of nonsymmetric Jacobi polynomials. In literature, there is a great concentration on first and second kinds of Chebyshev polynomials $T_n(x)$ and $U_n(x)$ and their various uses in numerous applications, (see for instance, [1–3]).

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However, there are few articles concentrate on the other two types of Chebyshev polynomials namely, third and fourth kinds $V_n(x)$ and $W_n(x)$, either from theoretical or practical points of view and their uses in various applications, (see, for example, [4–6]). This motivates our interest in such polynomials. The interested readers in Chebyshev polynomials of third and fourth kinds are refereed to the excellent book of Mason and Handscomb [7].

If we were asked for "a pecking order" of these four Chebyshev polynomials $T_n(x), U_n(x), V_n(x)$ and $W_n(x)$, then we would say that $T_n(x)$ is the most important and versatile. Moreover $T_n(x)$ generally leads to the simplest formulae, whereas results for the other polynomials may involve slight complications. However, all four polynomials have their role. For example, $U_n(x)$ is useful in numerical integration (see, Mason [8]), while $V_n(x)$ and $W_n(x)$ can be useful in situations in which singularities occur at one end point (+1 or -1) but not at the other (see, Mason and Handscomb [7]).

Due to their great importance in several applications, high even-order and high odd-order boundary value problems have been investigated by a large number of authors. Theorems which discuss the conditions for the existence and uniqueness of solutions of such problems are contained in a comprehensive survey in a book by Agarwal [9].

Spectral methods (see, for instance, Boyd [1] and Canuto et al. [10]) are a class of techniques used extensively in applied mathematics and scientific computing to numerically solving ordinary and partial differential equations. The numerical solution is written as an expansion in terms of certain "basis functions", which may be expressed in terms of various orthogonal polynomials. Spectral methods have advantage that they take on a global approach while finite element methods use a local approach, and for this reason, spectral methods have excellent error properties, and converge exponentially.

The study of odd-order equations is of mathematical and physical interest. As an example, third-order equation contains a type of operator which appears in many commonly occurring partial differential equations such as the Kortweg-de Vries equation. Also, fifth-order boundary value problems are of interest as they arise in the mathematical modelling of viscoelastic flows (see, [11,12]). Abd-Elhameed in [13] and Doha and Abd-Elhameed in [14] have constructed efficient spectral-Galerkin algorithms using compact combinations of ultraspherical polynomials for solving the differentiated forms of elliptic equations of high odd-order boundary value problems. Recently, in the two papers of Abd-Elhameed et al. in [15] and Doha et al. in [16], some algorithms for solving numerically the differentiated and integrated forms of third and fifth-order differential equations based on a dual Petrov–Galerkin method using two new families of general parameters generalized Jacobi polynomials, are analyzed.

Of the important high-order differential equations are the singular and singularly perturbed problems (SPPs). These kinds of problems usually appear in quantum mechanics, optimal control, etc. The presence of small parameter in these problems prevents one to obtain satisfactory numerical solutions. It is a well-known fact that the solutions of SPPs have a multi-scale character, that is, there are thin layer(s) where the solution varies very rapidly, while away from the layer(s) the solution behaves regularly and varies slowly. The existence and uniqueness of singularly purturbed boundary value problems was discussed by Howers [17], Kelevedjiev [18], and Roos et al. [19].

As an alternative approach to differentiating solution expansions is to integrate the differential equation q times, where q is the order of the equation. An advantage of this approach is that the resulted algebraic system contains a finite number of terms and hence they are cheaper in solving than those obtained by the differentiated forms. Doha et al. in [16] followed this approach for solving the integrated forms of third- and fifth-order elliptic differential equations. Moreover, Doha and Abd-Elhameed in [4] obtained new formulae for the repeated integrals of Chebyshev polynomials of third and fourth kinds and they used these formulae for solving the integrated forms of sixth-order boundary value problems.

The main objective of the present article is to develop some efficient spectral algorithms based on shifted Chebyshev third and fourth kinds-Galerkin methods for solving the integrated forms of high odd-order differential equations.

The contents of this article are organized as follows. In Section 2, some properties and relations of Chebyshev polynomials of third and fourth kinds and their shifted ones are presented. In Section 3, we discuss some algorithms for solving the integrated forms of high odd-order elliptic differential equations governed by homogeneous and nonhomogeneous boundary conditions using shifted Chebyshev third kind Petrov–Galerkin method (SC3PGM). In Section 4, we are concerned with the same equations but by using shifted Chebyshev fourth kind Petrov–Galerkin method (SC4PGM). Section 5 is concerned with discussing the condition numbers resulted from the application of the two proposed algorithms in Sections 3 and 4. In Section 6, we present three numerical examples including comparisons with some other methods aiming to exhibit the accuracy and the efficiency of our proposed algorithms. Some concluding remarks are presented in Section 7.

2. Some properties of third and fourth kinds of Chebyshev polynomials and their shifted ones

2.1. Chebyshev polynomials of third and fourth kinds

Chebyshev polynomials $V_i(x)$ and $W_i(x)$ of third and fourth kinds are polynomials in x, which can be defined by one of the two following equivalent forms (see, [7]):

$$V_i(x) = \frac{\cos\left(i+\frac{1}{2}\right)\theta}{\cos\frac{\theta}{2}} = \frac{2^{2i}}{\binom{2i}{i}} P_i^{(-\frac{1}{2},\frac{1}{2})}(x),$$

and

$$W_i(x) = \frac{\sin\left(i+\frac{1}{2}\right)\theta}{\sin\frac{\theta}{2}} = \frac{2^{2i}}{\binom{2i}{i}} P_i^{(\frac{1}{2},-\frac{1}{2})}(x),$$

where $x = \cos\theta$, and $P_i^{(\alpha,\beta)}(x)$ is the Jacobi polynomial of degree i. It is clear that

$$W_i(x) = (-1)^i\, V_i(-x). \tag{1}$$

The polynomials $V_i(x)$ and $W_i(x)$ are orthogonal on $(-1,1)$ with respect to the weight functions $\sqrt{\frac{1+x}{1-x}}$ and $\sqrt{\frac{1-x}{1+x}}$, respectively, i.e.,

$$\int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}\,V_i(x)\;V_j(x)\;dx=\int_{-1}^{1}\sqrt{\frac{1-x}{1+x}}\,W_i(x)\;W_j(x)\;dx=\begin{cases}\pi, & i=j,\\ 0, & i\neq j,\end{cases}$$

and they may be generated by using the two recurrence relations

$$V_i(x)=2x\;V_{i-1}(x)-V_{i-2}(x),\quad i=2,3,\ldots,$$

with the initial values

$$V_0(x)=1,\quad V_1(x)=2x-1,$$

and

$$W_i(x)=2x\;W_{i-1}(x)-W_{i-2}(x),\quad i=2,3,\ldots,$$

with the initial values

$$W_0(x)=1,\quad W_1(x)=2x+1.$$

The following special values are of important use later:

$$V_i(1)=1,\quad V_i(-1)=(-1)^i\;(2i+1),$$
$$W_i(1)=2i+1, W_i(-1)=(-1)^i.$$

Also, for all $m\geqslant 1$, we have

$$D^m V_i(x)=\frac{\sqrt{\pi}\;(i+m)!}{2^m\;(i-m)!}\begin{cases}\frac{1}{\Gamma(m+\frac{1}{2})}, & \text{if } x=1,\\ \frac{(-1)^{i+m}\;(2\;i+1)}{2\,\Gamma\left(m+\frac{3}{2}\right)}, & \text{if } x=-1,\end{cases}$$

and

$$D^m W_i(x)=\frac{\sqrt{\pi}(i+m)!}{2^m\;(i-m)!}\begin{cases}\frac{(2\;i+1)}{2\,\Gamma\left(m+\frac{3}{2}\right)}, & \text{if } x=1,\\ \frac{(-1)^{i+m}}{\Gamma\left(m+\frac{1}{2}\right)}, & \text{if } x=-1.\end{cases}$$

The following theorem is needed in obtaining the desired spectral numerical solutions.

Theorem 1. *If the m times repeated integration of $V_i(x)$ is denoted by*

$$I_i^{(m)}(x)=\overbrace{\int\int\cdots\int}^{m\;\;times}V_i(x)\overbrace{dx\;dx\ldots dx}^{m\;\;times},$$

then

$$I_i^{(m)}(x)=\sum_{k=0}^{2m}G_{k,i,m}\;V_{i+m-k}(x)+\sigma_{m-1}(x),\tag{2}$$

where

$$G_{k,i,m}=\frac{m!}{2^m}\begin{cases}\frac{(-1)^{\frac{k}{2}}\left(i-\frac{k}{2}\right)!}{\left(\frac{k}{2}\right)!\;\left(m-\frac{k}{2}\right)!\;\left(m+i-\frac{k}{2}\right)!}, & k\text{ even},\\ \frac{(-1)^{\frac{k+1}{2}}\left(i-\left(\frac{k+1}{2}\right)\right)!}{\left(\frac{k-1}{2}\right)!\;\left(m-\left(\frac{k+1}{2}\right)\right)!\;\left(m+i-\left(\frac{k-1}{2}\right)\right)!}, & k\text{ odd},\end{cases}\tag{3}$$

and $\sigma_{m-1}(x)$ is a polynomial in x of degree at most $(m-1)$.

(For the proof of Theorem 1, see Doha and Abd-Elhameed [4]).

Theorem 1 with the aid of relation (1), enables one to get immediately the following corollary.

Corollary 1. *The m times repeated integration of $W_i(x)$ is given by*

$$J_i^{(m)}(x)=\sum_{k=0}^{2m}\overline{G}_{k,i,m}\;W_{i+m-k}(x)+\rho_{m-1}(x),$$

where

$$\overline{G}_{k,i,m}=(-1)^k\;G_{k,i,m},\tag{4}$$

$\rho_{m-1}(x)$ is a polynomial in x of degree at most $(m-1)$ and $G_{k,i,m}$ is as defined in (3).

2.2. Shifted Chebyshev polynomials of third and fourth kinds

The shifted Chebyshev polynomials of third and fourth kinds are defined on $[a,b]$, respectively as

$$V_i^*(x)=V_i\left(\frac{2\;x-a-b}{b-a}\right),\quad W_i^*(x)=W_i\left(\frac{2\;x-a-b}{b-a}\right).\tag{5}$$

All results of Chebyshev polynomials of third and fourth kinds, can be easily transformed to give the corresponding results for their shifted ones. The orthogonality relations of $V_i^*(x)$ and $W_i^*(x)$ on $[a,b]$ with respect to the weight functions $\sqrt{\frac{x-a}{b-x}}$ and $\sqrt{\frac{b-x}{x-a}}$, are given by

$$\int_a^b\sqrt{\frac{x-a}{b-x}}\,V_i^*(x)\;V_j^*(x)\;dx=\int_a^b\sqrt{\frac{b-x}{x-a}}\,W_i^*(x)\;W_j^*(x)\;dx=\begin{cases}(b-a)\;\frac{\pi}{2}, & i=j,\\ 0, & i\neq j.\end{cases}\tag{6}$$

Based on (5) and with the aid of formula (2), we have the following theorem.

Theorem 2. *If the m times repeated integration of $V_i^*(x)$ is denoted by*

$$\bar{I}_i^{(m)}(x)=\overbrace{\int\int\cdots\int}^{m\;\;times}V_i^*(x)\overbrace{dx\;dx\;\;\ldots dx}^{m\;\;times},$$

then

$$\bar{I}_i^{(m)}(x)=\left(\frac{b-a}{2}\right)^m\sum_{k=0}^{2m}G_{k,i,m}\;V_{i+m-k}^*(x)+\bar{\sigma}_{m-1},$$

where $G_{k,i,m}$ is defined as in (3) and $\bar{\sigma}_{m-1}(x)$ is a polynomial in x of degree $(m-1)$ at most.

3. Solution of high odd-order differential equations by using shifted third kind Chebyshev polynomials

In this section, we are interested in using SC3PGM to solve the integrated forms of high odd-order elliptic linear differential equations governed by the homogeneous and nonhomogeneous boundary conditions.

3.1. High odd-order two point boundary value problems

Let us consider the following one dimensional high odd-order differential equations:

$$(-1)^{n+1}y^{(2n+1)}(x)+\sum_{k=0}^{2n}\mu_k\;y^{(k)}(x)=f(x),\quad x\in(a,b),\;n\geqslant 1,\tag{7}$$

governed by the homogeneous boundary conditions

$$y^{(m)}(a) = y^{(m)}(b) = y^{(n)}(a) = 0, \quad m = 0, 1, \ldots, n-1, \tag{8}$$

where $\{\mu_k, k = 0, 1, \ldots, 2n\}$ are known constant coefficients.

It is to be noted here that the main differential operator in (7) is not symmetric, so it is convenient to use a Petrov–Galerkin method. The difference between the Galerkin and Petrov–Galerkin methods, is that the test and trial functions in Galerkin method are the same, however, in case of Petrov–Galerkin method, the trial functions are chosen to satisfy the boundary conditions of the differential equation, and the test functions are chosen to satisfy the dual boundary conditions.

Now, the integral equation of (7)–(8) is:

$$\left.\begin{aligned}
&(-1)^{n+1}\; y(x) + \textstyle\sum_{k=0}^{2n}\mu_k \int^{(2n+1-k)} y(x)(dx)^{(2n+1-k)} \\
&\quad = F(x) + \textstyle\sum_{k=0}^{2n}\delta_k\; V_k^*(x), \quad x \in (a,b), \\
&y^{(m)}(a) = y^{(m)}(b) = y^{(n)}(a) = 0, \quad m = 0, 1, \ldots, n-1, \\
&F(x) = \textstyle\int^{(2n+1)} f(x)(dx)^{(2n+1)},
\end{aligned}\right\} \tag{9}$$

where

$$\int^{(m)} y(x)(dx)^m = \overbrace{\int\int\cdots\int}^{m\ times} y(x)\overbrace{dx\; dx \ldots dx}^{m\ times}.$$

If we define the following spaces

$$S_N = \mathrm{span}\{V_0^*(x), V_1^*(x), V_2^*(x), \ldots, V_N^*(x)\},$$

$$\Phi_N = \{\phi(x) \in S_N : \; \phi^{(m)}(a) = \phi^{(m)}(b) = \phi^{(n)}(a) = 0, \quad m = 0, 1, \ldots, n-1\},$$

$$\Psi_N = \{\psi(x) \in S_N : \; \psi^{(m)}(a) = \psi^{(m)}(b) = \psi^{(n)}(b) = 0, \quad m = 0, 1, \ldots, n-1\},$$

then the spectral shifted Chebyshev third kind Galerkin procedure for solving (9) is to find $y_N^n \in \Phi_N$ such that

$$\begin{aligned}
&((-1)^{n+1}\; y_N^n(x), \psi(x))_{w_1} \\
&\quad + \sum_{k=0}^{2n}\mu_k\left(\int^{(2n+1-k)} y_N^n(x)(dx)^{(2n+1-k)}, \psi(x)\right)_{w_1} \\
&\quad = \left(F(x) + \sum_{k=0}^{2n}\delta_k\; V_k^*(x), \psi(x)\right)_{w_1}, \quad \forall\; \psi(x) \in \Psi_N,
\end{aligned} \tag{10}$$

where $(y(x), \psi(x))_{w_1} = \int_a^b w_1\; y(x)\; \psi(x)\; dx$, is the scalar inner product in the weighted space $L^2_{w_1}(a, b)$ and $w_1 = \sqrt{\frac{x-a}{b-x}}$.

3.2. The choice of trial and test functions

First we consider the case $[a, b] = [-1, 1]$, and we aim to construct suitable basis functions and their dual basis. For this purpose, we set

$$\phi_{i,n}(x) = V_i(x) + \sum_{k=1}^{2n+1} p_{k,i} V_{i+k}(x), \quad x \in [-1, 1], \quad i = 0, 1, 2, \ldots, N-2n-1, \quad n \geqslant 1, \tag{11}$$

$$\psi_{i,n}(x) = V_i(x) + \sum_{k=1}^{2n+1} q_{k,i} V_{i+k}(x), \quad x \in [-1, 1], \quad i = 0, 1, 2, \ldots, N-2n-1, \quad n \geqslant 1. \tag{12}$$

The coefficients $\{p_{k,i}\}$ and $\{q_{k,i}\}$ are chosen such that $\phi_{i,n}(x) \in \Phi_{i+2n+1}$ and $\psi_{i,n}(x) \in \Psi_{i+2n+1}$, respectively. The $(n+1)$ boundary conditions $\phi_{i,n}^{(m)}(-1) = 0$, $m = 0, 1, 2, \ldots, n$, and the n boundary conditions $\phi_{i,n}^{(m)}(1) = 0$, $m = 0, 1, 2, \ldots, n-1$, lead respectively to the following system of $(2n+1)$ equations:

$$\sum_{k=1}^{2n+1}\frac{(-1)^k(2i+2k+1)(i+k+m)!}{(i+k-m)!}\; p_{k,i} = \frac{-(2i+1)(i+m)!}{(i-m)!}, \quad m = 0, 1, 2, \ldots, n,$$

$$\sum_{k=1}^{2n+1}\frac{(i+k+m)!}{(i+k-m)!}\; p_{k,i} = \frac{-(i+m)!}{(i-m)!}, \quad m = 0, 1, 2, \ldots, n-1.$$

The determinant of the above system of equations is different from zero, hence the coefficients $\{p_{k,i}\}$ can be uniquely determined to give

$$p_{2k,i} = \frac{(-1)^k\binom{n}{k}(i+1)_k(2i+4k+2n+3)}{(i+n+2)_k(2i+2n+3)}, \quad k = 1, 2, \ldots, n, \quad 0 \leqslant i \leqslant N-2n-1,$$

$$p_{2k+1,i} = \frac{(-1)^k\binom{n}{k}(i+1)_k(2i+4k-2n+1)}{(i+n+2)_k(2i+2n+3)}, \quad k = 0, 1, \ldots, n, \quad 0 \leqslant i \leqslant N-2n-1,$$

where $(z)_k$ denotes the Pochhammer symbol, i.e., $(z)_k = \frac{\Gamma(z+k)}{\Gamma(z)}$. Similarly, it can be easily verified that the coefficients $\{q_{k,i}\}$ can be uniquely determined to give

$$q_{2\,k,i} = \frac{(-1)^k\binom{n}{k}(i+1)_k}{(i+n+2)_k}, \quad k = 1, 2, \ldots, n, \quad 0 \leqslant i \leqslant N-2n-1,$$

$$q_{2\,k+1,i} = \frac{(-1)^{k+1}\binom{n}{k}(i+1)_k}{(i+n+2)_k}. \quad k = 0, 1, \ldots, n, \quad 0 \leqslant i \leqslant N-2n-1,$$

Now, if x in (11) and (12) is replaced by $\left(\frac{2\,x-a-b}{b-a}\right)$ for $x \in [a, b]$, and if we define the following basis functions:

$$\phi_{i,n}\left(\frac{2\,x-a-b}{b-a}\right) = \phi_{i,n}^*(x), \quad \psi_{i,n}\left(\frac{2\,x-a-b}{b-a}\right) = \psi_{i,n}^*(x), \quad x \in [a, b],$$

then it is clear that the basis and dual basis functions given by

$$\phi_{i,n}^*(x) = V_i^*(x) + \sum_{k=1}^{2n+1} p_{k,i} V_{i+k}^*(x),$$

and

$$\psi_{i,n}^*(x) = V_i^*(x) + \sum_{k=1}^{2n+1} q_{k,i} V_{i+k}^*(x),$$

satisfy the boundary conditions (8), i.e. $\phi_{i,n}^*(x) \in \Phi_{i+2n+1}$ and $\psi_{i,n}^*(x) \in \Psi_{i+2n+1}$, respectively. From now on, it would be convenient to write the basis functions $\phi_{i,n}^*(x)$ and their dual basis $\psi_{i,n}^*(x)$ in the two alternative forms:

$$\phi_{i,n}^*(x) = \sum_{k=0}^{2n+1} p_{k,i} V_{i+k}^*(x), \tag{13}$$

$$\psi_{i,n}^*(x) = \sum_{k=0}^{2n+1} q_{k,i} V_{i+k}^*(x), \tag{14}$$

where the coefficients $p_{k,i}$ and $q_{k,i}$ are given respectively by:

$$p_{k,i}=\begin{cases}\dfrac{(-1)^{\frac{k}{2}}\binom{n}{\frac{k}{2}}(i+1)_{\frac{k}{2}}(2i+2k+2n+3)}{(i+n+2)_{\frac{k}{2}}(2i+2n+3)}, & k=0,2,\ldots,2n,\\[2ex] \dfrac{(-1)^{\frac{k-1}{2}}\binom{n}{\frac{k-1}{2}}(i+1)_{\frac{k-1}{2}}(2i+2k-2n-1)}{(i+n+2)_{\frac{k-1}{2}}(2i+2n+3)}, & k=1,3,\ldots,2n+1,\end{cases} \tag{15}$$

$$q_{k,i}=\begin{cases}\dfrac{(-1)^{\frac{k}{2}}\binom{n}{\frac{k}{2}}(i+1)_{\frac{k}{2}}}{(i+n+2)_{\frac{k}{2}}}, & k=0,2,\ldots,2n,\\[2ex] \dfrac{(-1)^{\frac{k+1}{2}}\binom{n}{\frac{k-1}{2}}(i+1)_{\frac{k-1}{2}}}{(i+n+2)_{\frac{k-1}{2}}}, & k=1,3,\ldots,2n+1.\end{cases} \tag{16}$$

It is obvious that the basis functions $\phi^*_{i,n}(x)$ and $\psi^*_{i,n}(x)$ are linearly independent. Therefore, we have

$$\Phi_N=\operatorname{span}\left\{\phi^*_{i,n}(x): i=0,1,2,\ldots,N-2n-1\right\},$$

and

$$\Psi_N=\operatorname{span}\left\{\psi^*_{i,n}(x): i=0,1,2,\ldots,N-2n-1\right\}.$$

Now, the variational formulation (10) is equivalent to

$$\begin{aligned}&\left((-1)^{n+1}\ y^n_N(x),\psi^*_{i,n}(x)\right)_{w_1}\\&\quad+\sum_{k=0}^{2n}\mu_k\left(\int^{(2n+1-k)}y^n_N(x)(dx)^{(2n+1-k)},\psi^*_{i,n}(x)\right)_{w_1}\\&\quad=\left(F(x)+\sum_{k=0}^{2n}\delta_k\ V^*_k(x),\psi^*_{i,n}(x)\right)_{w_1},\quad 0\leqslant i\leqslant N-2n-1.\end{aligned} \tag{17}$$

The constants δ_k, $0\leqslant k\leqslant 2n$, would disappear if we take $i\geqslant 2n+1$ in (17), then we have

$$\begin{aligned}&\left((-1)^{n+1}\ y^n_N(x),\psi^*_{i,n}(x)\right)_{w_1}\\&\quad+\sum_{k=0}^{2n}\mu_k\left(\int^{(2n+1-k)}y^n_N(x)(dx)^{(2n+1-k)},\psi^*_{i,n}(x)\right)_{w_1}\\&\quad=\left(F(x),\psi^*_{i,n}(x)\right)_{w_1},\quad 2n+1\leqslant i\leqslant N.\end{aligned} \tag{18}$$

Let us denote

$$\begin{aligned}&A_n=\left(a^n_{ij}\right)_{2n+1\leqslant i,j\leqslant N},\\&a^n_{ij}=(-1)^{n+1}\left(\phi^*_{j,n}(x),\psi^*_{i,n}(x)\right)_{w_1},\\&B_{2n+1-m,n}=\left(b^{2n+1-m,n}_{ij}\right)_{2n+1\leqslant i,j\leqslant N},\\&b^{2n+1-m,n}_{ij}=\left(\int^{(2n+1-m)}\phi^*_{j,n}(x)(dx)^{(2n+1-m)},\psi^*_{i,n}(x)\right)_{w_1},\\&\mathbf{F}^n=\left(F^n_0,F^n_1,\ldots,F^n_{N-2n-1}\right)^T,\\&F^n_i=\left(F(x),\psi^*_{i,n}(x)\right)_{w_1},\\&y^n_N(x)=\sum_{k=0}^{N-2n-1}c^n_k\phi^*_{k,n}(x),\\&\mathbf{c}^n=\left(c^n_0,c^n_1,\ldots,c^n_{N-2n-1}\right)^T.\end{aligned}$$

Now Eq. (18), can be transformed into the following linear system

$$\left(A_n+\sum_{m=0}^{2n}\mu_m\ B_{2n+1-m,n}\right)\mathbf{c}^n=\mathbf{F}^n, \tag{19}$$

where the nonzero elements of the matrices A_n and $B_{2n+1-m,n}$ $(0\leqslant m\leqslant 2n)$ are given explicitly in the following theorem.

Theorem 3. *If the trial and test functions $\phi^*_{i,n}(x)$ and $\psi^*_{i,n}(x)$ are chosen as in (13) and (14), respectively, and if we set $a^n_{ij}=(-1)^{n+1}\left(\phi^*_{j-2n-1,n}(x),\psi^*_{i,n}(x)\right)_{w_1}$, and $b^{2n+1-m,n}_{ij}=\left(\int^{(2n+1-m)}\phi^*_{j-2n-1,n}(x)(dx)^{(2n+1-m)},\psi^*_{i,n}(x)\right)_{w_1}$, $0\leqslant m\leqslant 2n$, then*

$$\Phi_{N+2n+1}=span\{\phi^*_{0,n}(x),\phi^*_{1,n}(x),\cdots,\phi^*_{N,n}(x)\},$$

$$\Psi_{N+2n+1}=span\{\psi^*_{0,n}(x),\psi^*_{1,n}(x),\cdots,\psi^*_{N,n}(x)\},$$

and the nonzero elements of the matrices A_n and $B_{2n+1-m,n}$ $(0\leqslant m\leqslant 2n)$ are given explicitly by:

$$a^n_{ii}=\frac{\pi(a-b)(i-2n)_n(2i-2n-1)}{2(i-n+1)_n(2i-2n+1)}, \tag{20}$$

$$a^n_{ij}=(-1)^{n+1}(b-a)\left(\frac{\pi}{2}\right)\sum_{k=0}^{2n+1}p_{k,j-2n-1}\ q_{j-i-2n-1+k,i},\quad j=i+s,s\geqslant 1, \tag{21}$$

$$\begin{aligned}b^{2n+1-m,n}_{ij}&=\pi\left(\frac{b-a}{2}\right)^{2n-m+2}\sum_{k=0}^{2n+1}\sum_{\ell=0}^{2n+1}p_{k,j-2n-1}\ q_{\ell,i}\\&\quad G_{j-i+k-\ell-m,j-2n-1+k,2n+1-m},\quad j=i+s,\\&\quad m-2n-1\leqslant s\leqslant 6n-m+3,\quad 0\leqslant m\leqslant 2n,\end{aligned} \tag{22}$$

where $G_{k,i,m},p_{k,i}$ and $q_{k,i}$ are as defined in (3), (15) and (16) respectively.

Proof. The basis functions $\phi^*_{i,n}(x)$ and their dual functions $\psi^*_{i,n}(x)$ are chosen such that $\phi^*_{i,n}(x)\in\Phi_{N+2n+1}$ and $\psi^*_{i,n}(x)\in\Psi_{N+2n+1}$ for $i=0,1,\ldots,N$. Moreover, it is clear that $\{\phi^*_{i,n}(x)\}_{0\leqslant i\leqslant N}$ and $\{\psi^*_{i,n}(x)\}_{0\leqslant i\leqslant N}$ are linearly independent and the dimension of both Φ_{N+2n+1} and Ψ_{N+2n+1} is equal to $(N+1)$. Hence,

$$\Phi_{N+2n+1}=\operatorname{span}\{\phi^*_{0,n}(x),\phi^*_{1,n}(x),\ldots,\phi^*_{N,n}(x)\},$$

and

$$\Psi_{N+2n+1}=\operatorname{span}\{\psi^*_{0,n}(x),\psi^*_{1,n}(x),\ldots,\psi^*_{N,n}(x)\}.$$

To prove (20) and (21), we make use of the two formulae (13) and (14), to get

$$a^n_{ij}=(-1)^{n+1}\sum_{k=0}^{2n+1}\sum_{\ell=0}^{2n+1}p_{k,j-2n-1}\ q_{\ell,i}\left(V^*_{j-2n-1+k}(x),V^*_{i+\ell}(x)\right)_{w_1},$$

which in turn, with the aid of the orthogonality relation (6), yields

$$a^n_{ij}=(-1)^{n+1}(b-a)\left(\frac{\pi}{2}\right)\sum_{k=0}^{2n+1}p_{k,j-2n-1}\ q_{j-i-2n-1+k,i},\quad j=i+s,s\geqslant 1.$$

If we make use of relations (15) and (16), then a^n_{ij} can be evaluated to give relations (20) and (21). To prove relation (22), we have for $0 \leqslant m \leqslant 2n$

$$b^{2n+1-m,n}_{ij} = \left(\int^{(2n+1-m)} \phi^*_{j-2n-1,n}(x)(dx)^{(2n+1-m)}, \psi^*_{i,n}(x)\right)_{w_1}$$
$$= \left(\sum_{k=0}^{2n+1} p_{k,j-2n-1}\ \bar{I}^{(2n+1-m)}_{j-2n-1+k}(x), \sum_{\ell=0}^{2n+1} q_{\ell,i}\ V^*_{i+\ell}(x)\right)_{w_1},$$

and the application of Theorem 2 yields

$$\bar{I}^{(2n+1-m)}_{j-2n-1+k}(x) = \left(\frac{b-a}{2}\right)^{2n+1-m} \sum_{\ell=0}^{2(2n+1-m)} G_{\ell,j-2n-1+k,2n+1-m}\ V^*_{j+k-m-\ell}(x), \tag{23}$$

where $G_{k,i,m}$ is as given in (3), which leads after making use of the orthogonality relation (6) and relation (23), to the relation

$$b^{2n+1-m,n}_{ij} = \pi\left(\frac{b-a}{2}\right)^{2n-m+2} \sum_{k=0}^{2n+1}\sum_{\ell=0}^{2n+1} p_{k,j-2n-1}\ q_{\ell,i}\ G_{j-i+k-\ell-m,j-2n-1+k,2n+1-m}, 0 \leqslant m \leqslant 2n, j = i+s, m-2n-1 \leqslant s \leqslant 6n-m+3,$$

and this completes the proof of Theorem 3. □

Now, and based on Theorem 3, it is clear that the case corresponds to $\mu_m = 0,\ 0 \leqslant m \leqslant 2n$, leads to linear system with nonsingular upper triangular matrices. This result is given in the following corollary.

Corollary 2. *If $\mu_m = 0,\ 0 \leqslant m \leqslant 2n$, then the linear system (19) takes the matrix form $A_n\ \mathbf{c}^n = \mathbf{F}^n$, where A_n is an upper triangular matrix whose solution can be immediately obtained by the backward substitution as:*

$$c^n_i = \left(F^n_{i+2n+1} - \sum_{j=i+2n+2}^{N} a^n_{i+2n+1,j}\ c^n_j\right)/a^n_{i+2n+1,i+2n+1},$$
$$i = 0, 1, \ldots, N-2n-1,$$

where the elements a^n_{ii} and a^n_{ij} are given by (20) and (21) respectively.

Remark 1. If we consider the equation

$$(-1)^{n+1} y^{(2n+1)}(x) + \sum_{k=0}^{2n} \mu_k\ y^{(k)}(x) = f(x), \quad x \in (a,b),\ n \geqslant 1, \tag{24}$$

governed by the nonhomogeneous boundary conditions

$$y^{(m)}(a) = \alpha_m,\ y^{(m)}(b) = \beta_m,\ y^{(n)}(a) = \gamma, \quad m = 0, 1, \ldots, n-1, \tag{25}$$

then, and with the aid of a suitable transformation, Eq. (24)–(25), can be converted to an equation similar to (7)–(8).

4. Solution of high odd-order differential equations by using fourth kind Chebyshev polynomials

In this section, we are concerned with the solution of the same problems discussed in Section 3, but by using SC4PGM. All results of this section are given without proofs.

4.1. High odd-order two point boundary value problems SC4PGM

Let us consider the same one dimensional high odd-order differential Eqs. (7) governed by the same homogeneous boundary conditions (8). If we define the following spaces

$$\overline{S}_N = \text{span}\{W^*_0(x), W^*_1(x), W^*_2(x), \ldots, W^*_N(x)\},$$

$$\bar{\Phi}_N = \{\bar{\phi}(x) \in \overline{S}_N :\ \bar{\phi}^{(m)}(a) = \bar{\phi}^{(m)}(b) = \bar{\phi}^{(n)}(a) = 0, \quad m = 0, 1, \ldots, n-1\},$$

$$\bar{\Psi}_N = \{\bar{\psi}(x) \in \overline{S}_N :\ \bar{\psi}^{(m)}(a) = \bar{\psi}^{(m)}(b) = \bar{\psi}^{(n)}(b) = 0, \quad m = 0, 1, \ldots, n-1\},$$

then, the shifted Chebyshev fourth kind spectral Petrov–Galerkin procedure for solving (7)–(8) is to find $\bar{y}^n_N \in \bar{\Phi}_N$ such that

$$\left((-1)^{n+1}\ \bar{y}^n_N(x), \bar{\psi}(x)\right)_{w_2} + \sum_{k=0}^{2n} \mu_k \left(\int^{(2n+1-k)} \bar{y}^n_N(x)(dx)^{(2n+1-k)}, \bar{\psi}(x)\right)_{w_2} = \left(F(x) + \sum_{k=0}^{2n} \delta_k\ W^*_k(x), \bar{\psi}(x)\right)_{w_2}, \quad \forall\ \bar{\psi}(x) \in \bar{\Psi}_N.$$

In such case, the basis functions and their dual basis functions are selected to take the following forms:

$$\bar{\phi}^*_{i,n}(x) = \sum_{k=0}^{2n+1} \bar{p}_{k,i} W^*_{i+k}(x), \tag{26}$$

$$\bar{\psi}^*_{i,n}(x) = \sum_{k=0}^{2n+1} \bar{q}_{k,i} W^*_{i+k}(x), \tag{27}$$

where the constants $\bar{p}_{k,i}$ and $\bar{q}_{k,i}$ can be computed to give:

$$\bar{p}_{k,i} = \begin{cases} q_{k,i}, & k = 0, 2, \ldots, 2n, \\ -q_{k,i}, & k = 1, 3, \ldots, 2n+1, \end{cases} \tag{28}$$

$$\bar{q}_{k,i} = \begin{cases} p_{k,i}, & k = 0, 2, \ldots, 2n, \\ -p_{k,i}, & k = 1, 3, \ldots, 2n+1, \end{cases} \tag{29}$$

and the coefficients $p_{k,i}$ and $q_{k,i}$ are given in (15) and (16), respectively.

Now, the main result of this section is given in the following theorem.

Theorem 4. *If the trial and test functions $\bar{\phi}^*_{i,n}(x)$ and $\bar{\psi}^*_{i,n}(x)$ are chosen as given in (26) and (27), respectively, and if $\bar{y}^n_N(x) = \sum_{k=0}^{N-2n-1} c^n_k\ \bar{\phi}^*_{k,n}(x)$, is the Petrov–Galerkin approximation to (9), then the expansion coefficients $\{c^n_k, k = 0, 1, \ldots, N-2n-1\}$ satisfy the matrix system:*

$$\left(\bar{A}_n + \sum_{m=0}^{2n} \mu_m\ \bar{B}_{2n+1-m,n}\right)\mathbf{c}^n = \bar{\mathbf{F}}^n, \tag{30}$$

where the nonzero elements of the matrices $\overline{A}_n$ and $\overline{B}_{2n+1-m,n}$ $(0 \leqslant m \leqslant 2n)$ are given explicitly as follows:

$$\bar{a}^n_{ii} = \frac{\pi(b-a)(i-2n)_n}{2(i-n+1)_n}, \quad (31)$$

$$\bar{a}^n_{ij} = (-1)^{n+1}(b-a)\left(\frac{\pi}{2}\right)\sum_{k=0}^{2n+1} \bar{p}_{k,j-2n-1}\,\bar{q}_{j-i-2n-1+k,i} \quad j=i+s, s \geqslant 1, \quad (32)$$

$$\bar{b}^{2n+1-m,n}_{ij} = \pi\left(\frac{b-a}{2}\right)^{2n-m+2}\sum_{k=0}^{2n+1}\sum_{\ell=0}^{2n+1} \bar{p}_{k,j-2n-1}\,\bar{q}_{\ell,i}\,\overline{G}_{j-i+k-\ell-m,j-2n-1+k,2n+1-m},$$

$$j = i+s, \quad m-2n-1 \leqslant s \leqslant 6n-m+3, \quad 0 \leqslant m \leqslant 2n,$$

where $\overline{G}_{k,i,m}, \bar{p}_{k,i}$ *and* $\bar{q}_{k,i}$ *are as defined in (4), (28) and (29) respectively.*

We note that the case corresponds to $\mu_m = 0,\ 0 \leqslant m \leqslant 2n$, leads to linear system with nonsingular upper triangular matrices. The result for such case is summarized in the following corollary.

Corollary 3. *If* $\mu_m = 0,\ 0 \leqslant m \leqslant 2n$, *then the system (30) reduces to the matrix equation* $\overline{A}_n\,\mathbf{c}^n = \bar{\mathbf{F}}^n$, *where* $\overline{A}_n$ *is an upper triangular matrix whose solution can directly obtained by the backward substitution*

$$c^n_i = \left(\overline{F}^n_{i+2n+1} - \sum_{j=i+2n+2}^{N} \bar{a}^n_{i+2n+1,j}\ c^n_j\right) \Big/ \bar{a}^n_{i+2n+1,i+2n+1},$$
$$i = 0, 1, \ldots, N-2n-1,$$

where $\bar{a}^n_{ii}$ and $\bar{a}^n_{ij}$ are given by (31) and (32) respectively.

5. Condition numbers of the two resulting matrices in systems (19) and (30)

In this section, we discuss the condition numbers of the two resulting matrices that appear in the two linear systems (19) and (30). If we apply the two spectral methods namely, SC3PGM and SC4PGM, then the two linear systems resulted from solving the integral equation of the equation $\left((-1)^{n+1}\ y^{(2n+1)}(x) = f(x)\right)$ are given respectively by $A_n\mathbf{c}^n = \mathbf{F}^n$ and $\bar{A}_n\mathbf{c}^n = \bar{\mathbf{F}}^n$. Thus, $\forall n \geqslant 1$, we note that the condition number of the matrix $\bar{A}_n$ behaves like $O(N)$ for large values of N, while the condition number of the matrix A_n is independent of N. Moreover, it has been noted that the combined matrices $\overline{D}_n = \overline{A}_n + \sum_{m=0}^{2n} \overline{B}_{2n+1-m,n}$ in (30) and $D_n = A_n + \sum_{m=0}^{2n} B_{2n+1-m,n}$ in (19) have the same conditions numbers of the matrices $\overline{A}_n$ and A_n, respectively. Hence the propagation of roundoff errors should not be very significant. To ascertain these observations, Table 1 illustrates the condition numbers for the matrices A_n and $D_n = A_n + \sum_{m=0}^{2n} B_{2n+1-m,n}$ in (19) for some values of the parameter N and $(a,b) = (-1,1), n = 1,2$, while Table 2 illustrates the condition numbers for the matrices $\overline{A}_n$ and $\overline{D}_n = \overline{A}_n + \sum_{m=0}^{2n} \overline{B}_{2n+1-m,n}$ in (30), $n = 1,2$, for the same values of N and in the same interval.

Remark 2. It should be noted here that the main advantage of following the integrated form approach over the differentiated one is that the condition numbers in the integrated approach are smaller than those obtained via the differentiated approach. It can be shown that if the two algorithms SC3PGM and SC4PGM are applied but on the differentiated forms (7) and (8), then the condition numbers of the matrices appear in

Table 1 Condition number of the matrices A_n and D_n.

N	n	$\kappa(A_n)$	$\kappa(D_n)$	n	$\kappa(A_n)$	$\kappa(D_n)$
16		4.093	8.611		8.472	7.434
24	1	4.389	8.849	2	10.179	9.034
32		4.539	8.880		11.084	9.819
40		4.629	8.881		11.643	10.279

Table 2 Condition number of the matrices $\overline{A}_n$ and $\overline{D}_n$.

N	n	$\kappa\left(\frac{\overline{A}_n}{N}\right)$	$\kappa\left(\frac{\overline{D}_n}{N}\right)$	n	$\kappa\left(\frac{\overline{A}_n}{N}\right)$	$\kappa\left(\frac{\overline{D}_n}{N}\right)$
16		3.031	2.293		0.663	0.395
24	1	4.561	3.544	2	0.951	0.600
32		6.087	4.797		1.241	0.809
40		7.613	6.052		1.531	1.021

Table 3 Error E for $N = 8, 10, 12, 14, 16, 18$ for Example 1.

N	ϵ	SC3PGM	SC4PGM
	1/16	1.3×10^{-6}	1.3×10^{-6}
8	1/32	5.5×10^{-7}	5.3×10^{-7}
	1/64	2.2×10^{-7}	2.1×10^{-7}
	1/16	1.4×10^{-8}	1.1×10^{-8}
10	1/32	5.8×10^{-9}	4.7×10^{-9}
	1/64	2.4×10^{-9}	1.9×10^{-9}
	1/16	9.8×10^{-11}	6.6×10^{-11}
12	1/32	4.2×10^{-11}	2.8×10^{-11}
	1/64	1.7×10^{-11}	1.1×10^{-11}
	1/16	4.9×10^{-13}	2.7×10^{-13}
14	1/32	2.1×10^{-13}	1.2×10^{-13}
	1/64	8.4×10^{-14}	4.6×10^{-14}
	1/16	6.5×10^{-16}	1.6×10^{-15}
16	1/32	7.4×10^{-16}	2.5×10^{-16}
	1/64	2.2×10^{-16}	1.7×10^{-16}

Table 4 Comparison between different methods for Example 1.

ϵ	Methods in [20]	SC3PGM	SC4PGM
1/16	5.4×10^{-6}	6.5×10^{-16}	1.6×10^{-15}
1/32	2.8×10^{-6}	7.4×10^{-16}	2.5×10^{-16}
1/64	1.4×10^{-7}	2.2×10^{-16}	1.7×10^{-16}

Table 5 Error E for $N = 8, 10, 12, 14, 16$ for Example 2.

N	SC3PGM	SC4PGM
8	1.2×10^{-6}	7.6×10^{-7}
10	3.4×10^{-9}	1.8×10^{-9}
12	5.1×10^{-12}	2.3×10^{-12}
14	4.8×10^{-15}	1.9×10^{-15}
16	1.1×10^{-16}	9.0×10^{-17}

Table 6 Comparison between different methods for Example 2.

x	Methods in [21,22]	Method in [23]	Method in [24]	Method in [25]	Method in [26]	SC3PGM	SC4PGM
0.1	1.0×10^{-9}	7.0×10^{-4}	2.3×10^{-7}	0.0	3.38×10^{-15}	5.6×10^{-17}	5.6×10^{-17}
0.2	2.0×10^{-9}	7.2×10^{-4}	1.6×10^{-6}	1.0×10^{-5}	4.47×10^{-15}	1.1×10^{-16}	1.1×10^{-16}
0.3	1.0×10^{-8}	4.1×10^{-4}	4.6×10^{-6}	1.0×10^{-5}	3.44×10^{-15}	0.0	5.6×10^{-17}
0.4	2.0×10^{-8}	4.6×10^{-4}	8.9×10^{-6}	1.0×10^{-4}	6.88×10^{-16}	5.6×10^{-17}	0.0
0.5	3.1×10^{-8}	4.7×10^{-4}	1.3×10^{-5}	3.2×10^{-4}	3.17×10^{-15}	0.0	5.6×10^{-17}
0.6	3.7×10^{-8}	4.8×10^{-4}	1.6×10^{-5}	3.6×10^{-4}	7.28×10^{-15}	0.0	0.0
0.7	4.1×10^{-8}	3.9×10^{-4}	1.6×10^{-5}	1.4×10^{-4}	1.05×10^{-14}	5.6×10^{-17}	5.6×10^{-17}
0.8	3.1×10^{-8}	3.1×10^{-4}	1.2×10^{-5}	3.1×10^{-4}	1.15×10^{-14}	0.0	5.6×10^{-17}
0.9	1.4×10^{-8}	1.6×10^{-4}	5.1×10^{-6}	5.8×10^{-4}	8.68×10^{-15}	5.6×10^{-17}	1.1×10^{-16}

the resulting two linear systems are both of $O(N^{2n+1})$, for large values of N.

6. Numerical results

In this section, we give two numerical results for the sake of testing the efficiency and the applicability of the proposed algorithms of Sections 3 and 4. We consider the following two examples.

Example 1. Consider the following singulary perturbed linear third-order boundary value problem (see, [20]):

$$-\epsilon\, y^{(3)}(x) + y(x) = 81\ \epsilon^2\ \cos(3x) + 3\ \epsilon \sin(3x), \quad 0 \leqslant x \leqslant 1,$$

subject to the boundary conditions

$$y(0) = 0, \quad y(1) = 3\ \epsilon \sin(3), \quad y^{(1)}(0) = 9\ \epsilon,$$

with the analytic solution $y(x) = 3\ \epsilon \sin(3x)$.

In Table 3, we list the maximum pointwise error $E = |y - y_N|$ using (SC3PGM and SC4PGM) with various values of N and ϵ, while in Table 4, we introduce a comparison between the best absolute errors obtained by (SC3PGM and SC4PGM) in case of $N = 18$ with the best absolute errors obtained by using method in [20]. This table shows that our two methods are more accurate if compared with the method developed in [20].

Example 2. Consider the following linear fifth-order boundary value problem (see, [21–26]):

$$u^{(5)}(x) = u(x) - 15\ e^x - 10\ x\ e^x, \quad 0 \leqslant x \leqslant 1,$$

governed by the boundary conditions

$$u(0) = 0, \quad u^{(1)}(0) = 1, \quad u^{(2)}(0) = 0, \quad u(1) = 0, \quad u^{(1)}(1) = -e.$$

The exact solution of this problem is $u(x) = x(1-x)e^x$.

Table 5 lists the maximum pointwise error $E = |u - u_N|$ using (SC3PGM and SC4PGM) with various values of N. In Table 6, we introduce a comparison between the best absolute errors obtained by (SC3PGM and SC4PGM) at $N = 16$ with the best error obtained by using the methods developed in [21–26] is displayed in Table 6. This table shows that our two methods are more accurate if compared with the methods obtained in all of these papers.

7. Concluding remarks

We have presented some efficient direct solvers for the integrated forms of high odd-order differential equations using shifted Chebyshev polynomials of third and fourth kinds based on applying Petrov–Galerkin spectral method. We have found that for some particular differential equations, the resulting systems of linear equations are upper triangular, and this is certainly reduces the computational cost for the numerical solutions for these special cases. The presented numerical examples exhibit the high accuracy and efficiency of the proposed algorithms.

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