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ORIGINAL ARTICLE

A new generalized Weibull distribution generated by gamma random variables

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Abstract We propose a new and simple representation for the probability density function of the gamma-G family of distributions as an absolutely convergent power series of the cumulative function of the baseline G distribution. Additionally, the special case the so-called gamma exponentiated Weibull model is introduced and studied in details.

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1. Introduction

The two-parameter Weibull distribution is a very popular distribution that has been extensively used over the past decades for modeling data in reliability, engineering and biological studies. It is well-known that the major weakness of the Weibull distribution is its inability to accommodate nonmonotone failure rates. The first generalization of the two-parameter Weibull distribution to accommodate nonmonotone failure rates was introduced by [1] and it is known as the exponentiated Weibull (EW) distribution. The three-parameter EW distribution has cumulative function in the form $G_{\rm EW}(x) = G_{\rm EW}(x; b, \alpha, \beta) = (1 - {\rm e}^{-\alpha x^{\beta}})^{b}$, $x > 0$, where $\beta > 0$ and $b > 0$ are shape parameters, and $\alpha > 0$ is the scale parameter. The EW density function is $g_{\rm EW}(x) = g_{\rm EW}(x; b, \alpha, \beta) = b\ \alpha\ \beta\ x^{\beta-1}\ {\rm e}^{-\alpha x^{\beta}}(1 - {\rm e}^{-\alpha x^{\beta}})^{b-1}$, $x > 0$. The reader is refereed to [2] for an overview of the EW distribution.

The recent literature has suggested several other ways of extending well-known distributions. The earliest is the class of distributions generated by a standard beta random variable introduced by [3]. The more recent ones are as follows: the class of distributions generated by [4]'s random variable introduced by [5]; the class of distributions generated by gamma random variables introduced by [6,7]; the class of distributions generated by [8]'s generalized beta random variable introduced by [9]; and the *T-X* family of distributions introduced in [10]. Some of the above methods were recently discussed in [11]. By using the generator approach suggested by [3], several generalized distributions have been proposed in the last few years. In particular, [3,12,13] defined the beta normal, beta Fréchet, beta Gumbel, beta exponential, beta Weibull and beta Pareto distributions by taking $G(x)$ to be the cumulative function of the normal, Fréchet, Gumbel, exponential, Weibull and Pareto distributions, respectively. More recently, [14–20] defined the

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beta generalized exponential, beta generalized half-normal, beta modified Weibull, beta Burr XII, beta Birnbaum–Saunders, beta Laplace and beta half-Cauchy distributions, respectively. Some generalized distributions generated by [4]'s random variable are proposed in [21–23]. Recently, a five-parameter continuous model generated by [8]'s generalized beta random variable was proposed by [24]. As can be observed from these references, several new generalized distributions were constructed from the logit of a beta random variable. On the other hand, the generator approaches in [5–7,9,10] have not been much explored for generating new classes of generalized distributions. We refer the reader to [25,26] for some generalized distributions constructed by using the generator approach of [10].

In this paper, we use the generator approach of [6] to introduce a new generalized Weibull family of distributions. The generator approach introduced by these authors is as follows. For any continuous baseline cumulative distribution function (cdf) $G(x) = G(x;\boldsymbol{\tau})$ and parameter vector $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_q)^\top$ of dimension q, the cumulative function of the new distribution is defined by $F(x) = F(x; a, \boldsymbol{\tau}) = \Gamma(a)^{-1}\gamma(a, -\log[1-G(x)])$, $x \in \mathbb{R}$, where $a > 0$ is an additional shape parameter to those in $\boldsymbol{\tau}$ that aims to introduce skewness and to provide greater flexibility of its tails. Also, $\Gamma(r) = \int_0^\infty t^{r-1}\mathrm{e}^{-t}\mathrm{d}t$ is the gamma function, and $\gamma(r,s) = \int_0^s t^{r-1}\mathrm{e}^{-t}\mathrm{d}t$ is the incomplete gamma function. From now on, the cdf $G(x)$ will be referred to as the parent distribution or baseline distribution. The probability density function (pdf) of the new distribution takes the form

$$f(x) = f(x; a, \boldsymbol{\tau}) = \frac{g(x)}{\Gamma(a)}\{-\log[1-G(x)]\}^{a-1}, \quad x \in \mathbb{R}, \tag{1}$$

where $g(x) = g(x;\boldsymbol{\tau}) = \mathrm{d}G(x)/\mathrm{d}x$ is the baseline pdf. For $a = 1$, $f(x) = g(x)$ and, therefore, $g(x)$ is a basic exemplar of (1). Further, if Z has a gamma distribution, $Z \sim \mathrm{Gamma}(a, 1)$ say, with density function $h(z) = \Gamma(a)^{-1}z^{a-1}\mathrm{e}^{-z}(z > 0)$, then the random variable $X = G^{-1}(1-\mathrm{e}^{-Z})$ has pdf given by Eq. (1). In this paper, we shall refer to (1) as the gamma G (Γ-G) distribution.

Recently, [7] used a similar approach presented in [6] to introduce a new family of distributions generated by gamma random variables. They define $F(x)$ in the form $F(x) = F(x; a, \boldsymbol{\tau}) = 1 - \Gamma(a)^{-1}\gamma(a, -\log[G(x)])$, for $x \in \mathbb{R}$, whereas the pdf is

$$f(x) = f(x; a, \boldsymbol{\tau}) = \frac{g(x)}{\Gamma(a)}\{-\log[G(x)]\}^{a-1}, \quad x \in \mathbb{R}. \tag{2}$$

Some interesting motivations for this new class of distributions are provided by [7]. In particular, if $Z \sim \mathrm{Gamma}(a, 1)$, then the random variable $X = G^{-1}(\mathrm{e}^{-Z})$ has pdf given by Eq. (2). Thus, accordingly to [7], the new family of distributions may be regarded as a dual family of the Zografos–Balakrishnan's family of distributions. Also, let Z be a random variable with log-gamma distribution with density function $h(z) = \Gamma(a)^{-1}\exp(az - \mathrm{e}^z)$, $z \in \mathbb{R}$. Then, the random variable $X = G^{-1}(\exp(-\mathrm{e}^Z))$ also has the pdf (2). We shall refer to (2) as the gamma dual G ($\Gamma 2$-G) distribution.

The purposes of the present paper are twofold. First, we propose a new representation for the pdf of the Γ-G model as an absolutely convergent power series of the cumulative function of the baseline distribution. Second, we use the generator approach suggested by [6] to define a new model, called the gamma exponentiated Weibull (Γ-EW) distribution, which generalizes the exponentiated exponential, Weibull and EW models. In addition, we investigate some structural properties of the new model and discuss maximum likelihood estimation of its parameters. The proposed model is much more flexible than the Weibull and EW distributions and can be used effectively for modeling positive real data in many areas. A real data example is presented to show the flexibility of the Γ-EW model over other lifetime models in practice.

Recently, a new four-parameter generalization of the Weibull distribution was introduced in [27] by using the generator approach of [7], named here as the *gamma dual exponentiated Weibull distribution* ($\Gamma 2$-EW). Unfortunately, the expansion for the $\Gamma 2$-EW density function derived by these authors, which is used to obtain some general properties of this model, is not a valid expansion, i.e. not convergent (see Appendix), and hence some properties of the $\Gamma 2$-EW distribution presented in their paper like moments, moment generating function, etc., do not work. The general expansion derived in this paper for the Γ-G density function, however, is a valid expansion (i.e. convergent). In particular, we use this general expansion to derive the moments, moment generating function, etc., of the new four-parameter Γ-EW distribution.

2. Expansion for the *G* density function

In what follows, we derive a very useful representation for the Γ-G density function, which can be used to derive general properties (moments, entropy, etc.) of this new class of distributions. It should be noticed that a representation for the $\Gamma 2$-G density function can be directly obtained from the representation for the Γ-G density function simply by replacing the baseline cdf $G(x)$ with the survival function of the baseline G distribution, that is, by replacing $G(x)$ with $S(x) = 1 - G(x)$.

It can be shown that

$$\left[-\frac{\log(1-z)}{z}\right]^{\delta} = 1 + \delta z \sum_{n=0}^{\infty} \psi_n(n+\delta) z^n, \tag{3}$$

where $\delta \in \mathbb{R}$, $|z| < 1$ and the coefficients $\psi_n(\cdot)$ are Stirling polynomials. These coefficients can be expressed in the form

$$\psi_{n-1}(w) = \frac{(-1)^{n-1}}{(n+1)!}\left[H_n^{n-1} - \frac{w+2}{n+2}H_n^{n-2} + \frac{(w+2)(w+3)}{(n+2)(n+3)}H_n^{n-3} - \cdots + (-1)^{n-1}\frac{(w+2)(w+3)\cdots(w+n)}{(n+2)(n+3)\cdots(2n)}H_n^0\right], \tag{4}$$

where H_n^m are positive integers defined recursively by $H_{n+1}^m = (2n+1-m)H_n^m + (n-m+1)H_n^{m-1}$, with $H_0^0 = 1$, $H_{n+1}^0 = 1 \times 3 \times 5 \times \cdots \times (2n+1)$, and $H_{n+1}^n = 1$. The first six polynomials are $\psi_0(w) = 1/2$, $\psi_1(w) = (2+3w)/24$, $\psi_2(w) = (w+w^2)/48$, $\psi_3(w) = (-8-10w+15w^2+15w^3)/5760$, $\psi_4(w) = (-6w-7w^2+2w^3+3w^4)/11520$ and $\psi_5(w) = (96+140w-224w^2-315w^3+63w^5)/2903040$.

Remark 1. According to another definition,[1] the polynomials $S_0(w) = 1$ and $S_n(w) = n!(w+1)\psi_{n-1}(w)$, $n \geqslant 1$, are also known as *Stirling polynomials*. In this article, we use this

[1] See, for example, http://mathworld.wolfram.com/StirlingPolynomial.html.

terminology to refer to the polynomials $\psi_n(w)$ in accordance with [28,29].

We have the following propositions.

Proposition 1. *The expansion (3) is absolutely convergent.*

Proof. The proof is given in details by [30] or [31] (see Theorem VI.2, page 385). □

Proposition 2. *The expansion (3) can be expressed as* $[-\log(1-z)]^{\delta} = z^{\delta}\sum_{m=0}^{\infty}\rho_m(\delta)\, z^m$, *where* $\delta \in \mathbb{R}$, $|z|<1$, $\rho_0(\delta)=1$, $\rho_m(\delta)=\delta\psi_{m-1}(m+\delta-1)$ *for* $m \geqslant 1$, *and the coefficients* $\psi_m(\cdot)$ *are Stirling polynomials given by* (4).

Proof. It follows from the results by [28,29]. □

We have the following theorem.

Theorem 1. *The Γ-G density function admits the expansion*

$$f(x) = g(x)\sum_{m=0}^{\infty}\varphi_m(a)\; G(x)^{m+a-1}, \quad x \in \mathbb{R}, \tag{5}$$

where $a>0$, $\varphi_0(a)=\Gamma(a)^{-1}$, $\varphi_m(a)=\Gamma(a)^{-1}\rho_m(a-1)=(a-1)\Gamma(a)^{-1}\psi_{m-1}(m+a-2)$ *for* $m \geqslant 1$, *and the coefficients* $\psi_m(\cdot)$ *are Stirling polynomials given by* (4).

Proof. It follows from Proposition 2. □

Remark 2. Theorem 1 shows that the Γ-G pdf can be expressed as an absolutely convergent power series of the distribution function of the baseline distribution. The general expansion in (5) is very useful to obtain properties of interest of the Γ-G distribution. In Section 5, we will use this expansion to derive some properties of the Γ-EW model.

3. The generalized Weibull distribution

The cdf of the new four-parameter Γ-EW distribution is given by $F(x)=\Gamma(a)^{-1}\,\gamma(a,-\log[1-(1-e^{-\alpha x^{\beta}})^{b}])$, $x>0$, where $a>0$, $b>0$ and $\beta>0$ are shape parameters, and $\alpha>0$ is the scale parameter. If X has the Γ-EW distribution, we use the notation $X \sim \Gamma\text{-EW}(a,b,\alpha,\beta)$. If $X \sim \Gamma\text{-EW}(a,b,\alpha,\beta)$, then $kX \sim \Gamma\text{-EW}(a,b,\alpha^*,\beta)$, where $\alpha^*=\alpha k^{-\beta}$ and $k>0$, i.e. the class of Γ-EW distributions is closed under scale transformations. The EW, Weibull and exponentiated exponential (EE) distributions are clearly the most important sub-models for $a=1$, $a=b=1$ and $a=\beta=1$, respectively. Other sub-models can be immediately defined from Table 1.

The Γ-EW pdf takes the form

$$f(x) = \frac{b\alpha\beta x^{\beta-1}e^{-\alpha x^{\beta}}(1-e^{-\alpha x^{\beta}})^{b-1}}{\Gamma(a)\{-\log[1-(1-e^{-\alpha x^{\beta}})^{b}]\}^{1-a}}, \quad x>0, \tag{6}$$

whereas the Γ-EW failure rate function is defined as $r(x)=f(x)/[1-F(x)]$, for $x>0$. Evidently, the new density function (6) does not involve any complicated function. Also, there is no functional relationship among the parameters and they vary freely in the parameter space. If $Z \sim \text{Gamma}(a,1)$, then $X \sim \Gamma\text{-EW}(a,b,\alpha,\beta)$ is given by $X=\alpha^{-1/\beta}\{-\log[1-(1-e^{-Z})^{1/b}]\}^{1/\beta}$, which can be used to generate Γ-EW random variates.

The new Γ-EW distribution has three shape parameters: $a>0$, $b>0$ and $\beta>0$. These parameters allow for a high degree of flexibility of the Γ-EW distribution. Fig. 1 illustrates some possible shapes of the density function (6) for selected parameter values. Notice that the additional shape parameter a allows for a high degree of flexibility of the Γ-EW distribution; that is, it introduces more flexibility to the new model both in terms of skewness and in terms of kurtosis. It is interesting to note that the Γ-EW pdf can also be approximately symmetric depending on the parameter values. It is worth emphasizing that the Γ-EW distribution can be applied in survival analysis, engineering, biological studies, hydrology, economics, among others, as the Weibull distribution and it can be used to model reliability problems.

We display in Fig. 2 some plots of the Γ-EW failure rate function for some parameter values to show the flexibility of this model. The parameter α does not change the shape of the failure rate function since it is a scale parameter. It is evident that the failure rate function of the proposed Γ-EW distribution can be increasing, decreasing, upside-down bathtub shaped (unimodal) or bathtub-shaped depending on the parameter values. So, the new distribution is quite flexible and can be used effectively in analyzing lifetime data. It should be mentioned that is difficult (or even impossible) to determine analytically the parameter spaces corresponding to the increasing, decreasing, upside-down bathtub shaped (unimodal) or bathtub-shaped failure rate functions for the Γ-EW distribution.

Next, we use the general results derived in Section 2 to write the Γ-EW density as a linear combination of EW densities. Note that $g_{EW}(x)G_{EW}(x)^{a+m-1}=b\;\alpha\;\beta x^{\beta-1}e^{-\alpha x^{\beta}}\;(1-e^{-\alpha x^{\beta}})^{b(m+a)-1}$. Hence, it follows from (5) that

$$f(x)=\sum_{m=0}^{\infty}p_m\; g_{EW}(x; b(a+m),\alpha,\beta), \quad x>0, \tag{7}$$

where $p_m=p_m(a)=\varphi_m(a)/(a+m)$. So, the Γ-EW pdf can be expressed as an infinite linear combination of EW pdfs with parameters $b(a+m)$, α and β. From (7), we can obtain several mathematical properties of the Γ-EW distribution directly from those properties of the EW distribution. It illustrates the utility and applicability of the general expansion for the Γ-G density function derived in Section 2.

Table 1 Sub-models of the Γ-EW distribution.

Distribution	a	b	β	α
Exponentiated Weibull	1	–	–	–
Γ-exponentiated Rayleigh	–	–	2	–
Γ-exponentiated exponential	–	–	1	–
Γ-Rayleigh	–	1	2	–
Γ-exponential	–	1	1	–
Exponentiated exponential	1	–	1	–
Exponentiated Rayleigh	1	–	2	–
Weibull	1	1	–	–
Rayleigh	1	1	2	–
Exponential	1	1	1	–

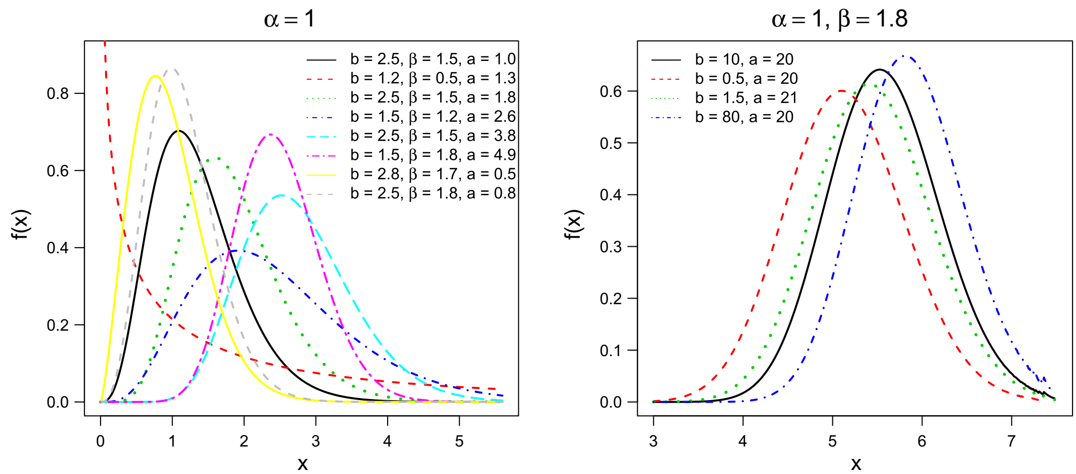

Figure 1 Plots of the Γ-EW density function for some parameter values.

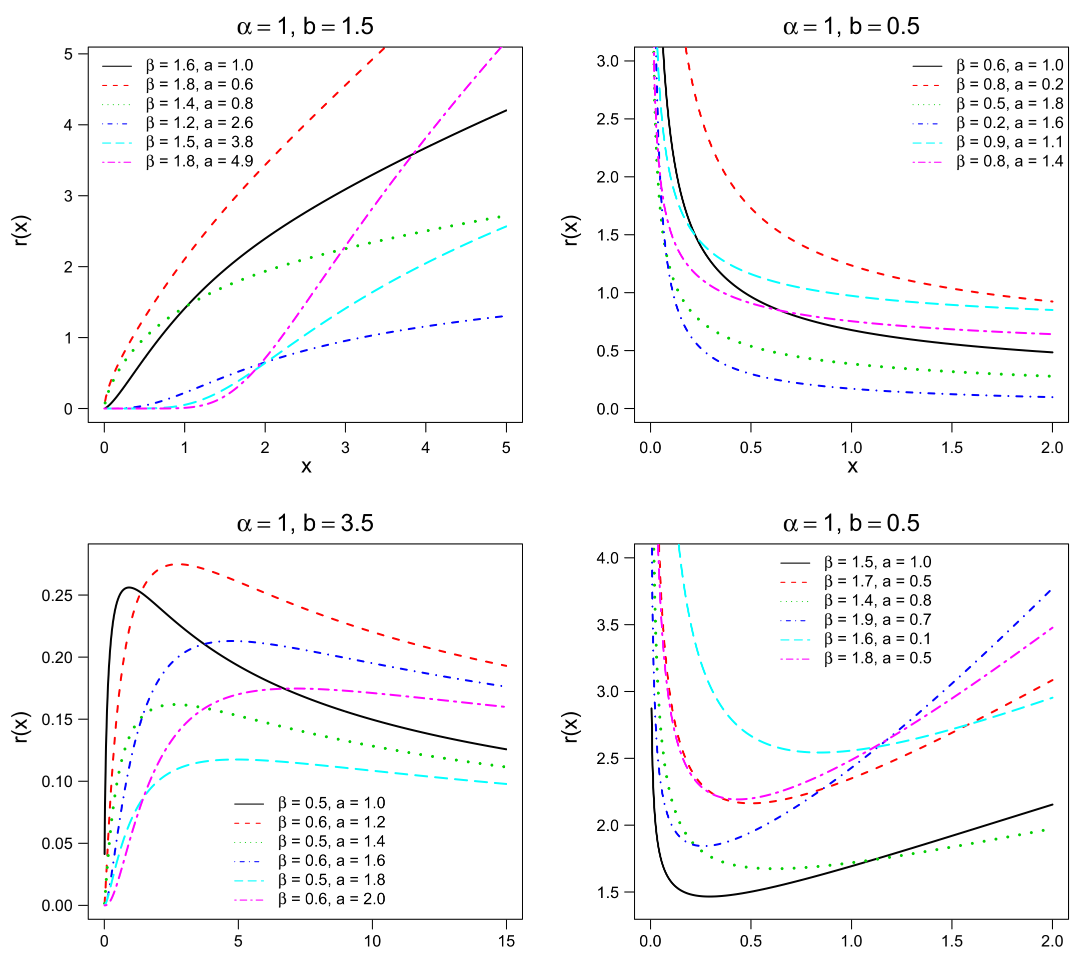

Figure 2 The Γ-EW failure rate function for some parameter values.

4. Numerical computation

Here, we briefly discuss the precision and convergence speed of (7). In order to evaluate numerically the polynomials $\psi_n(\cdot)$ as defined in (4), we use the procedure below to obtain the H_p^r numbers. This algorithm calculates $\left\{H_{p+1}^0,\ldots,H_{p+1}^p\right\}$ given $\left\{H_p^0,\ldots,H_p^{p-1}\right\}$, for $p>0$. The algorithm is as follows:

H_p^r numbers, $0\leqslant r\leqslant p-1$. *Remark*: $H_1^0=1$
input: $v^{(p)}=\left[H_p^0,H_p^1,,\ldots H_p^{p-1}\right]$
output: $v^{(p+1)}=\left[H_{p+1}^0,H_{p+1}^1,,\ldots H_{p+1}^p\right]$
if $p=1$ **do** $v^{(2)}=[3,1]$ and **stop**.
$v^{(p+1)}[p]\leftarrow 1$
$v^{(p+1)}[0]\leftarrow(2p+1)v^{(p)}[0]$
for $1\leqslant i\leqslant p-1$ **do**
  $v^{(p+1)}[i]\leftarrow(2p-i+1)v^{(p)}[i]+(p-i+1)v^{(p)}[i-1]$
end **for**
return $v^{(p+1)}$

This algorithm is very simple and similar to the Pascal's triangle. One can use the following R script [32] to compute the H_p^r numbers:

```
Stnumbers <- function(v1) {
  p <- length(v1)+1
  if (p==2) return(c(3,1))
  v2 <- c(rep(0,p-1),1)
  v2[1] <- v1[1]*(2*p-1)
  for (i in 2:length(v1))
    v2[i] <- (2*p-i)*v1[i]+(p-i+1)*v1[i-1]
  return(v2)
}
v1 <- c(3,1)
Stnumbers(v1)
15 10 1
```

Hence, to evaluate numerically the polynomials $\psi_n(\cdot)$ in (4), one can use the following R script:

```
psi <- function(x,p) {
  if (p==0) return(0.5)
  p <- p+1
  X <- rep(1,p)
  for(i in 2:p)
    X[i] <- -X[i-1]*(x+i)/(p+i)
  H <- 1
  while(length(H)<p) H <- Stnumbers(H)
  psi <- rev(H)*X
  psi <- (-1)^(p-1)*sum(psi)/factorial(p+1)
  return(psi)
}
psi(1,1)
0.2083333
```

Fig. 3 displays comparisons between the Γ-EW density function obtained from (6) and the Γ-EW density function obtained from expansion (7). Note that only the first 25 terms in (7) are needed to come a good agreement of (6) and (7). We observe that, in general, the expansion (7) for the Γ-EW density function provides very fast convergence with very good accuracy for $0<a<1$. For $a>1$, the expansion also converges fast and the speed of convergence becomes slow if x is far of 0. In this case, a large amount of terms is required to achieve good accuracy.

5. Moment properties

Let $X\sim\Gamma\text{-EW}(a,b,\alpha,\beta)$. The sth moment of X follows immediately from (7) and is given by $\mu_s'=\mathbb{E}(X^s)=\int_0^\infty x^s f(x)\mathrm{d}x=\sum_{m=0}^\infty p_m\int_0^\infty x^s g(x;b(a+m),\alpha,\beta)\mathrm{d}x$. We have the following lemma.

Lemma 1. *Let* $Z\sim\text{EW}(b,\alpha,\beta)$. *The sth moment of Z takes the form* $\mathbb{E}(Z^s)=b\ \alpha^{-s/\beta}\Gamma(s/\beta+1)\sum_{n=1}^\infty(1-b)_{n-1}/(n!\ n^{s/\beta})$, $s>-\beta$, *where* $(q)_m=q(q+1)\cdots(q+m-1)$, $q\in\mathbb{R}$ *and* $(q)_0=1$.

Proof. It follows from the expansion of the EW density as a linear combination of Weibull densities and from the moments of the Weibull distribution. □

We have the following proposition.

Proposition 3. *The sth moment of* $X\sim\Gamma\text{-EW}(a,b,\alpha,\beta)$ *takes the form* $\mu_s'=b\ \alpha^{-s/\beta}\Gamma(s/\beta+1)\ \sum_{m=0}^\infty\sum_{n=1}^\infty\varphi_m(a)\ (1-b(a+m))_{n-1}/(n!\ n^{s/\beta})$, $s>-\beta$.

Proof. By using the expansion (7) and Lemma 1 the result holds. □

The moment generating function (mgf) of the Γ-EW model, $M(t)$ say, can be directly obtained from (7) as $M(t)=\mathbb{E}(\mathrm{e}^{tX})=\int_0^\infty \mathrm{e}^{tx}f(x)\mathrm{d}x=\sum_{m=0}^\infty p_m M_m(t)$, where $M_m(t)$ is the mgf of the EW model with parameters $b(m+a)$, α and β. We have the following lemma.

Lemma 2. *Let* $Z\sim\text{EW}(b,\alpha,\beta)$. *The mgf of the Z takes the form* $\mathbb{E}(\mathrm{e}^{tZ})=b\sum_{n=1}^\infty\sum_{k=0}^\infty\Gamma(k/\beta+1)(1-b)_{n-1}\ t^k\ \alpha^{-k/\beta}/(n!\ k!)$, $\beta>1$, *where* $(q)_m=q(q+1)\cdots(q+m-1)$, $q\in\mathbb{R}$ *and* $(q)_0=1$.

Proof. It follows from the expansion of the EW density as a linear combination of Weibull densities and from the mgf of the Weibull distribution. □

We have the following proposition.

Proposition 4. *If* $X\sim\Gamma\text{-EW}(a,b,\alpha,\beta)$, *then* $M(t)=b\sum_{m=0}^\infty\sum_{n=1}^\infty\sum_{k=0}^\infty\Gamma(k/\beta+1)\varphi_m(a)\ (1-b)_{n-1}\ t^k\ \alpha^{-k/\beta}/\ (n!\ k!)$, $\beta>1$.

Proof. The result holds from the mgf of the EW model derived in Lemma 2. □

Let $\phi(t)$ be the characteristic function of $X\sim\Gamma\text{-EW}(a,b,\alpha,\beta)$. We have the following result.

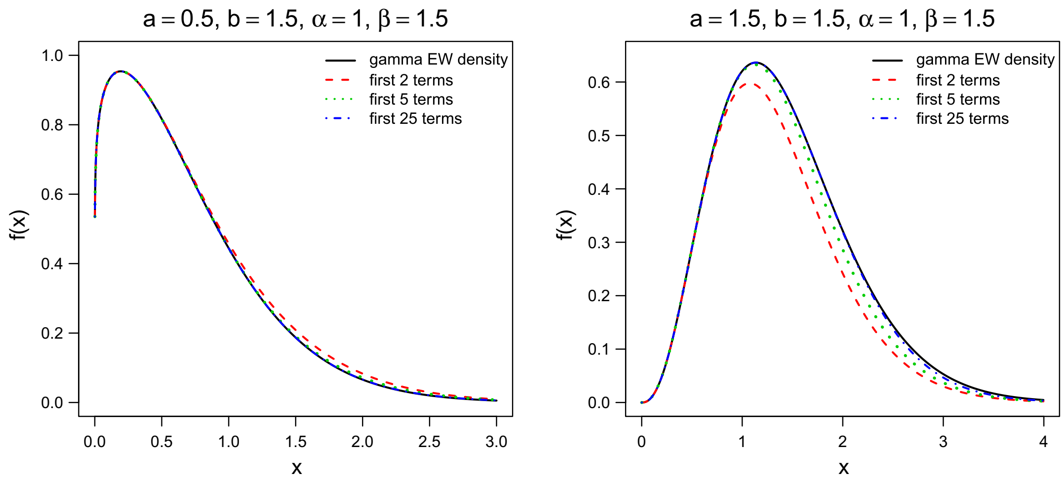

Figure 3 The Γ-EW density function computed from Eq. (6) and from expansion (7) over the first 2, 5 and 25 terms.

Theorem 2. *If* $X \sim \Gamma\text{-EW}(a, b, \alpha, \beta)$, *then* $\phi(t) = b\sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\sum_{k=0}^{\infty}\Gamma(k/\beta+1)\varphi_m(a)\ (1-b)_{n-1}\ (\mathrm{i}t)^k\ \alpha^{-k/\beta}/(n!\ k!)$.

Proof. The result follows from the characteristic function of the EW model. □

One of the popular entropy measure is the Shannon entropy. It plays a similar role as the kurtosis measure in comparing the shapes of various densities and measuring heaviness of tails. It is defined by $\Im_{Sh} = \mathbb{E}\{-\log[f(X)]\} = -\int_{-\infty}^{\infty} f(x)\log[f(x)]\mathrm{d}x$. We have the following theorem.

Theorem 3. *For the general density function (1), the Shannon entropy can be expressed as* $\Im_{Sh} = \log[\Gamma(a)] - (a-1)\Psi(a) - \mathbb{E}_Z\{\log[g(Q_G(\mathrm{e}^{-V}))]\}$, *where* $\Psi(\cdot)$ *is the digamma function,* $Q_G(\cdot)$ *is the quantile function of the G distribution and the expectation* $\mathbb{E}_Z\{\cdot\}$ *is calculated with respect to the random variable* $Z \sim Gamma(a, 1)$.

Table 2 Empirical means and $\sqrt{\text{MSE}}$ in parentheses; $\alpha = 1$.

n	$\hat{a}$	$\hat{b}$	$\hat{\alpha}$	$\hat{\beta}$
$a = 1.5,\ b = 1.0$ *and* $\beta = 1.8$				
250	1.804(1.695)	1.450(1.610)	1.248(0.951)	1.704(0.515)
350	1.739(1.622)	1.419(1.487)	1.203(0.894)	1.679(0.500)
500	1.767(1.590)	1.311(1.265)	1.203(0.863)	1.676(0.471)
$a = 2.5,\ b = 1.5$ *and* $\beta = 1.2$				
250	2.622(2.100)	2.807(4.161)	1.249(1.132)	1.101(0.423)
350	2.662(2.106)	2.490(3.274)	1.236(1.076)	1.092(0.399)
500	2.559(1.963)	2.313(2.693)	1.158(0.903)	1.094(0.378)
$a = 2.0,\ b = 1.2$ *and* $\beta = 1.0$				
250	2.289(1.988)	2.024(2.685)	1.282(1.066)	0.930(0.309)
350	2.283(1.955)	1.822(2.099)	1.241(0.969)	0.932(0.284)
500	2.174(1.729)	1.715(1.891)	1.168(0.841)	0.942(0.268)
$a = 1.0,\ b = 1.5$ *and* $\beta = 2.5$				
250	1.149(0.856)	1.853(1.914)	1.146(0.767)	2.167(0.948)
350	1.131(0.782)	1.755(1.683)	1.104(0.637)	2.159(0.915)
500	1.097(0.756)	1.680(1.576)	1.071(0.596)	2.091(0.955)

Proof. It follows immediately. □

We have the following corollary.

Corollary 1. *If* $X \sim \Gamma\text{-EW}(a, b, \alpha, \beta)$, *then the Shannon entropy becomes* $\Im_{Sh} = \log[\Gamma(a)] - (a-1)\Psi(a) - \log(b\ \beta) - \ \beta^{-1}\log(\alpha) + a\ b^{-1}(b-1) - \beta^{-1}(\beta-1) + a\ \beta^{-1}(\beta-1) + \sum_{k=1}^{\infty}k^{-1}(1+k/b)^{-a} - b^a\ \beta^{-1}(\beta-1)\sum_{n=0}^{\infty}\psi_n(n)(n+b)^{-a}$, *where* $\psi_n(\cdot)$ *was defined in Section 2, and* $a \geqslant 1$.

Proof. It follows from Theorem 3. □

Several other properties like reliability, mean deviations, other kinds of entropy, etc., can be derived in a similar fashion, but we consider only the above properties to save space.

6. Maximum likelihood estimation

Let $x_1,\ldots,x_n$ be a random sample of size n of the Γ-EW(a,b,α,β) distribution. The log-likelihood function for $\boldsymbol{\theta} = (a,b,\alpha,\beta)^\top$ based on a given random sample is $\ell(\boldsymbol{\theta}) = n\log(b\,\alpha\,\beta) - n\log[\Gamma(a)] + (\beta-1)\sum_{i=1}^{n}\log(x_i) - \sum_{i=1}^{n}v_i + (b-1)\sum_{i=1}^{n}\log(1-z_i) + (a-1)\sum_{i=1}^{n}\log\{-\log[S(x_i)]\}$, where $v_i = \alpha\,x_i^{\beta}$, $z_i = \exp(-v_i)$, $G(x_i) = (1-z_i)^b$ and $S(x_i) = 1 - G(x_i)$, for $i = 1,\ldots,n$. The maximum likelihood estimates of the unknown parameters are obtained by maximizing the log-likelihood function $\ell(\boldsymbol{\theta})$ with respect to $\boldsymbol{\theta}$. The likelihood equations, which are obtained from the partial derivatives of $\ell(\boldsymbol{\theta})$ with respect to the parameters, become

$$\frac{\partial\ell(\boldsymbol{\theta})}{\partial a} = -n\ \Psi(a) + \sum_{i=1}^{n}\log\{-\log[S(x_i)]\},$$

$$\frac{\partial\ell(\boldsymbol{\theta})}{\partial b} = \frac{n}{b} + \frac{1}{b}\sum_{i=1}^{n}\log[G(x_i)] - \frac{(a-1)}{b}\sum_{i=1}^{n}\frac{G(x_i)\log[G(x_i)]}{S(x_i)\log[S(x_i)]},$$

$$\frac{\partial\ell(\boldsymbol{\theta})}{\partial \alpha} = \frac{n}{\alpha} - \frac{1}{\alpha}\sum_{i=1}^{n}v_i + \frac{(b-1)}{\alpha}\sum_{i=1}^{n}\frac{v_i\ z_i}{1-z_i} - \frac{b\ (a-1)}{\alpha}\sum_{i=1}^{n}\frac{v_i\ z_i\ (1-z_i)^{b-1}}{S(x_i)\log[S(x_i)]},$$

$$\frac{\partial \ell(\boldsymbol{\theta})}{\partial \beta} = \frac{n}{\beta} + \sum_{i=1}^{n}(1-v_i)\log(x_i) + (b-1)\sum_{i=1}^{n}\frac{v_i\ z_i \log(x_i)}{1-z_i} - b\ (a-1)\sum_{i=1}^{n}\frac{v_i\ z_i\ (1-z_i)^{b-1}\log(x_i)}{S(x_i)\log[S(x_i)]},$$

where $\Psi(\cdot)$ is the digamma function. The maximum likelihood estimator $\hat{\boldsymbol{\theta}} = (\hat{a}, \hat{b}, \hat{\alpha}, \hat{\beta})^\top$ of $\boldsymbol{\theta} = (a, b, \alpha, ,\beta)^\top$ can be obtained by solving simultaneously the likelihood equations $\partial\ell(\boldsymbol{\theta})/\partial a = \partial\ell(\boldsymbol{\theta})/\partial b = \partial\ell(\boldsymbol{\theta})/\partial\alpha = \partial\ell(\boldsymbol{\theta})/\partial\beta = 0$. There is no closed-form expression for the maximum likelihood estimator and its computation has to be performed numerically using a nonlinear optimization algorithm. The observed information matrix used for computing asymptotic confidence intervals for the parameters a, b, α and β can be determined numerically from standard maximization routines, which now provide the observed information matrix as part of their output; e.g., one can use the R functions optim or nlm, the Ox function MaxBFGS, the SAS procedure NLMixed, among others, to compute the observed information matrix numerically.

Next, a small Monte Carlo simulation is conducted to evaluate the estimations of the Γ-EW distribution parameters. The simulation was performed using the Ox matrix programming language. The number of Monte Carlo replications was $R = 2,000$. For maximizing the log-likelihood function, we use the subroutine MaxBFGS with analytical derivatives. The evaluation of point estimation was performed based on the following quantities for each sample size: the empirical mean and the root mean squared error $(\sqrt{\text{MSE}})$, where MSE is the mean squared error estimated from R Monte Carlo replications. We set the sample size at $n = 250, 350$ and 500. We consider different values for the shape parameters a, b and β, whereas the scale parameter α was fixed at 1.0 without loss of generality. It can be seen from Table 2 that the estimates are quite stable and, more important, are close to the true values for the sample sizes considered. Additionally, as the sample size increases, the $\sqrt{\text{MSE}}$ decreases, as expected.

7. Real data illustration

In this section, we present an application of the proposed Γ-EW distribution to real data for illustrative purposes. We also consider some sub-models of the new four-parameter Γ-EW distribution to fit this real data set for the sake of comparison: Weibull distribution, EE distribution, gamma Weibull $(\Gamma - \text{W})$ distribution, gamma exponentiated exponential (Γ-EE) distribution, and EW distribution. Additionally, three recent four-parameter generalizations of the Weibull distribution will also be considered to fit these data: Γ2-EW distribution [27], beta Weibull (BW) distribution [34], and Kumaraswamy Weibull (KwW) distribution [21]. We shall consider the real data set presented by [33], which represents the remission times (in months) of a random sample of 128 bladder cancer patients. For each model, we estimate the unknown parameters by the maximum likelihood method. Table 3 lists the maximum likelihood estimates (and the corresponding standard errors in parentheses) of the unknown parameters of all lifetime models for the remission times data. All the computations were performed using the Ox matrix programming language.

Now, we shall apply formal goodness-of-fit tests in order to verify which distribution fits better these real data sets. We consider the Cramér–von Mises (W^*) and Anderson–Darling (A^*) statistics, which are described in details by [35]. In general, the smaller the values of these statistics, the better the fit to the data. The statistics W^* and A^* for all the models are listed in Table 4. Note that the new Γ-EW distribution outperforms all their sub-models as well as the four-parameter Γ2-EW, BW and KwW distributions. Notice that the Γ-EW distribution is clearly a competitive model for the Γ2-EW, BW and KwW distributions, since they have the same number of parameters. Therefore, the new model may be an interesting alternative to the other models available in the literature for modeling positive real data.

Table 3 Maximum likelihood estimates (standard errors in parentheses).

Model	Estimates			
Γ-EW(a,b,α,β)	0.4243	6.7574	0.4450	0.6261
	(0.4919)	(9.5571)	(0.2730)	(0.1612)
Γ2-EW(a,b,α,β)	0.7590	2.5262	0.5023	0.6884
	(1.0878)	(1.6978)	(0.3452)	(0.2209)
BW(a,b,α,β)	2.7348	0.9083	0.4697	0.6661
	(1.6355)	(1.5443)	(0.3728)	(0.2495)
KwW(a,b,α,β)	4.1178	2.9414	0.4949	0.4589
	(5.8731)	(8.2214)	(0.5063)	(0.5193)
EW(b,α,β)	2.7960	0.4537	0.6544	
	(1.2603)	(0.2384)	(0.1342)	
Γ-W(a,α,β)	3.7479	1.3099	0.5201	
	(2.6699)	(1.5159)	(0.1984)	
Γ-EE(a,b,α)	0.5915	2.0405	0.1078	
	(0.3942)	(1.3036)	(0.0182)	
EE(b,α)	1.2180	0.1212		
	(0.1486)	(0.0136)		
Weibull(α,β)	0.0939	1.0478		
	(0.0191)	(0.0676)		

Table 4 Statistics W^* and A^*.

Model	W^*	A^*
Γ-EW	**0.03945**	**0.25992**
Γ2-EW	0.04338	0.28654
BW	0.04362	0.28825
KwW	0.04149	0.27322
EW	0.04367	0.28848
Γ-W	0.04788	0.31425
Γ-EE	0.10540	0.63199
EE	0.11221	0.67412
Weibull	0.13137	0.78648

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Appendix A

We will show that the expansion for the Γ2-EW density function derived in [27] cannot be used to compute some general properties of the Γ2-EW model as, for example, moments, generating function, etc. The Γ2-EW density can be expressed as $f(x) = (b^a \alpha\, \beta/\Gamma(a))\, x^{\beta-1} y\, (1-y)^{b-1}[-\log(1-y)]^{\delta-1}$, where $0 < y = \mathrm{e}^{-\alpha x^\beta} < 1$ for $x > 0$. We have that $-\log(1-y)$ can be expanded (for $0 < y < 1$) in the form $-\log(1-y) = y(1 + y/2 + y^2/3 + y^3/4 + \cdots) = y[1 + y\sum_{s=0}^{\infty} y^s/(s+2)]$. Hence, we obtain

$$[-\log(1-y)]^{\delta-1} = y^{\delta-1}\left[1 + y\sum_{s=0}^{\infty}\frac{y^s}{s+2}\right]^{\delta-1}, \qquad \delta > 0.$$

We have that the binomial expansion $(1+z)^{\delta-1} = \sum_{m=0}^{\infty}\binom{\delta-1}{m} z^m$ is convergent if and only if $|x| < 1$; that is, the expansion

$$\left[1 + y\sum_{s=0}^{\infty}\frac{y^s}{s+2}\right]^{\delta-1} = \sum_{m=0}^{\infty}\binom{\delta-1}{m} y^m \left(\sum_{s=0}^{\infty}\frac{y^s}{s+2}\right)^m,$$

which was used in [27], will be convergent if and only if

$$0 < y\sum_{s=0}^{\infty}\frac{y^s}{s+2} < 1 \qquad (8)$$

for all values of $y \in (0,1)$, since $0 < y = \mathrm{e}^{-\alpha x^\beta} < 1$ for $x > 0$. However, the inequality (8) is not satisfied for all values of $y \in (0,1)$. This can be proved noting that $y^{-1}[-\log(1-y)] - 1 = y\sum_{s=0}^{\infty} y^s/(s+2)$. So, we have to show that the inequality $0 < y^{-1}[-\log(1-y)] - 1 < 1$ is not valid for all values of $y \in (0,1)$. After some algebra, we have that $\mathrm{e}^{-y} > 1 - y$ and $1 - y > \mathrm{e}^{-2y}$, and the solution for these system of equations is in the interval $0 < y < K$, where K is the solution of the equation $1 - y = \mathrm{e}^{-2y}$ and it is given by $K \approx 0.7968121$. It implies that the inequality (8) is not validy for all values of $0 < y = \mathrm{e}^{-\alpha x^\beta} < 1$ (for $x > 0$) and therefore the expansion derived in [27] for the Γ2-EW pdf is not valid (convergent) for all values of $y \in (0,1)$.

References

[1] G.S. Mudholkar, D.K. Srivastava, Exponentiated Weibull family for analyzing bathtub failure-rate data, IEEE Trans. Reliab. 42 (1993) 299–302.
[2] S. Nadarajah, G.M. Cordeiro, E.M.M. Ortega, The exponentiated Weibull distribution: a survey, Stat. Papers 54 (2013) 839–877.
[3] N. Eugene, C. Lee, F. Famoye, Beta-normal distribution and its applications, Commun. Stat. – Theory Methods 31 (2002) 497–512.
[4] P. Kumaraswamy, Generalized probability density-function for double-bounded random-processes, J. Hydrol. 46 (1980) 79–88.
[5] G.M. Cordeiro, M. de Castro, A new family of generalized distributions, J. Stat. Comput. Simul. 81 (2011) 883–898.
[6] K. Zografos, N. Balakrishnan, On families of beta-and generalized gamma-generated distribution and associate inference, Stat. Methodol. 6 (2009) 344–362.
[7] M.M. Ristić, N. Balakrishnan, The gamma-exponentiated exponential distribution, J. Stat. Comput. Simul. 82 (2012) 1191–1206.
[8] J.B. McDonald, Some generalized functions for the size distribution of income, Econometrica 52 (1984) 647–663.
[9] C. Alexander, G.M. Cordeiro, E.M.M. Ortega, J.M. Sarabia, Generalized beta-generated distributions, Comput. Stat. Data Anal. 56 (2012) 1880–1897.
[10] A.Y. Alzaatreh, C. Lee, F. Famoye, A new method for generating families of continuous distributions, METRON 71 (2013) 63–79.
[11] C. Lee, F. Famoye, A.Y. Alzaatreh, Methods for generating families of univariate continuous distributions in the recent decades, WIREs Comput. Stat. 5 (2013) 219–238.
[12] F. Famoye, C. Lee, O. Olumolade, The beta-Weibull distribution, J. Stat. Theory Appl. 4 (2005) 121–136.
[13] A. Akinsete, F. Famoye, C. Lee, The beta-Pareto distribution, Statistics 42 (2008) 547–563.
[14] W. Barreto–Souza, A. Santos, G.M. Cordeiro, The beta generalized exponential distribution, J. Stat. Comput. Simul. 80 (2010) 159–172.
[15] R.R. Pescim, C.G.B. Demétrio, G.M. Cordeiro, E.M.M. Ortega, M.R. Urbano, The beta generalized half-normal distribution, Comput. Stat. Data Anal. 54 (2010) 945–957.
[16] G.O. Silva, E.M.M. Ortega, G.M. Cordeiro, The beta modified Weibull distribution, Lifetime Data Anal. 16 (2010) 409–430.
[17] P.F. Paranaíba, E.M.M. Ortega, G.M. Cordeiro, R.R. Pescim, The beta Burr XII distribution with application to lifetime data, Comput. Stat. Data Anal. 55 (2011) 1118–1136.
[18] G.M. Cordeiro, A.J. Lemonte, The β-Birnbaum–Saunders distribution: an improved distribution for fatigue life modeling, Comput. Stat. Data Anal. 55 (2011) 1445–1461.
[19] G.M. Cordeiro, A.J. Lemonte, The beta Laplace distribution, Stat. Probab. Lett. 81 (2011) 973–982.
[20] G.M. Cordeiro, A.J. Lemonte, The beta-half-Cauchy distribution, J. Probab. Stat. 2011 (2011) 1–18.
[21] G.M. Cordeiro, E.M.M. Ortega, S. Nadarajah, The Kumaraswamy Weibull distribution with application to failure data, J. Franklin Inst. 347 (2011) 1399–1429.
[22] P.F. Paranaíba, E.M.M. Ortega, G.M. Cordeiro, M. de Pascoa, The Kumaraswamy Burr XII distribution: theory and practice, J. Stat. Comput. Simul. 82 (2012) 1–27.
[23] G.M. Cordeiro, S. Nadarajah, E.M.M. Ortega, The Kumaraswamy Gumbel distribution, Stat. Methods Appl. 21 (2012) 139–168.
[24] G.M. Cordeiro, A.J. Lemonte, The McDonald inverted beta distribution, J. Franklin Inst. 349 (2012) 1174–1197.
[25] A.Y. Alzaatreh, F. Famoye, C. Lee, Gamma-Pareto distribution and its applications, J. Modern Appl. Stat. Methods 11 (2012) 78–94.
[26] A.Y. Alzaatreh, F. Famoye, C. Lee, The gamma-normal distribution: properties and applications, Comput. Stat. Data Anal. 69 (2014) 67–80.
[27] L.G.B. Pinho, G.M. Cordeiro, J.S. Nobre, The gamma-exponentiated Weibull distribution, J. Stat. Theory Appl. 11 (2012) 379–395.
[28] N. Nielsen, Handbuch der Theorie der Gammafunktion, Leipzig, 1906.
[29] M. Ward, The representation of Stirling's numbers and Stirling's polynomials as sums of factorial, Am. J. Math. 56 (1934) 87–95.
[30] P. Flajonet, A. Odlyzko, Singularity analysis of generating function, SIAM: SIAM J. Discr. Math. 3 (1990) 216–240.
[31] P. Flajonet, R. Sedgewick, Analytic Combinatorics, Cambridge University Press, 2009.
[32] R Development Core Team, R: A Language and Environment for Statistical Computing, R Foundation for Statistical Computing, Vienna, Austria, 2012.
[33] E.T. Lee, J.W. Wang, Statistical Methods for Survival Data Analysis, third ed., Wiley, New York, 2003.

[34] C. Lee, F. Famoye, O. Olumolade, Beta-Weibull distribution: some properties and applications to censored data, J. Modern Appl. Stat. Methods 6 (2007) 173–186.

[35] G. Chen, N. Balakrishnan, A general purpose approximate goodness-of-fit test, J. Qual. Technol. 27 (1995) 154–161.